A REMARK ON A UNIFORM LYAPUNOV DIMENSION OF CASCADE SYSTEMS

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Abstract. In this paper we obtain sharp upper estimates of the uniform Lyapunov dimension of a cascade systems in terms of the corresponding Lyapunov exponents of their components. The obtained result is applied for estimating the Lyapunov and fractal dimensions of the attractors of nonautonomous dissipative systems generated by PDEs of mathematical physics.

Introduction

We study the iterations of the following cascade map defined on an invariant set $\Phi\subset H:=H_1\times H_2$:

$$\mathbb{A}:\Phi\to\Phi,\quad \mathbb{A}\begin{pmatrix}x\\y\end{pmatrix}:=\begin{pmatrix}A(x,y)\\B(y)\end{pmatrix},\quad \begin{pmatrix}x\\y\end{pmatrix}\in\Phi, \tag{0.1}$$

where H_1 and H_2 are Hilbert spaces and A and B are the nonlinear maps. This map generates a discrete dynamical system on a space $\Phi\subset H$ by the following standard expression:

$$\xi(n+1)=\mathbb{A}(\xi(n)),\quad \xi(0)=\xi_0\in\Phi,\quad \xi(n):=\begin{pmatrix}x(n)\\y(n)\end{pmatrix},\quad n\in\mathbb{Z}_+. \tag{0.2}$$

It is worth to note that system (0.2) can be interpreted as a nonautonomous dynamical system acting on H_1:

$$x(n+1)=A_n(x(n)),\quad x(0)=x_0,\quad A_n(x):=A(x,y(n)), \tag{0.3}$$

where the sequence $y(n)$ is then interpreted as nonautonomous 'external forces' which satisfy the following autonomous equation:

$$y(n+1)=B(y(n)),\quad y(0)=y_0. \tag{0.4}$$

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Equations of the form (0.3) and (0.4) often arise under the dynamical study of dissipative systems generated by nonautonomous PDEs of mathematical physics. For instance, if the external forces are quasiperiodic in time, then equation (0.4) generates a linear (discrete) flow on a k-dimensional torus $\mathbb{T}^k$ (k is a number of rationally independent frequencies). In this case, the space H_2 is finite-dimensional, the space H^1 is the (infinite-dimensional) phase space of the PDE under consideration (e.g., reaction-diffusion equation, damped wave equation or 2D Navier-Stokes system), $\Phi = H^1 \times \mathbb{T}^k$ and $A_n(x) := U(n+1,n)x$ is a solving operator of this PDE during the time period $t \in [n, n+1]$, see [3] and [4] for the details. Then, reducing this nonautonomous system to the autonomous one by the skew-product technique, one obtains the cascade system of the form (0.1) and (0.2).

It is well known that, in many cases, the longtime behavior of dissipative systems mentioned above can be described in terms of the global or/and uniform attractors. Moreover, although the initial phase space of these systems is infinite-dimensional (usually, $H_1 = L^2(\Omega)$), very often their attractors $\mathcal{A}$ have finite Hausdorff and fractal dimension, so the limit dynamics can be described by the finite number of parameters, see [1], [4], [9] and the references therein. The crucial question here is, of course, to obtain the realistic bounds of the dimension of the attractor in terms of the physical parameters of the systems considered.

We recall that, for the most part of autonomous dissipative PDEs, the best known estimates for the attractor's dimension have been obtained using the classical inequality between the fractal and Lyapunov dimensions:

$$\dim_F(\mathcal{A}) \le \dim_L(\mathcal{A}) \tag{0.5}$$

and the so-called volume contraction method for estimating the Lyapunov dimension in the right-hand side of (0.5), see [2], [5-9] and the references therein. Thus, in order to extend this machinery to nonautonomous equations, we need to know how to obtain good estimates for the Lyapunov dimension of a cascade system of the form (0.1) and (0.2).

To the best of our knowledge, the first attempt to extend this machinery to the nonautonomous PDEs with quasiperiodic in time external forces was made in [3] where the authors obtained the estimates for the dimensions of uniform attractors for wide class of nonautonomous equations of mathematical physics applying the volume contraction method to the extended autonomous system (0.1) and (0.2). We however note that the method applied does not take into account the special structure of map (0.1) and (as a result) some additional term which depend on the norms of $D_yA(x,y)$ (and which is in a sense irrelevant) appears in all these estimates.

An alternative approach which does not use the extended system (0.1) and (0.2) and directly generalizes the proof of estimate (0.5) to the nonautonomous case has been recently suggested in [4]. Using this method, the authors obtain the sharp formula for the quasiperiodic case which does not contain the additional irrelevant term and has the following form:

$$\dim_F(\mathcal{A}) \le \dim_L\{(0.3)\} + k, \tag{0.6}$$

where $\dim_L\{(0.3)\}$ is the Lyapunov dimension associated with the nonautonomous problem in the form (0.3) (see Section 1 for its rigorous definition) and k is a number of the rationally independent frequencies (we also note that in applications the first term in the right-hand side usually possesses the same estimates as the Lyapunov dimension of the attractor in the autonomous case, see [4] for the details). Moreover, the authors also give a general formula for estimating the Kolmogorov's ε-entropy $\mathbb{H}_\varepsilon(\mathcal{A})$ of such attractors (= the logarithm of the minimal number of ε-balls which cover $\mathcal{A}$) in terms of the corresponding entropy of the external forces (which are now not necessarily quasiperiodic and even can be infinite-dimensional) which has the following form:

$$\mathbb{H}_\varepsilon(\mathcal{A}) \le C + (\dim_L\{(0.3)\} + \delta) \ln \frac{1}{\varepsilon} + \mathbb{H}_{\varepsilon/L}\left(\mathcal{H}\big|_{[0, M_\delta \ln \frac{1}{\varepsilon}]}\right), \tag{0.7}$$

where the last term in the right-hand side of (0.7) is an entropy of the so-called hull $\mathcal{H}$ of the external forces restricted to the interval $[0, M_\delta \ln \frac{1}{\varepsilon}]$, see [4] for the details, $\delta > 0$ is an arbitrary small number and the constants L and M_δ depend on δ. For instance, if equation (0.4) for the external forces is finite-dimensional, then we can extract from general estimate (0.7) the following estimate for the fractal dimension of the attractor:

$$\dim_F(\mathcal{A}) \le (\dim_L\{(0.3)\} + \delta) + k + CM_\delta \cdot k\mu, \tag{0.8}$$

where μ is the first Lyapunov exponent of (0.4), k is the dimension of system (0.4) and C is some positive constant. In particular, this estimate immediately gives (0.6) since all of the Lyapunov exponents of the linear flow on the torus $\mathbb{T}^k$ equal zero. We however note that, for general (non quasiperiodic) external forces which have positive first Lyapunov exponent μ, this estimate is also can be far from the optimal since the constant M_δ is usually very large (and tends to $+\infty$ as $\delta \to 0$).

