then the coefficients $H_1, H_2, \ldots$ are the densities of conservation laws [6]. Thus, the Lax equations (3), and hence system (2), are hydrodynamically integrable.

The hodograph method is a local procedure which does not provide an answer to the question as to whether the solution of system (2) exists for all t.

It is known that the one-dimensional Lax equation (5) with an initial condition periodic in x is not uniquely determined on the entire range of t. Apparently, owing to the absence of dispersion, for any N there exist initial conditions for the system such that the solution is only defined on a finite interval of time.

References

1. E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge Univ. Press, Cambridge (1937).
2. G. D. Birkhoff, *Dynamical Systems* Colloquium Publ., Vol. 9, Amer. Math. Soc. (1927).
3. V. V. Kozlov, *Mat. Zametki* [*Math. Notes*], **45**, No. 4, 46–52 (1989).
4. B. A. Dubrovin and S. P. Novikov, *Dokl. Akad. Nauk SSSR* [*Soviet Math. Dokl.*], **270**, No. 4, 781–785 (1983).
5. S. P. Tsarev, *Dokl. Akad. Nauk SSSR* [*Soviet Math. Dokl.*], **282**, No. 3, 534–537 (1985).
6. Y. Kodama and J. Gibbons, *Phys. Lett. A.*, **135**, No. 3, 167–170 (1989).

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Boundedness of the Solutions of a Nonlinear Elliptic System in a Cylindrical Domain

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Key words: nonlinear elliptic systems, bounded solutions, boundary value problems.

In the half-cylinder $\Omega_+ = \mathbb{R}_+ \times \omega$, where ω is a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary, we consider the elliptic system

$$\begin{aligned} &a(\partial_t^2 u + \Delta u) + \gamma \partial_t u - f(u,t) = g(t), \\ &u\big|_{t=0} = u_0, \qquad u\big|_{\partial\omega} = 0. \end{aligned} \tag{1}$$

Here $u = u(t,x) = (u_1, \ldots, u_k)$, $g = g(t,x)$, and $f(u,t)$ are vector functions, $(t,x) \in \Omega_+$, Δ is the Laplace operator with respect to the variable $x = (x_1, \ldots, x_n)$, and γ and a are constant matrices ($\gamma, a \in L(\mathbb{R}^k, \mathbb{R}^k)$); moreover, $a = a^* > 0$. It is assumed that g belongs to the space $\left[L_p^{\mathrm{loc}}(\mathbb{R}_+, L_p(\omega))\right]^k$ and, for some $p > n+1$, has a finite norm

$$|g|_a = \sup_{t \in \mathbb{R}_+} \|g, \Omega_t\|_{L_p} < \infty, \qquad \text{where} \quad \Omega_t = [t, t+1] \times \omega.$$

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The nonlinear function f satisfies the following conditions:

$$\begin{aligned} &f \in C(\mathbb{R}^k \times \mathbb{R}_+, \mathbb{R}^k), \\ &f(u,t) . u \geq -C_1 + C_2|u|^{2+\varepsilon}, \qquad \varepsilon > 0, \quad C_2 > 0, \\ &|f(u,t)| \leq Q(|u|), \qquad \text{where} \quad Q \colon \mathbb{R}_+ \to \mathbb{R}_+ \text{ is a monotone function.} \end{aligned} \tag{2}$$

Here and in the following, by $u . v$ we denote the inner product on $\mathbb{R}^k$. The initial condition u_0 is assumed to belong to the space

$$V_0 = \left[H_{2-1/p,p}(\omega) \cap H^0_{1,p}(\omega)\right]^k \qquad (u_0 \in V_0)$$

(see [1]).

A *solution* of problem (1) is understood as a function u that belongs to the space $\left[H_{2,p}(\Omega_t)\right]^k$ for all $t > 0$, where $\Omega_t = [t, t+1] \times \omega$, and satisfies (1) in $\left[L_p^{\mathrm{loc}}(\Omega_+)\right]^k$. (Here and in the following by $H_{l,p}$ we denote the Sobolev space of functions whose generalized derivatives of order $\leq l$ belong to L_p [2].) The set of all solutions of problem (1) in this sense will be denoted by $\mathcal{V}(u_0)$.

