THE TRAJECTORY ATTRACTOR FOR A NONLINEAR ELLIPTIC SYSTEM IN A CYLINDRICAL DOMAIN WITH PIECEWISE SMOOTH BOUNDARY

Introduction

In the half–cylinder $\Omega_+ = \mathbb{R}_+ \times \omega$, where ω is a bounded polyhedral domain in $\mathbb{R}^n$, we consider the following elliptic system

$$\begin{cases} a(\partial_t^2 u + \Delta u) + \gamma \partial_t u - f(u) = g(t) \\ \partial_n u|_{\partial\omega} = 0 \ \ ; \ \ u|_{t=0} = u_0 \end{cases} \tag{0.1}$$

Here $(t, x) \in \Omega_+$, Δ - is Laplacian with respect to the variable $x = (x^1, \cdots, x^n)$, $u = u(t, x) = (u^1, \cdots, u^k)$ – is unknown vector function, $f = (f^1, \cdots, f^k)$ and $g = (g^1, \cdot, g^k)$ are given functions, a – is a given positive selfadjoint matrix ($a \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^k)$)

$$a = a^* > 0$$

and γ is an arbitrary constant matrix.

Recall that ω is plyhedral if any of its boundary points b is either regular or there are a polyhedron $H \subset \mathbb{R}^n$, a non regular boundary point b_1 of H, open subsets U, V of $\mathbb{R}^N$ with $b \in U$, $b_1 \in V$, and a diffeomorphizm $\chi : U \to V$ such that $\chi(b) = b_1$ and $\chi(\overline{\omega} \cap U) = \overline{H} \cap V$.

We suppose that the nonlinear term $f(u)$ satisfies the following conditions

$$\begin{cases} 1. \ \ f \in C(\mathbb{R}^k, \mathbb{R}^k) \\ 2. \ \ f(u).u \geq -C_1 + C_2|u|^p \ , \ 2 + \frac{4}{n-3} > p > 2 \\ 3. \ \ |f(u)| \leq C(1 + |u|^{p-1}) \end{cases} \tag{0.2}$$

Here and below we denote by $u.v$ the usual scalar product of vectors u and v in the space $\mathbb{R}^k$.

The right-hand side g is supposed to belong to the space $[L_2(\Omega_T)]^k$ for all $T \geq 0$ where $\Omega_T = (T, T+1) \times \omega$ and to have a finite norm

$$|g|_b = \sup_{T \geq 0} ||g, \Omega_T||_{0,2} < \infty \tag{0.3}$$

We suppose also that the initial data u_0 belongs to the space V_0 of restrictions with respect to $\{t = 0\}$ of functions from the space $F_0^+ = [H_{\mathbb{Q},b}(\Omega_+)]^k$ (see Appendix 1) and the right-hand side g is translation compact in L_2 (see §3).

The function $u(t, x)$ is said to be a solution of the problem (0.1) if u belongs to the space $[H_{\mathbb{Q},b}(\Omega_+)]^k$ and satisfies the equation (0.1) in a sense of distributions.

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§1 A PRIORI ESTIMATES

In this Section we obtain some a priori estimates for the solutions of our problem (0.1). We will use these estimates in the next Sections in order to prove the solutions existense and construct the trajectory attractor of the problem (0.1).

Theorem 1.1. *Let u – be a solution of the problem* (0.1). *Then the following estimate is valid*

$$\|u,\Omega_T\|^2_{1,2} \le C(\|u_0\|^p_{V_0}\chi(1-T)+1+\|g,\Omega_{T-1,T+2}\|^2_{0,2}) \tag{1.1}$$

Here $\Omega_{T-1,T+2}=[\max\{0,T-1\},T+2]\times\omega$, $\chi(z)$ is Heviside function ($\chi(z)=0$ for $z<0$ and $\chi(z)=1$ for $z\ge 0$) and C does not depend on u.