In the present paper, we give a sharp estimate for the Lyapunov dimension of the cascade system (0.2) in terms of the Lyapunov exponents μ_i^{int} and μ_i^{ext} associated with problems (0.3) and (0.4) respectively. This estimate has the following form:

$$\dim_L\{(0.2)\} \le d_0 + \frac{\sum_{i=1}^{d_0} \tilde\mu_i}{|\tilde\mu_{d_0+1}|}, \tag{0.9}$$

where $\{\tilde\mu_i\}_{i=1}^\infty$ is the union of all exponents μ_i^{int} and μ_j^{ext} renumerated in the non-increasing order and d_0 is the first integer such that $\sum_{i=1}^{d_0+1} \tilde\mu_i < 0$. In particular, for the quasiperiodic external forces, this formula implies the analogue of estimate (0.6) with the Lyapunov dimension in the left-hand side and give its natural (and, in a sense, sharp) generalization to the case where the external forces have positive Lyapunov exponents.

The paper is organized as follows. Some definitions which are necessary in order to formulate and proof our main result are given in Section 1. The estimate of the Lyapunov dimension of system (0.2) in terms of the corresponding Lyapunov exponents of systems (0.3) and (0.4) is formulated and proved in Section 2. And finally, in Section 3, we give some examples illustrating the above theory.

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§1 Uniform Lyapunov exponents and Lyapunov dimension.

In this Section, we briefly recall the definition of uniform Lyapunov exponents and Lyapunov dimension and formulate their simplest properties which are important for what follows (see e.g. [9] for the detailed exposition). As usual, we start by introducing the volume expanding coefficients $\omega_d(L)$ of a linear operator L in a Hilbert space.

Definition 1.1. Let L be a linear operator in a Hilbert space H. Then, by definition, the volume expanding coefficient $\omega_d(L)$ is the norm of dth exterior power $\Lambda^d L$ ($d\in\mathbb N$) of the operator L in the space $\Lambda^d H$:

$$\omega_d(L):=\|\Lambda^dL\|_{\mathcal L(\Lambda^dH)}=\sup_{\substack{h_1,\cdots,h_d\in H\\ \|h_i\|_H\le1}}\det\{(Lh_i,h_j)_H\}_{1\le i,j\le d}, \tag{1.1}$$

where $(h_1,h_2)_H$ is a scalar product in H.

We now recall the basic properties of the introduced numbers:
1) Let L_1 and L_2 be linear operators in H. Then

$$\omega_d(L_1\cdot L_2)\le\omega_d(L_1)\omega_d(L_2),\quad d\in\mathbb N. \tag{1.2}$$

2) The ω-numbers $\omega_d(L)$ can be expressed in terms of the α-numbers of the operator L via

$$\omega_d(L)=\alpha_1(L)\cdot\alpha_2(L)\cdots\alpha_d(L), \tag{1.3}$$

where the α-numbers $\alpha_m(L)$ are defined by the following min-max formula

$$\alpha_m(L):=\sup_{\substack{F\subset H\\ \dim F=m}}\ \inf_{\substack{h\in F\\ \|h\|_H=1}}\|Lh\|_H, \tag{1.4}$$

where the supremum in the right-hand side of (1.4) is taken over all m-dimensional linear subspaces of H. We also recall that, in the case where L is compact, the α-numbers coincide with the eigenvalues of the operator $(LL^*)^{1/2}$:

$$\alpha_m(L)=\lambda_m((LL^*)^{1/2}),\quad \lambda_m\in\sigma((LL^*)^{1/2}) \tag{1.5}$$

and analogous (but slightly more complicated) relation holds in general (non-compact) case as well (see [9] the for details).
3) Let the linear operators L and T satisfy

$$(Lh,Lh)_H\le(Th,Th)_H,\ \ \forall h\in H,\quad \text{then}\quad \omega_d(L)\le\omega_d(T),\ \forall d\in\mathbb N. \tag{1.6}$$

The detailed proof of the above assertions can be found in [9].

We are now ready to define the Lyapunov exponents and Lyapunov dimensions for dynamical systems (0.2)–(0.4). To this end, we assume that we are given a compact invariant set $\mathcal A$ of map (0.1):

$$\mathbb A(\mathcal A)=\mathcal A \tag{1.7}$$

(which is usually a global attractor in applications) and assume also that $\mathbb A\in C^1(\mathcal A)$ (the latter means that the map $\mathbb A$ is uniformly quasidifferentiable on $\mathcal A$, see [9], and that its (quasi)differential depends continuously on the point $\xi_0 \in \mathcal{A}$). We also denote by $\mathbb{S}_{\mathbb{A}}(n)$ the nth iteration of the map $\mathbb{A}$. Then, obviously, $\mathbb{S}_{\mathbb{A}}(n)$ is a solving operator for equation (0.2), i.e.

$$\xi(n) = \mathbb{S}_{\mathbb{A}}(n)\xi_0, \quad \forall \xi_0 \in \mathbb{A}. \tag{1.8}$$

We first define the uniform volume expanding coefficients $\omega_d(\mathbb{A}, n)$ as follows:

$$\omega_d(\mathbb{A}, n) := \sup_{\xi_0 \in \mathcal{A}} \omega_d(D_\xi \mathbb{S}_{\mathbb{A}}(n)(\xi_0)), \tag{1.9}$$

where $D_\xi \mathbb{S}_{\mathbb{A}}(n)(\xi_0)$ denotes the derivative with respect to the initial data of the solving operator $\mathbb{S}_{\mathbb{A}}(n)$ at point $\xi_0 \in \mathcal{A}$ (this derivative exists since $\mathbb{A} \in C^1(\mathcal{A})$ and the supremum in the right-hand side exists since $\mathcal{A}$ is assumed to be compact). Then, these numbers are submultiplicative with respect to n:

$$\omega_d(\mathbb{A}, n+m) \le \omega_d(\mathbb{A}, n)\omega_d(\mathbb{A}, m). \tag{1.10}$$