A *bounded solution* of problem (1) is understood as a solution, u, that additionally satisfies the condition

$$\|u\|_a = \sup_{t \in \mathbb{R}_+} \|u, \Omega_t\|_{H_{2,p}} < \infty.$$

The set of all bounded solutions of problem (1) will be denoted by $\mathcal{V}_a(u_0)$.

Bounded solutions of problem (1) under various conditions imposed on the nonlinear function f were studied by many authors [3–5].

It was proved in [5] that under the cited conditions the set $\mathcal{V}_a(u_0)$ is not empty and the estimate

$$\|u, \Omega_T\|_{2,p} \leq R(\|u_0\|_{V_0}) e^{-\alpha T} + R(|g|_a) \tag{3}$$

is valid, where $\alpha > 0$ and R is a monotone function independent of u_0.

The main result of the present paper is as follows.

Theorem 1. *Suppose that the above-mentioned conditions are satisfied. Then $\mathcal{V}(u_0) = \mathcal{V}_a(u_0)$ and the estimate*

$$\|u, \Omega_T\|_{2,p} \leq R(\|u_0\|_{V_0}) \chi(T_0 - T) + R(|g|_a), \tag{4}$$

stronger than (3), is valid. Here $\chi(z)$ is the Heaviside function, which is equal to zero for $z \leq 0$ and is equal to 1 for $z > 0$, and $T_0 = T_0(f, |g|_a) > 0$ is a number independent of u_0.

The proof of Theorem 1 is based on a series of auxiliary statements, which are of some interest in themselves.

Lemma 1. *Let $u_0 \in \mathcal{V}(u_0)$ and $y(t) = (|u|^p, 1)$, where (u, v) is the inner product on $L_2(\omega)$ and $|u| = [au . u]^{1/2}$. Then $y''(t) \in L_1^{\mathrm{loc}}(\mathbb{R}_+)$ and*

$$y''(t) - \beta^2 y(t) |y(t)|^\alpha \geq h(t) \tag{5}$$

for some $\alpha > 0$ and $\beta > 0$ and for $h(t) = -C(1 + \|g(t)\|^p_{0,p})$.

Proof. Consider the inner product of Eq. (1) by $u|u|^{p-2}$ in $\left[L_2(\omega)\right]^k$. After simple manipulations, we obtain

$$\begin{aligned} &\frac{1}{p}\partial_t^2 (|u|^p, 1) - \frac{4(p-2)}{p^2}\left(\left\|\partial_t |u|^{p/2}\right\|^2_{0,2} + \left\|\nabla |u|^{p/2}\right\|^2_{0,2}\right) \\ &\quad = -(\gamma \partial_t u, u|u|^{p-2}) + (g, u|u|^{p-2}) + (f(u), u|u|^{p-2}) + (|\partial_t u|^2 + |\nabla u|^2, |u|^{p-2}). \end{aligned} \tag{6}$$

Let us obtain a lower bound for the right-hand side of (6) with the help of the Hölder inequality and condition (2). Then we obtain

$$\partial_t^2(|u(t)|^p, 1) - C(|u(t)|^{p+\varepsilon}, 1) \geq h(t). \tag{7}$$

To estimate the second summand in (7), we use the Jensen inequality

$$(|u(t)|^{p+\varepsilon}, 1) = \int_\omega |u(t)|^{p+\varepsilon}\,dx = \int_\omega \left[|u(t)|^p\right]^{1+\varepsilon/p} dx \geq C(\omega)\left(\int_\omega |u(t)|^p\,dx\right)^{1+\varepsilon/p}. \tag{8}$$

On substituting (8) into (7), we arrive at the estimate (5). □

Lemma 2. *Suppose that $y(t) \geq 0$, $y'' \in L_1^{\mathrm{loc}}(\mathbb{R}_+)$, and inequality (5) is satisfied. Then*

$$y(t) \leq C\left(y(0)\chi(t_0 - t) + 1 + \sup_{T\geq 0}\int_T^{T+1} |h(t)|\,dt\right),$$

where C and t_0 are independent of y.

The purely technical proof of this statement is omitted.