Remark 1.2. *Due to the results of Appendix 1 the nonlinear term $f(u)$ belongs to the space $\left[L^2_{\text{loc}}(\Omega_+)\right]^k$ and so the equation* (0.1) *can be considered as equality in this space.*

Proof. By the definition of the space V_0 there exists a function $v\in H_{\mathbb{Q},b}(\Omega_+)$ such that $\operatorname{supp} v\subset\Omega_0$, $v\big|_{t=0}=u_0$ and

$$\|v,\Omega_0\|_{\mathbb{Q}} \le C\|u_0\|_{V_0} \tag{1.2}$$

where the constant C does not depend on u_0.

Let us rewrite our problem with respect to a new unknown function $w=u-v$

$$\begin{cases} a(\partial_t w+\Delta_x w)-\gamma\partial_t w-f(w+v)=g(t)+a(\partial_t v+\Delta_x v)-\gamma\partial_t v\equiv h(t)\\ w\big|_{t=0}=0\end{cases} \tag{1.3}$$

It follows from the choice of v that

$$\|h,\Omega_T\|_{0,2}\le C(\|g,\Omega_T\|_{0,2}+\chi(1-T)\|u_0\|_{V_0}) \tag{1.4}$$

Let $\phi(t)=\phi_T(t)$ be the following cut–of function

$$\phi(t)=\begin{cases}(|t-T+1/2|-1)^{\frac{2p}{p-2}} &, \quad \text{for } t\in[T-1/2,T+3/2]\\ 0 &, \quad \text{for } t\notin[T-1/2,T+3/2]\end{cases}$$

It is very easy to calculate that $\phi'\in L_\infty(\mathbb{R})$ and the following estimate is valid

$$|\phi'(t)|\le C\phi(t)^{\frac12+\frac1p}\ ,\ t\in\mathbb{R} \tag{1.5}$$

Let us multiply the equation (1.3) in $\mathbb{R}^k$ by the function ϕw and integrate over Ω_+

$$\left\langle a\partial_t^2 w,\phi w\right\rangle+\langle a\Delta_x w,\phi w\rangle-\langle\gamma\partial_t w,\phi w\rangle-\langle f(v+w),\phi w\rangle=\langle h,\phi w\rangle \tag{1.6}$$

It follows from the positivness of a and from the estimate (1.5) that

$$\begin{aligned} -\left\langle a\partial_t^2 w,\phi w\right\rangle &\ge C_1\left\langle\phi|\partial_t w|^2,1\right\rangle-\langle|\phi'||\partial_t w|,|w|\rangle\ge\\ &\ge C_1\left\langle\phi|\partial_t w|^2,1\right\rangle-\frac{C_1}{2}\left\langle\phi|\partial_t w|^2,1\right\rangle-C\left\langle\phi^{2/p}|w|^2,1\right\rangle\ge\\ &\ge C_2\left\langle\phi|\partial_t w|^2,1\right\rangle-C\left\langle\phi^{2/p}|w|^2,1\right\rangle\end{aligned} \tag{1.7}$$

Applying Holder inequality to the second term of (1.6) we obtain

$$(1.8)\quad |\langle\gamma\partial_t w,\phi w\rangle| \le \mu\left\langle\phi|\partial_t w|^2,1\right\rangle + C_\mu\left\langle\phi|w|^2,1\right\rangle \le \\ \le \mu\left\langle\phi|\partial_t w|^2,1\right\rangle + C_\mu\left\langle\phi^{2/p}|w|^2,1\right\rangle$$

This estimate is valid for any positive μ.

Due to the conditions to our nonlinear function $f(u)$

$$(1.9)\quad \langle f(w+v),\phi w\rangle = \langle f(w+v).(w+v),\phi\rangle - \langle f(w+v),v\phi\rangle \ge \\ \ge -C + C_1\langle\phi|w+v|^p,1\rangle - C\left\langle 1+|w+v|^{p-1},\phi v\right\rangle \ge \\ \ge -C_2(1+\langle\phi|v|^p,1\rangle) + C_3\langle\phi|w|^p,1\rangle \ge -C_4(1+\chi(1-T)\|u_0\|^p_{V_0}) + C_3\langle\phi|w|^p,1\rangle$$

Here we've used the embedding (A.11) and the estimate (1.2).