Indeed, since $\mathbb{S}_{\mathbb{A}}(n)$ is a semigroup with respect to n then, differentiating the identity $\mathbb{S}_{\mathbb{A}}(n+m) = \mathbb{S}_{\mathbb{A}}(n) \circ \mathbb{S}_{\mathbb{A}}(m)$ by ξ_0, we have

$$D_\xi \mathbb{S}_{\mathbb{A}}(n+m)(\xi_0) = D_\xi \mathbb{S}_{\mathbb{A}}(n)(\mathbb{S}_{\mathbb{A}}(m)\xi_0) \circ D_\xi \mathbb{S}_{\mathbb{A}}(m)(\xi_0)$$

and, consequently, according to (1.2),

$$\omega_d(D_\xi \mathbb{S}_{\mathbb{A}}(n+m)(\xi_0)) \le \omega_d(D_\xi \mathbb{S}_{\mathbb{A}}(n)(\mathbb{S}_{\mathbb{A}}(m)\xi_0)) \cdot \omega_d(D_\xi \mathbb{S}_{\mathbb{A}}(m)(\xi_0)). \tag{1.11}$$

Applying the supremum over $\xi_0 \in \mathcal{A}$ to the both parts of this inequality, we deduce (1.10). Inequality (1.10) implies the existence of the following limit:

$$\Lambda_d(\mathbb{A}) := \lim_{n\to\infty} (\omega_d(\mathbb{A}, n))^{1/n} = \inf_{n\in\mathbb{N}} (\omega_d(\mathbb{A}, n))^{1/n}. \tag{1.12}$$

Moreover, for the case of non-integer $d = k+s$, $k \in \mathbb{N}$ and $0 \le s \le 1$, the Λ-numbers (1.12) can be defined by the following interpolation formula:

$$\Lambda_d(\mathbb{A}) = \Lambda_n(\mathbb{A})^{1-s}\Lambda_{n+1}(\mathbb{A})^s, \tag{1.13}$$

see [9] for the details.

Definition 1.2. A uniform Lyapunov dimension of the set $\mathcal{A}$ with respect to the map $\mathbb{A}$ is the following number:

$$\dim_L\{(0.2)\} = \dim_L(\mathbb{A}) = \dim_L(\mathbb{A}, \mathcal{A}) := \inf\{d \in \mathbb{R}_+,\ \Lambda_d(\mathbb{A}) < 1\} \tag{1.14}$$

and the uniform Lyapunov exponents $\mu_i = \mu_i(\mathbb{A})$ of equation (0.2) are defined via

$$\mu_1 = \ln \Lambda_1(\mathbb{A}), \quad \mu_i := \ln \Lambda_i(\mathbb{A}) - \ln \Lambda_{i-1}(\mathbb{A}), \quad i > 1. \tag{1.15}$$

Definition (1.14) can be rewritten in terms of the Lyapunov exponents μ_i:

$$\dim_L(\mathbb{A}) = n_0 + \frac{\sum_{i=1}^{n_0} \mu_i}{|\mu_{n_0+1}|}, \tag{1.16}$$

where n_0 is the minimal integer such that $\sum_{i=1}^{n_0+1}\mu_i<0$, see [9] for the details.

Thus, we have defined the Lyapunov exponents for the extended system (0.2). The Lyapunov exponents $\Lambda_d(B)$ and $\mu_i(B)$ for system (0.4) (which describes the evolution of the 'external forces') can be defined completely analogously. So, it only remains to define the uniform Lyapunov exponents for the nonautonomous equation (0.3). To this end, we first define the solving semigroup $S_B(n)$ of equation (0.4), i.e. $y(n)=S_B(n)y_0$ and then, for every 'external forces' $y(n)=S_B(n)y_0$, we define the solving operator $S_A(n,y_0)$ for problem (0.3) by the expression

$$x(n)=S_A(n,y_0)x_0,\quad \forall(x_0,y_0)\in\mathcal{A}, \tag{1.17}$$

where $x(n)$ solves problem (0.3) with the external forces $y(n)=S_B(n)y_0$. We however note that, in contrast to $\mathbb{S}_{\mathbb{A}}(n)$ and $S_B(n)$ the operators $S_A(n,y_0)$ do not satisfy the semigroup identity (since equation (0.3) is nonautonomous), which should be now replaced by the so-called cocycle identity

$$S_A(n+m,y_0)=S_A(n,S_B(m)y_0)\circ S_A(m,y_0). \tag{1.18}$$

Moreover, we also mention the following obvious, but useful, relation between $\mathbb{S}_{\mathbb{A}}(n)$ and $S_A(n,y_0)$

$$S_A(n,y_0)x_0=\Pi_1\mathbb{S}_{\mathbb{A}}(n)(x_0,y_0) \tag{1.19}$$

(where Π_1 is a projector to the first component of a Cartesian product $H=H_1\times H_2$) which follows from the fact that $(x(n),y(n))$ solves equation (0.2).

We are now ready to define the uniform volume expanding coefficients for the nonautonomous equation (0.3) as follows:

$$\omega_d(A,n):=\sup_{(x_0,y_0)\in\mathcal{A}}\omega_d(D_xS_A(n,y_0)(x_0)), \tag{1.20}$$

where $D_xS_A(n,y_0)(x_0)$ is the x-derivative of the operator $S_A(n,y_0)$ computed at point x_0. We note that, in contrast to (1.9), only the x-differentiation is involved in definition (1.20) and the variable y_0 plays the role of a parameter.