Corollary. *Let $u \in \mathcal{V}(u_0)$. Then the estimate*

$$\|u(t)\|_{0,p} \leq C_1\left(\chi(t_0 - t)\|u_0\|_{V_0} + 1 + |g|_a + 1\right), \qquad C_1 > 0,$$

is valid.

Lemma 3. *Let $u \in \mathcal{V}(u_0)$. Then for $T \geq 1$ one has the estimate*

$$\|u, \Omega_T\|_{0,\infty} \leq C\left(1 + \|u, \Omega_{T-1,T+2}\|_{0,p} + |g|_a\right), \tag{9}$$

where $\Omega_{T-1,T+2} = [T-1, T+2] \times \omega$.

Proof. Consider the inner product of (1) by $u|u|^{q-2}$ in $\left[L_2(\omega)\right]^k$ $(q \geq 2)$. Arguing just as in the derivation of (6) and (7), we obtain

$$\left\|\partial_t|u|^{q/2}\right\|_{0,2}^2 + \left\|\nabla|u|^{q/2}\right\|_{0,2}^2 \leq Cq\left(\partial_t^2(|u|^q, 1) + (g, u|u|^{q-2}) + (|u|^p, 1)\right). \tag{10}$$

Now consider the family of domains $G^m = [T - 2^{-m}, T + 1 + 2^{-m}]$, $m \geq 0$, $\Omega^m = G^m \times \omega$, and a family of cutoff functions $\varphi_m(t) \in C_0^\infty(\mathbb{R})$ such that $\varphi_m(t) \equiv 1$ for $t \in G^{m+1}$ and $\varphi_m(t) \equiv 0$ for $t \notin G^m$. Obviously, the family φ_m can be chosen so that $|\varphi_m''(t)| \leq C2^{2m}$, where C is independent of m. We multiply (10) by $\varphi_m(t) \geq 0$, integrate with respect to t, and use the elementary formula

$$\varphi_m\partial_t^2 F = \partial_t(\varphi_m\partial_t F) - \partial_t(\varphi_m' F) + \varphi_m'' F, \qquad F = (|u|^q, 1),$$

to obtain the estimate

$$\left\||u|^{q/2}, \Omega^{m+1}\right\|_{1,2}^2 \leq C_1 q 2^{2m}\left(1 + \|u, \Omega^m\|_{0,q}^q + \int_{\Omega^m} g \,.\, u|u|^{q-2}\,dx\,dt\right). \tag{11}$$

By Sobolev's embedding theorem [1], we have

$$\|u, \Omega^{m+1}\|_{q(n+1)/(n-1)}^q \leq C\left\||u|^{q/2}, \Omega^{m+1}\right\|_{1,2}^2. \tag{12}$$

Estimating the last integral in (11) with the help of the Hölder inequality, we obtain

$$\int_{\Omega^m} g \,.\, u|u|^{q-2}\,dx\,dt \le |g|_a^q + \|u, \Omega^m\|_{0,q(n+1)/n}^q. \tag{13}$$

We substitute the estimates (12) and (13) into inequality (11) and find that

$$\|u, \Omega^{m+1}\|_{q(n+1)/(n-1)}^q \le C_2 q 2^{2m}\Big(\|u, \Omega^m\|_{0,q(n+1)/n}^q + \big(1+|g|_a\big)^q\Big). \tag{14}$$

Let

$$W_l^m(u) = \max\Big\{\|u, \Omega^m\|_{0,l}, \big(1+|g|_a\big)\Big\}. \tag{15}$$

In view of (15), extracting the qth root of (14) yields

$$W_{l(\delta+1)}^{m+1}(u) \le [C_3 l 2^{2m}]^{(n+1)/(ln)} W_l^m(u). \tag{16}$$

Here $l = nq/(n+1)$ and $\delta = 1(n-1)$.