Using the positiveness of a we obtain after integrating by part

$$(1.10)\qquad -\langle a\Delta_x w,\phi w\rangle \ge C\left\langle\phi|\nabla w|^2,1\right\rangle$$

And due to the estimate (1.4) and Holder inequality

$$(1.11)\quad |\langle h,\phi w\rangle| \le \left\langle\phi|h|^2,1\right\rangle + \left\langle\phi|w|^2,1\right\rangle \le \\ \le C(\left\langle\phi|g|^2,1\right\rangle + \chi(1-T)\|u_0\|^2_{V_0}) + C_1\left\langle\phi^{2/p}|w|^2,1\right\rangle$$

Replacing all terms of equality (1.6) by their estimates (1.7)– (1.11) we get after simple calculations

$$(1.12)\quad \left\langle\phi|\partial_t w|^2,1\right\rangle + \left\langle\phi|\nabla w|^2,1\right\rangle + \langle\phi|w|^p,1\rangle - C\left\langle\phi^{2/p}|w|^2,1\right\rangle \le \\ \le C_1(1+\left\langle\phi|g|^2,1\right\rangle + \chi(1-T)\|u_0\|_{V_0})$$

Let us estimate the last term at the left–hand side of (1.12) by Holder inequality

$$\left\langle\phi^{2/p}|w|^2,1\right\rangle = \left\langle|\phi^{1/p}w|^2,1\right\rangle \le C\langle\phi|w|^p,1\rangle^{2/p} \le \mu\langle\phi|w|^p,1\rangle + C_\mu$$

for any positive μ. Let us take $\mu > 0$ sufficiently small and apply this estimate to the inequality (1.12)

$$(1.13)\quad \left\langle\phi|\partial_t w|^2,1\right\rangle + \left\langle\phi|\nabla w|^2,1\right\rangle + \langle\phi|w|^p,1\rangle \le C_2(1+\left\langle\phi|g|^2,1\right\rangle + \chi(1-T)\|u_0\|_{V_0})$$

Recall that by definition $\phi(t) > C_0 > 0$ for $t \in [T,T+1]$. Hence it follows from the estimate (1.13) that

$$(1.14)\qquad \|w,\Omega_T\|^2_{1,2} \le C(1+\chi(1-T)\|u_0\|^p_{V_0} + \|g,\Omega_{T-1,T+1}\|^2_{0,2})$$

Theorem 1.1 is proved.

Remark 1.3. *It follows also from the estimate* (1.13) *that*

$$(1.15)\qquad \|u,\Omega_T\|^p_{0,p} \le C(\chi(1-T)\|u_0\|^p_{V_0} + \|g,\Omega_{T-1,T+2}\|^2_{0,2})$$

Theorem 1.4. *Let u be a solution of the equation* (0.1) *then for every $T \geq 0$ the following estimate is valid*

$$(1.16)\quad \|u, \Omega_T\|_{0,2(p-1)}^{2(p-1)} \leq \\ \leq C(1 + \|g, \Omega_{T-1,T+2}\|_{0,2}^2 + \|u, \Omega_{T-1,T+2}\|_{0,p}^p + \chi(1-T)\|u_0\|_{V_0}^{2(p-1)})$$

Here the exponent p were defined in (0.2).

Proof.