Moreover, differentiating the cocycle identity (1.18) by x and using inequality (1.2), we derive (analogously to (1.10)) that

$$\omega_d(A,m+n)\le\omega_d(A,m)\cdot\omega_d(A,n) \tag{1.21}$$

which guarantees the existence of the limit

$$\Lambda_d(A):=\lim_{n\to\infty}\left(\omega_d(A,n)\right)^{1/n}=\inf_{n\in\mathbb{N}}\left(\omega_d(A,n)\right)^{1/n} \tag{1.22}$$

and, for the non-integer d, the quantities $\Lambda_d(A)$ can be defined by the interpolation formula (1.13). Furthermore, the uniform Lyapunov dimension $\dim_L\{(0.3)\}$ of the nonautonomous system (0.3) and its Lyapunov exponents $\mu_i(A)$ can be defined by (1.14) and (1.15) (where, obviously, one should replace $\Lambda_d(\mathbb{A})$ by $\Lambda_d(A)$).

Definition 1.3. In order to simplify the notations, we will use from now on the symbols $\Lambda_d^{un}:=\Lambda_d(\mathbb{A})$, $\Lambda_d^{int}:=\Lambda_d(A)$ and $\Lambda_d^{ext}:=\Lambda_d(B)$ for the uniform Λ-numbers of equations (0.2), (0.3) and (0.4) respectively. Moreover, we denote by μ_i^{un}, μ_i^{int} and μ_i^{ext} the corresponding Lyapunov exponents.

The following standard result which shows that the Λ-numbers and Lyapunov exponents are independent of the concrete form of the inner product in H is crucial for what follows.

Proposition 1.1. *Let the above assumptions hold and let K be a self-adjoint positive and invertible operator in H which generates a new inner product in H via $(\xi_1,\xi_2)_K := (K\xi_1,\xi_2)_H$. Then,*

$$\Lambda_{d,K}(\mathbb{A}) = \Lambda_d(\mathbb{A}) \quad \text{and} \quad \mu_{i,K}(\mathbb{A}) = \mu_i(\mathbb{A}), \tag{1.23}$$

where $\Lambda_{d,K}(\mathbb{A})$ and $\mu_{i,K}(\mathbb{A})$ are computed using the new inner product $(\cdot,\cdot)_K$.

Proof. Indeed, it is not difficult to verify, using e.g. formulae (1.3) and (1.4), that

$$\omega_{d,K}(L) = \omega_d(K^{1/2}LK^{-1/2})$$

and, consequently, using (1.2), we find

$$\omega_{d,K}(\mathbb{A},n) \le [\omega_d(K)\cdot\omega_d(K^{-1})]^{1/2}\omega_d(\mathbb{A},n). \tag{1.24}$$

Taking the n^{-1}th power from the both sides of this inequality and passing to the limit $n\to\infty$, we verify that $\Lambda_{d,K}(\mathbb{A}) \le \Lambda_d(\mathbb{A})$. The inverse inequality can be proven analogously. Thus, the first identity of (1.23) is verified and the second one is an immediate corollary of the first one. Proposition 1.1 is proven.

§2 The main result.

In this section, we prove the following theorem which estimates Λ_d^{un} in terms of Λ_d^{int} and Λ_d^{ext}.

Theorem 2.1. *Let the above assumptions hold. Then the Λ-numbers Λ_d^{un} of the extended system (0.2) can be estimated in terms of the Λ-numbers Λ_d^{int} of the internal nonautonomous problem (0.3) and the Λ-numbers Λ_d^{ext} of problem (0.4) (which describes the evolution of the external forces) as follows:*

$$\Lambda_d^{un} \le \max\nolimits_{s=0,\cdots,d}\{\Lambda_s^{int}\cdot\Lambda_{d-s}^{ext}\}, \tag{2.1}$$

where $d\in\mathbb{N}$ and, by definition, $\Lambda_0^{int}=\Lambda_0^{ext}=1$.

Proof. For simplicity, we below prove the theorem in the case where the derivative $D_\xi\mathbb{A}(\xi_0)$ is a compact operator for every $\xi_0\in\mathcal{A}$ (the general case is completely analogous, but slightly more complicated since, instead of (1.5), we need to use slightly more complicated description of the α-numbers in a general non-compact case, see [9, Proposition V.3.1.1]).

We first note that, due to (1.19) the the matrix of the derivative $D_\xi\mathbb{S}_{\mathbb{A}}(n)(\xi_0)$ at point $\xi_0 := (x_0,y_0)$ has the following structure:

$$\mathbb{T}_n(\xi_0) := D_\xi\mathbb{S}_{\mathbb{A}}(n)(\xi_0) = \begin{pmatrix} D_xS_A(n,y_0)(x_0) & , & T_n(\xi_0) \\ 0 & , & D_yS_B(n)(y_0)\end{pmatrix}, \tag{2.2}$$

where $T_n(\xi_0)\in\mathcal{L}(H_2,H_1)$ is a bounded linear operator. We see that matrix (2.2) differs on a direct sum of operators $D_xS_A(n,y_0)(x_0)$ and $D_yS_B(n)(y_0)$ (which are involved into the definition of Λ_d^{int} and Λ_d^{ext}) by the presence of the upper-diagonal term $T_n(\xi_0)$ and this is the main difficulty in the proof of estimate (2.1). In order to eliminate the influence of this term, we use below a special scalar product in the space $H = H_1 \times H_2$, namely, we set

$$(\xi_1, \xi_2)_\varepsilon := (x_1, x_2)_{H_1} + \varepsilon^{-2}(y_1, y_2)_{H_2}, \quad \xi_i = (x_i, y_i) \in H, \quad i = 1, 2, \tag{2.3}$$

where ε is a small positive number. Then, for all $\xi = (x, y) \in H$, we have

$$\begin{aligned}(\mathbb{T}_n(\xi_0)\xi, \mathbb{T}_n(\xi_0)\xi)_\varepsilon = (D_x S_A(n, y_0)(x_0)[D_x S_A(n, y_0)(x_0)]^* x, x)_{H_1} + \\ + \varepsilon^{-2}(D_y S_B(n)(y_0)[D_y S_B(n)(y_0)]^* y, y)_{H_2} + 2(T_n(\xi_0)y, D_x S_A(n, y_0)(x_0)x)_{H_1}.\end{aligned} \tag{2.4}$$