Consider the sequence $l_m = p(\delta+1)^m$. Then (16) implies the estimate

$$W_{l_{m+1}}^{m+1}(u) \le A_m W_{l_m}^m(u), \qquad A_m = [C_3 l_m 2^{2m}]^{(1+1/n)/l_m}. \tag{17}$$

The mth iteration of (17) gives

$$\|u, \Omega^m\|_{0,l_m} \le P_m\big(1+|g|_a + \|u, \Omega^0\|_{0,p}\big), \qquad P_m = A_0 \times \cdots \times A_m.$$

One can readily verify that

$$\lim_{m\to\infty} P_m = C < \infty;$$

consequently,

$$\|u, \Omega_T\|_{0,l_m} \le C\big(1+|g|_a + \|u, \Omega^0\|_{0,p}\big). \tag{18}$$

Since $H_{2,p}(\Omega_T) \subset C(\Omega_T)$ by the embedding theorem, we obtain the estimate (9) by passing to the limit in (18). □

Corollary. *Let $u \in \mathcal{V}(u_0)$. Then one has the estimate*

$$\|u, \Omega_T\|_{0,\infty} \le C\big(\chi(t_0 - T)\|u_0\|_{V_0} + 1 + |g|_a\big). \tag{19}$$

Lemma 4. *Let $u \in \mathcal{V}(u_0)$. Then the estimate*

$$\|u, \Omega_T\|_{2,p} \le R\big(\|u, \Omega_{T-1,T+2}\|_{0,\infty}\big) + C|g|_a \tag{20}$$

is valid, where $R\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a monotone function.

The proof can be found in [5].

Proof of Theorem 1. Let $u \in \mathcal{V}(u_0)$. Then the substitution of the estimate (19) into (20) gives the estimate (4). Consequently, $\mathcal{V}(u_0) \subset \mathcal{V}_a(u_0)$. The opposite inclusion is obvious. The proof of Theorem 1 is complete. □

In conclusion, let us present an analog of the estimate (4) for an elliptic system in a bounded cylinder.

Theorem 2. *Let $\Omega = [0, M] \times \omega$, and let $u \in [H_{2,p}(\Omega)]^k$ be a solution of the problem*

$$a(\partial_t^2 u + \Delta u) + \gamma\partial_t u - f(u, t) = g(t),$$
$$u\big|_{t=0} = u_0, \quad u\big|_{\partial\omega} = 0, \quad u\big|_{t=M} = u_M.$$

Then one has the estimate

$$\|u, \Omega_T\|_{2,p} \le R(\|u_0\|_{V_0})\chi(T_0 - T) + R(\|u_M\|_{V_0})\chi(T - M + T_0) + R(|g|_a).$$

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References

1. H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, Akademische Verlag Geest und Portig, Leipzig (1983).
2. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics* [in Russian], Nauka, Moscow (1988).
3. A. Calsina, X. Mora, and J. Sola-Morales, *J. Differential Equations.*, **102**, 244–304 (1993).
4. A. V. Babin, *Izv. Ross. Akad. Nauk Ser. Mat.* [*Russian Acad. Sci. Izv. Math.*], **58**, No. 2, 3–18 (1994).
5. M. I. Vishik and S. V. Zelik, *Mat. Sb.* [*Russian Acad. Sci. Sb. Math.*], **187**, No. 12, 21–56 (1996).

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On Minimum Modulus of Trigonometric Polynomials With Random Coefficients

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KEY WORDS: random trigonometric polynomials, Littlewood hypothesis.

For independent random variables $\xi_0, \dots, \xi_{n-1}$ each of which equals $+1$ or -1 with probability $1/2$, denote by $P(u)$ the probability

$$P(u) = P_n(u) = \Pr\left(\min_{x \in T}\left|\sum_{j=0}^{n-1} \xi_j \exp(ijx)\right| > u\right) \qquad (u \ge 0).$$

Littlewood [1] conjectured that $P(\varepsilon\sqrt{n}) \to 0$ as $n \to \infty$ for any $\varepsilon > 0$. Kashin [2] proved this conjecture and found that $P(n^{1/2}(\log n)^{-1/3}) \to 0$ as $n \to \infty$. Odlyzhko showed that $P(n^{1/3+\varepsilon}) \to 0$ as $n \to \infty$ for any $\varepsilon > 0$ and conjectured that for large n and any $\varepsilon > 0$ most of the polynomials

$$T(x) = \sum_{j=0}^{n-1} \pm\exp(ijx)$$

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