Let us fix some $T \geq 0$ and define another cut–off function $\varphi(t) \in C_0^\infty(\mathbb{R})$, such that $\varphi(t) = 1$ for $t \in [T, T+1]$ and $\varphi(t) = 0$ for $t \notin [T-1, T+2]$, $0 \leq \varphi(t) \leq 1$. Multiplying the equation (1.3) by the function $\varphi w|w|_a^{p-2}$, where $|w|_a \equiv (aw.w)^{1/2}$ and integrating over Ω_+ we obtain the following equality

$$(1.17)\quad \left\langle a(\partial_t^2 w + \Delta_x w), \phi w|w|_a^{p-2}\right\rangle = \\ = \left\langle \varphi\gamma\partial_t w, w|w|_a^{p-2}\right\rangle + \left\langle \varphi f(w+v).w, |w|_a^{p-2}\right\rangle + \left\langle \varphi h, w|w|_a^{p-2}\right\rangle$$

Recall that due to the the space $H_{\mathbb{Q}}$ definition $\partial_t^2 w + \Delta_x w \in L_2$ and due to the embedding (A.11) functions $w|w|_a^{p-2}$ and $f(w+v)$ are also from the space L^2 hence all of the integrals in (1.17) are correctly defined. Moreover due to Theorem A.7 $w|w|_a^{p-2} \in H^{1,2}(\Omega_{T-1,T+2})$ hence we can integrate by part the left–hand side of (1.17).

$$(1.18)\quad \left\langle a\partial_t^2 w, \phi w|w|_a^{p-2}\right\rangle = -\left\langle a\partial_t w, \partial_t(\phi w|w|_a^{p-2})\right\rangle = \\ = -\frac{1}{p}\left\langle \phi', \partial_t(|w|_a^p)\right\rangle - \left\langle \phi|\partial_t w|_a^2, |w|_a^{p-2}\right\rangle - (p-2)\left\langle \phi(a\partial_t w.w)^2, |w|_a^{p-4}\right\rangle = \\ = \frac{1}{p}\left\langle \phi'', |w|_a^p\right\rangle - \left\langle \phi|\partial_t w|_a^2, |w|_a^{p-2}\right\rangle - \frac{4(p-2)}{p^2}\left\langle \phi\partial_t(|w|_a^{p/2}), \partial_t(|w|_a^{p/2})\right\rangle \leq \\ \leq C_1\|w, \Omega_{T-1,T+2}\|_{0,p}^p - C_2\left\langle \phi\partial_t(|w|_a^{p/2}), \partial_t(|w|_a^{p/2})\right\rangle$$

Analogously

$$\left\langle a\Delta_x w, \phi w|w|_a^{p-2}\right\rangle \leq -C_2\left\langle \phi\nabla_x(|w|_a^{p/2}), \nabla_x(|w|_a^{p/2})\right\rangle$$

Hence

$$(1.19)\quad -\left\langle a(\partial_t^2 w + \Delta_x w), \phi w|w|_a^{p-2}\right\rangle \geq C_1\|w, \Omega_{T-1,T+2}\|_{0,p}^p + \\ + C_2\left(\left\langle \phi\partial_t(|w|_a^{p/2}), \partial_t(|w|_a^{p/2})\right\rangle + \left\langle \phi\nabla_x(|w|_a^{p/2}), \nabla_x(|w|_a^{p/2})\right\rangle\right)$$

It follows from Holder inequality that

$$(1.20)\quad |\left\langle \gamma\partial_t w, \phi w|w|_a^{p-2}\right\rangle| \leq \mu\left\langle \phi\partial_t(|w|_a^{p/2}), \partial_t(|w|_a^{p/2})\right\rangle + C_\mu\left\langle \phi|w|^p, 1\right\rangle$$

and

$$(1.21)\quad |\left\langle h, \phi w|w|^{p-2}\right\rangle| \leq \mu\left\langle \phi|w|^{2(p-1)}, 1\right\rangle + C_\mu\left\langle \phi|h|^2, 1\right\rangle \leq \\ \leq \mu\left\langle \phi|w|^{2(p-1)}, 1\right\rangle + C_\mu(\|g, \Omega_{T-1,T+2}\|_{0,2}^2 + \chi(1-T)\|u_0\|_{V_0}^2)$$

Here μ is an arbitrary positive number.