Since the operators $T_n(\xi_0)$ and $D_x S_A(n, y_0)(x_0)$ are uniformly bounded on $\mathcal{A}$, we can estimate the last term in the right-hand side of (2.4) as follows:

$$\begin{aligned}2(T_n(\xi_0)y, D_x S_A(n, y_0)(x_0)x)_{H_1} \le 2C_n\|x\|_{H_1}\|y\|_{H_2} \le \\ \le C_n\varepsilon\|x\|^2_{H_1} + C_n\varepsilon^{-1}\|y\|^2_{H_2} = C_n\varepsilon\|\xi\|^2_\varepsilon,\end{aligned} \tag{2.5}$$

where C_n depends on n, but independent of $\xi_0 \in \mathcal{A}$. Combining (2.4) and (2.5), we have

$$(\mathbb{T}_n(\xi_0)\xi, \mathbb{T}_n(\xi_0)\xi)_\varepsilon \le ((T_{A,B}(n, \xi_0)\xi, \xi)_\varepsilon, \tag{2.6}$$

where

$$T_{A,B}(n, \xi_0) := \begin{pmatrix} T_A(n, \xi_0)[T_A(n, \xi_0)]^* + C_n\varepsilon \operatorname{Id} & , & 0 \\ 0 & , & T_B(n, y_0)[T_B(n, y_0)]^* + C_n\varepsilon \operatorname{Id} \end{pmatrix},$$

$T_A(n, x_0) := D_x S_A(n, y_0)(x_0)$ and $T_B(n, y_0) := D_y S_B(n)(y_0)$.

We now note that the spectrum of the self-adjoint operator $T_{A,B}(n, \xi_0)$ is, obviously, a union of the spectrums (in H_1 and H_2 respectively) of its components $T_A(n, \xi_0)[T_A(n, xi_0)]^* + C_n\varepsilon \operatorname{Id}$ and $T_B(n, y_0)[T_B(n, y_0)]^* + C_n\varepsilon \operatorname{Id}$ respectively. Furthermore, due to our compactness assumption, these operators have the complete orthonormal basises of eigenvectors and, consequently, their α-numbers possess the following description

$$\begin{aligned}\alpha_k(T_A(n, \xi_0)[T_A(n, \xi_0)]^* + C_n\varepsilon) = \alpha_k(T_A(n, \xi_0))^2 + C_n\varepsilon \\ \alpha_k(T_B(n, y_0)[T_B(n, y_))]^* + C_n\varepsilon) = \alpha_k(T_B(n, y_0))^2 + C_n\varepsilon\end{aligned} \tag{2.7}$$

and, in order to obtain the α-numbers of $T_{A,B}(n, \xi_0)$, it is sufficient to renumerate the union of the α-numbers of these operators in the non-increasing order. Thus, due to (1.3) and (2.7), we infer

$$\begin{aligned}\omega_{d,\varepsilon}(T_{A,B}(n, \xi_0)) = \\ = \max_{k=0,\cdots,d}\{\Pi^k_{i=1}(\alpha_i(T_A(n, \xi_0))^2 + C_n\varepsilon)\Pi^{d-k}_{j=1}(\alpha_j(T_B(n, y_0))^2 + C_n\varepsilon)\},\end{aligned}$$

where $\omega_{d,\varepsilon}(L)$ are the ω-numbers of the operator L computed in metric (2.3). Applying (1.3) again and using the fact that all the α-numbers are uniformly bounded on the set $\mathcal{A}$, we have

$$\begin{aligned}\omega_{d,\varepsilon}(T_{A,B}(n, \xi_0)) \le \\ \le \max_{k=0,\cdots,d}\{(\omega_k(T_A(n, \xi_0))^2 + C'_n\varepsilon)(\omega_{d-k}(T_B(n, y_0))^2 + C'_n\varepsilon)\},\end{aligned} \tag{2.8}$$

where, by definition, $\omega_0(\cdot) = 1$ and the constant C_n' depends on d and n, but is independent of $\xi_0 \in \mathcal{A}$. Using now (1.6) and (2.6), we deduce from (2.8) that

$$\begin{aligned} (2.9)\quad & \omega_{d,\varepsilon}(D_\xi \mathbb{S}_{\mathbb{A}}(n)(\xi_0))^2 \le \\ & \le \max_{k=0,\cdots,d}\{(\omega_k(D_x S_A(n,y_0)(x_0))^2 + C_n'\varepsilon)(\omega_{d-k}(D_y S_B(n)(y_0))^2 + C_n'\varepsilon)\}. \end{aligned}$$

Applying the supremum over $\xi_0 \in \mathcal{A}$ to the both sides of this inequality, we deduce the analogous relation for the uniform volume expanding coefficients

$$\omega_{d,\varepsilon}(\mathbb{A},n)^2 \le \max_{k=0,\cdots,d}\{(\omega_k(A,n)^2 + C_n'\varepsilon)(\omega_{d-k}(B,n)^2 + C_n'\varepsilon)\}. \tag{2.10}$$

We now recall that the Λ-numbers Λ_d^{un} are independent of the choice of the inner product in H, consequently, due to (1.12) and Proposition 1.1,

$$\begin{aligned} (\Lambda_d^{un})^{2n} &= (\Lambda_{d,\varepsilon}^{un})^{2n} \le \\ & \le \omega_{d,\varepsilon}(\mathbb{A},n)^2 \le \max_{k=0,\cdots,d}\{(\omega_k(A,n)^2 + C_n'\varepsilon)(\omega_{d-k}(B,n)^2 + C_n'\varepsilon)\}. \end{aligned}$$

Passing to the limit $\varepsilon \to 0$ in this relation, we have

$$(\Lambda_d^{un})^n \le \max_{k=0,\cdots,d}\{\omega_k(A,n)\cdot\omega_{d-k}(B,n)\}. \tag{2.11}$$

Taking the n^{-1}th power from the both sides of (2.11) and passing to the limit $n \to \infty$, we finally obtain estimate (2.1) and finish the proof of Theorem 2.1.

Our next task is to clarify estimate (2.1). To this end, we reformulate it in terms of the Lyapunov exponents μ_i^{int} and μ_i^{ext} associated with equations (0.3) and (0.4) respectively.

Corollary 2.1. *Let $\{\tilde\mu_i\}_{i=1}^\infty$ be the union $\{\mu_1^{int},\mu_2^{int},\cdots\}\cup\{\mu_1^{ext},\mu_2^{ext},\cdots\}$ of all of the Lyapunov exponents associated with equations* (0.3) *and* (0.4) *renumerated in the non-increasing order. Then, under the assumptions of Theorem 2.1, the following estimate holds:*

$$\ln \Lambda_d^{un} \le \sum_{i=1}^d \tilde\mu_i, \quad d \in \mathbb{N}. \tag{2.12}$$

Indeed, taking the logarithm from both sides of inequality (2.1) and using the definition of the Lyapunov exponents, we have

$$\ln \Lambda_d^{un} \le \max_{k=0,\cdots,d}\left\{\sum_{i=1}^k \mu_i^{int} + \sum_{j=1}^{d-k}\mu_j^{ext}\right\}. \tag{2.13}$$

It remains to note that estimate (2.12) is an elementary corollary of (2.13).

We now ready to estimate the Lyapunov dimension of the extended system (0.2) in terms of the Lyapunov exponents of (0.3) and (0.4).

Corollary 2.2. *Let the above assumptions hold and let the exponents $\tilde\mu_i$ be the same as in Corollary 2.1. Then, the Lyapunov dimension of the invariant set $\mathcal{A}$ of equation* (0.2) *can be estimated as follows:*

$$\dim_L\{(0.2)\} = \dim_L(\mathbb{A},\mathcal{A}) \le d_0 + \frac{\sum_{i=1}^{d_0}\tilde\mu_i}{|\tilde\mu_{d_0+1}|}, \tag{2.14}$$

where d_0 is the first integer such that $\sum_{i=1}^{d_0+1}\tilde\mu_i < 0$.

Indeed, according to (2.12), we have the following estimate for the numbers Λ_d^{un}, for every $d = n+s$, $n\in\mathbb{N}$ and $0\le s\le 1$:

$$\ln\Lambda_d^{un} = s\ln\Lambda_n^{un} + (1-s)\ln\Lambda_{n+1}^{un} \le \sum_{i=1}^n \tilde\mu_i + s\tilde\mu_{n+1}. \tag{2.15}$$

This estimate, together with definition (1.14) of the Lyapunov dimension imply estimate (2.14).

§3 Examples and applications.

In this Section, we give several examples illustrating the general theory. We start with the most simple case of direct sum of two semigroups which shows that estimates (2.1) and (2.14) are, in a sense, sharp.

Example 3.1. Let the map A in formula (0.1) be independent of y:

$$\mathbb{A}(\xi) := \begin{pmatrix} A(x)\\ B(y)\end{pmatrix}$$

and let the compact invariant set $\mathcal{A} := \mathcal{A}_1\times\mathcal{A}_2$ where $\mathcal{A}_1\subset H_1$ and $\mathcal{A}_2\in H_2$ are invariant with respect to the maps A and B respectively. In that case the non-diagonal operator $T_{A,B}$ vanishes and inequality (2.10) becomes a strict equality with $C_n = C_n' = 0$ for every $\varepsilon > 0$. This, in turn, gives a strict equality in inequalities (2.11) and (2.1).

Assume, in addition, that μ_i^{int} and μ_i^{ext} are monotone decreasing with respect to i (it will be so, eg, for the case of linear maps A and B). Then, the Lyapunov exponents μ_i^{un} of equation (0.2) *coincide* with the exponents $\tilde\mu_i$, introduced in Corollary 3.2:

$$\mu_i^{un} = \tilde\mu_i,\quad i\in\mathbb{N} \tag{3.1}$$

and we also have strict equalities in (2.12) and (2.14).

The next obvious example is the so-called *quasiperiodic* external forces.

Example 3.2. Let the map B be a linear map on the k-dimensional torus $\mathbb{T}^k := \mathbb{R}^k\backslash\mathbb{Z}^k$:

$$B(y_1,\cdots,y_k) := (y_1+\alpha_1,\cdots,y_k+\alpha_k)(\bmod 1),\quad y=(y_1,\cdots,y_k)\in\mathbb{T}^k, \tag{3.2}$$

where $\alpha = (\alpha_1,\cdots,\alpha_k)$ are the rationally independent frequencies and we also assume that the projection $\Pi_2\mathcal{A}$ of the invariant set $\mathcal{A}$ to the second component of the Cartesian product $H_1\times H_2$ coincides with the torus $\mathbb{T}^k$.

It is well-known (see e.g. [3-4]) that the study of the longtime behavior of general nonautonomous PDEs with the *quasiperiodic* external forces can be reduced (using the so-called skew-product technique) to the study of iterations of the map $\mathbb{A}$ in which the second component has the form of (3.2).

In this case, all the Lyapunov exponents μ_i^{ext} of the map B are, obviously, equal zero:

$$\mu_1^{ext} = \cdots = \mu_k^{ext} = 0. \tag{3.3}$$

Then, inequality (2.1) shows that $\Lambda_{d+k}^{un} \le \Lambda_d^{int}$ if d is such that $\Lambda_{d+1} \le 1$. Using this fact and the definition of the Lyapunov dimension, we conclude that

$$\dim_L\{(0.2)\} \le \dim_L\{(0.3)\} + k. \tag{3.4}$$

We recall that, due to the classical inequality (0.5) between the fractal and Lyapunov dimension, estimate (3.4) can be considered as a generalization of estimate (0.6).

We now consider the applications of formulae (2.1) and (2.14) to some concrete equations of mathematical physics. We start with the reaction-diffusion system.

Example 3.3. Let us consider the following reaction-diffusion system in a bounded domain $\Omega \subset \mathbb{R}^m$:

$$\partial_t u = \nu\Delta_x u - f(u) + g(t), \quad u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0. \tag{3.5}$$

Here $u = (u_1, \cdots, u_N)$ is an unknown vector-valued function, $\nu > 0$ is a given parameter, $f(u)$ is a given nonlinear interaction function which satisfies the standard dissipativity and growth assumptions:

$$\begin{cases} 1. \ \ f \in C^1(\mathbb{R}^k, \mathbb{R}^k), \\ 2. \ \ f(u).u \ge -C, \quad f'(u) \ge -K, \end{cases} \tag{3.6}$$

where $u.v$ denotes the standard inner product in $\mathbb{R}^N$. We also assume that the external forces $g(t)$ are finite-dimensional and have the following structure:

$$g(t) := g(t, y_0) := G(y_1(t), \cdots, y_k(t)), \tag{3.7}$$

where $G \in C^1(\mathbb{R}^k, [L^\infty(\Omega)]^N)$ is a given function and the vector

$$y(t) := (y_1(t), \cdots, y_k(t))$$

solves the following autonomous ODE:

$$\frac{d}{dt}y(t) = B(y(t)), \quad y(0) = y_0 \tag{3.8}$$

and $B \in C^1(\mathbb{R}^k, \mathbb{R}^k)$ is a given function. To be more precise, we assume that there exists a compact invariant set $\mathcal{A}_0 \subset \mathbb{R}^k$ of system (3.8) and consider a family of equations (3.5) with the external forces $\{g(t, y_0)\}_{y_0 \in \mathcal{A}_0}$.

It is well known that, under the above assumptions, problem (3.5), (3.7), (3.8) is uniquely solvable in $[L^2(\Omega)]^N \times \mathbb{R}^k$ for every $u_0 \in [L^2(\Omega)]^N$ and $y_0 \in \mathcal{A}_0$ and generates a dissipative semigroup $\mathbb{S}_{\mathbb{A}}(t)$ on the space $\Phi := [L^2(\Omega)]^N \times \mathcal{A}_0$ via

$$(u(t), y(t)) = \mathbb{S}_{\mathbb{A}}(t)(u_0, y_0), \quad t \geq 0, \tag{3.9}$$

where $(u(t), y(t))$ is a unique solution of (3.5), (3.7), (3.8). Moreover, this semigroup possesses a global attractor $\mathcal{A}$ (=uniform attractor of the initial nonautonomous problem (3.5)) in the space Φ (which is, by definition, a compact invariant set of this semigroup which attracts the images of all bounded subsets of Φ as $t \to \infty$) and $\mathbb{S}_{\mathbb{A}}(t) \in C^1(\mathcal{A})$ for every fixed $t \geq 0$, see e.g., [1], [4] and [9] for the details.

Moreover, obviously, the map

$$\mathbb{A} := \mathbb{S}_{\mathbb{A}}(1) \tag{3.10}$$

has the structure of (0.1), so we can apply our theory for estimating the Lyapunov dimension of the attractor $\mathcal{A}$. To this end, we first need to estimate the numbers Λ_d^{int} for problem (0.3) or (which is the same) for the nonautonomous equation (3.5). In order to do so, we recall that the operator $D_x S_A(n, y_0)(u_0)$ is now determined as a solving operator of the following variation equation associated with (3.5):

$$\partial_t U = \nu \Delta_x U - f'(u(t))U, \quad U\big|_{\partial\Omega} = 0, \\ U\big|_{t=0} = U_0, \quad (u(t), y(t)) = \mathbb{S}_{\mathbb{A}}(t)(u_0, y_0). \tag{3.11}$$

To be more precise, for every $U_0 \in [L^2(\Omega)]^N$, we have $D_x S_A(n, y_0)(u_0)U_0 := U(n)$, where $U(t)$ solves (3.11). Then, applying the standard generalized Liouville theorem for estimating the volume expanding coefficients of the linear equation (3.11), we have

$$\ln \Lambda_d^{int} \leq C_1 d - C_2 \nu d^{1+2/m}, \quad d \in \mathbb{R}_+, \tag{3.12}$$

where the positive constants C_1 and C_2 depend only on K, N, m and Ω (but are independent of ν and g, see [4], [9] for the details). Thus, the Lyapunov dimension of the non-autonomous equation (3.5) can be estimated as follows:

$$\dim_L\{(0.3)\} \leq d_{int} := \left(\frac{C_1}{C_2}\right)^{m/2} \nu^{-m/2}. \tag{3.13}$$

Let us now return to the estimating the Lyapunov dimension of the extended map (3.10). To this end, we first note that if we replace the Lyapunov exponents μ_i^{ext} by large ones, then the right-hand side of (2.14) also becomes larger and the inequality preserves. For instance, we may replace μ_i^{ext} by $(\mu_i^{int})_+ := \max\{\mu_i^{ext}, 0\}$. Then, in order to obtain the exponents $\tilde{\mu}_i$, we need to insert all these exponents into the sequence μ_i^{int} before the first negative exponent. Moreover, we need to compensate the sum of these exponents by the negative Lyapunov exponents of of the nonautonomous system. This arguments, together with (2.14) lead to the following estimate:

$$\dim_L(\mathcal{A}) \leq k + \inf\{d, \ \ln \Lambda_d^{int} < -\sum_{i=1}^{k} (\mu_i^{ext})_+\}. \tag{3.14}$$

Thus, according to (3.12), the infunum in the right-hand side of (3.14) is not larger that $d_{un} := d_{int} + d_{add}$, where

$$C_1(d_{int} + d_{add}) - C_2\nu(d_{int} + d_{add})^{1+2/m} = -\sum_{i=1}^{k}(\mu_i^{ext})_+. \tag{3.15}$$

Using now the standard inequality $(1+x)^\kappa \geq 1 + \kappa x$ and the explicit formula for d_{int}, we deduce that

$$d_{add} \leq C_1^{-1}\frac{m}{2}\sum_{i=1}^{k}(\mu_i^{ext})_+. \tag{3.16}$$

Combining the estimates obtained, we finally infer

$$\dim_L(\mathcal{A}) \leq d_{int} + k + C_1^{-1}\frac{m}{2}\sum_{i=1}^{k}(\mu_i^{ext})_+. \tag{3.17}$$

Thus, the first term in the right-hand side coincide with estimate (3.13) of the internal Lyapunov dimension, the second one is exactly the same as for the quasiperiodic external forces and the third one describes the influence of the positive Lyapunov exponents of the external forces.

Our next example is the non-autonomous 2D Navier-Stokes system.

Example 3.4. Let us consider the following nonautonomous version of 2D Navier-Stokes system in a bounded domain Ω:

$$\partial_t u + (u, \nabla_x u)u - \nu\Delta_x u = \nabla_x p + g(t), \quad u\big|_{\partial\Omega} = 0, \ u\big|_{t=0} = u_0, \ \operatorname{div} u = 0, \tag{3.18}$$

where the external forces $g(t)$ have exactly the same structure as in the previous example (obviously, now $N = m = 2$).

It is also well-known, that, for every $y_0 \in \mathcal{A}_0 \subset \mathbb{R}^k$ and every $u_0 \in H_0 := \{u \in [L^2(\Omega)]^2, \ \operatorname{div} u = 0\}$ problem (3.18), (3.7), (3.8) possesses a unique solution $(u(t), y(t))$ for all $t \geq 0$ and thus, a dissipative semigroup $\mathbb{S}_{\mathbb{A}}(t)$ is well defined in a phase space $\Phi := H_0 \times \mathcal{A}_0$ (analogously to (3.9)). Moreover, this semigroup possesses a global (uniform) attractor $\mathcal{A}$ in this phase space and $\mathbb{S}_{\mathbb{A}}(t) \in C^1(\mathcal{A})$ for any $t \geq 0$, see [4], [9] for the details.

It is also clear that the operator $\mathbb{A} := \mathbb{S}_{\mathbb{A}}(1)$ has the structure of (0.1), so we can again apply our theory for estimating the Lyapunov dimension of the attractor $\mathcal{A}$. To this end, we first recall the standard estimates for the internal volume contraction exponents of the 2D Navier-Stokes equation, namely,

$$\ln \Lambda_d^{int} \leq C_1\nu^{-3} - C_2\nu d^2, \quad d \in \mathbb{R}_+, \tag{3.19}$$

where the coefficient C_2 depends only on Ω and the coefficient C_1 also depends (in a linear way) on the norm $\sup_{y_0\in\mathcal{A}_0}\|G(y_0)\|^2_{L^\infty}$, see e.g. [4], [9]. Thus, in this case the estimate for the internal Lyapunov dimension of system (0.3) or (which is the same) system (3.18) has the following form:

$$\dim_L\{(0.3)\} \leq d_{int} := \left(\frac{C_1}{C_2}\right)^{1/2}\nu^{-2}. \tag{3.20}$$

Let us now return to the Lyapunov dimension of the extended system (0.2). Then, arguing as in the previous example, we deduce the following equation for the number d_{add}

$$C_1\nu^{-3} - C_2\nu(d_{int} + d_{add})^2 = -\sum_{i=1}^{k}(\mu_i^{ext})_+. \tag{3.21}$$

Using now the explicit value for d_{int} (and dropping out the term with d_{add}^2), we have

$$d_{add} \le \frac{\nu}{\sqrt{C_1C_2}}\sum_{i=1}^{k}(\mu_i^{ext})_+ \tag{3.22}$$

and, consequently,

$$\dim_L(\mathcal{A}) \le d_{int} + k + \frac{\nu}{\sqrt{C_1C_2}}\sum_{i=1}^{k}(\mu_i^{ext})_+. \tag{3.23}$$

We note that, estimating $(\mu_i^{ext})_+ \le \mu_{max} := \mu \ge 0$, we infer

$$\dim_L(\mathcal{A}) \le d_{int} + k + \frac{\nu}{\sqrt{C_1C_2}}k\mu \tag{3.24}$$

which has the similar form to (0.8), but in contrast to (0.8) (where we need to take $M_\delta \ge C_\delta\nu^{-1}$ in order to preserve the asymptotic $\sim \nu^{-2}$ of the first term), we now have the *small* constant $\nu/\sqrt{C_1C_2}$ (which is proportional to $\nu \ll 1$) in the third term.

We conclude our exposition by the example where both of the spaces H_1 and H_2 are infinite-dimensional.

Example 3.5. Let us consider the following system of reaction-diffusion equations in a bounded domain $\Omega \subset \mathbb{R}^m$:

$$\begin{cases} \partial_t u = \Delta_x(\nu u + L(v)) - f_1(u), \ u\big|_{\partial\Omega} = 0, \ \ u\big|_{t=0} = u_0, \\ \partial_t v = \nu\Delta_x v - f_2(v), \ v\big|_{\partial\Omega} = 0, \ v\big|_{t=0} = v_0, \end{cases} \tag{3.25}$$

where the functions f_1 and f_2 satisfy conditions (3.6) and $L \in C_0^\infty(\mathbb{R})$ is a given coupling function.

Then, it is not difficult to verify, arguing in a standard way (see [1], [4]), that, for every $(u_0, v_0) \in H := [L^2(\Omega)]^2$, this problem possesses a unique solution $(u(t), v(t))$ for every $t \ge 0$ and thus, generates a dissipative semigroup $\mathbb{S}_{\mathbb{A}}(t)$ in the phase space H. Moreover, this semigroup possesses a global attractor $\mathcal{A}$ in H and $\mathbb{S}_{\mathbb{A}}(t) \in C^1(\mathcal{A})$ for every $t \ge 0$.

Thus, since $\mathbb{A} := \mathbb{S}_{\mathbb{A}}(1)$ has the structure of (0.1), we can apply our theory in order to estimate the Lyapunov dimension of this global attractor. To this end, we note that the variation equations for computing both of the quantities Λ_d^{int} and Λ_d^{ext} now have the form of equation (3.11). Consequently, using the generalized Liouville theorem and arguing in a standard way (see [4], [9]), we deduce that

$$\max\{\ln\Lambda_d^{int}, \ln\Lambda_d^{ext}\} \le C_1 d - C_2\nu d^{1+2/m}, \tag{3.26}$$

where the constants C_1 and C_2 depend only on m, K and Ω. Then, due to estimate (2.1), we have

$$\ln \Lambda_d^{un} \le C_1 d - C_2 \nu \min_{k=0,\cdots,d} \{k^{1+2/m} + (d-k)^{1+2/m}\} \le C_1 d - C_2' \nu d^{1+2/m}, \tag{3.27}$$

for the appropriate new constant C_2' and, consequently,

$$\dim_L(\mathcal{A}) \le \left(\frac{C_1}{C_2'}\right)^{m/2} \nu^{-m/2} \tag{3.28}$$

which has the same form as (3.13).

We however not that the direct application of the volume contraction machinery to equation (3.25) (with the most natural 'naive' inner product in $H = [L^2(\Omega)]^2$) fails, since the main part of the generator of the variation equation associated with (3.25) has the form

$$\begin{pmatrix} \nu\Delta_x & , & L'(v(t))\Delta_x \\ 0 & , & \nu\Delta_x \end{pmatrix} \tag{3.29}$$

and the quadratic form associated with this operator (using this 'naive' scalar product) is not bounded from above (if $L'(v(t)$ is large enough).

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