Arguing as in redusing (1.9) we obtain

$$\left\langle f(w+v),\phi w|w|_a^{p-2}\right\rangle \geq -C_1(1+\left\langle \phi|v|^{2(p-1)},1\right\rangle)+C_2\left\langle \phi|w|^{2(p-1)},1\right\rangle \geq \\ \geq -C_3(1+\chi(1-T)\|u_0\|_{V_0}^{2(p-1)})+C_2\left\langle \phi|w|^{2(p-1)},1\right\rangle \tag{1.22}$$

Replacing all of the terms in equality (1.17) by their estimates (1.19)–(1.22) and taking sufficiently small $\mu>0$ we obtain after simple calculations

$$\left\langle \phi|w|^{2(p-1)},1\right\rangle \leq \\ \leq C(1+\|w,\Omega_{T-1,T+2}\|_{0,p}^p+\|g,\Omega_{T-1,T+2}\|_{0,2}^2+\chi(1-T)\|u_0\|_{V_0}^{2(p-1)}) \tag{1.23}$$

Theorem 1.4 is proved.

Remark 1.5. *It follows immediately from the estimates* (1.16) *and* (1.15) *that*

$$\|f(u),\Omega_T\|_{0,2}\leq C\left(1+\|g,\Omega_{T-1,T+2}\|_{0,2}+\chi(1-T)\|u_0\|_{V_0}^{p-1}\right) \tag{1.24}$$

Theorem 1.6 (The main estimate). *Let u–be a solution of the problem* (0.1)*. Then the following estimate is valid*

$$\|u,\Omega_T\|_{\mathbb{Q}}\leq C(\|u_0\|_{V_0}^{p-1}\chi(1-T)+1+\|g,\Omega_{T-1,T+2}\|_{0,2}) \tag{1.25}$$

Proof. Let us rewrite the equation (1.3) in the following form

$$\begin{cases} \partial_t^2(\varphi w)+\Delta(\varphi w)=h_w(t) \\ \varphi w\big|_{\max\{T-1,0\}}=0\ ;\ \varphi w\big|_{T+2}=0\ ;\ \partial_n(\varphi w)\big|_{\partial\omega}=0 \end{cases} \tag{1.26}$$

Here φ is the same as in Theorem 1.4 and

$$h_w(t)=\varphi''w+2\varphi'\partial_t w-\mathrm{a}^{-1}(\varphi h(t)+\varphi f(u)-\gamma\partial_t w) \tag{1.27}$$

Due to the (1.24) and (1.1) we have the following estimate

$$||h_w,\Omega_{T-1,T+2}||_{0,2}\leq C(\|u_0\|_{V_0}^{p-1}\chi(2-T)+1+\|g,\Omega_{T-2,T+3}\|_{0,2}) \tag{1.28}$$

Due to the L^2–regularuty Theorem (see Appendix 1)

$$\|w,\Omega_+\cap\Omega_T\|_{\mathbb{Q}}\leq C_1\|\varphi w,\Omega_{T-1,T+2}\|_{\mathbb{Q}}\leq \\ \leq C\|h_w,\Omega_{T-1,T+2}\|_{0,2}\leq C_2(\|u_0\|_{V_0}\chi(2-T)+1+\|g,\Omega_{T-2,T+3}\|_{0,2}) \tag{1.29}$$

Theorem 1.6 is proved.

Remark 1.7. *Let the condition* (0.3) *be valid. Then any solution u of the problem* (0.1) *from the space $H^{\mathrm{loc}}_{2,\mathbb{Q}}(\Omega_+)$ is automatically bounded with respect to $t\to\infty$ i.e.*

$$\|u\|_b\equiv\sup_{T\geq 0}\|u,\Omega_T\|_{\mathbb{Q}}\leq C(1+\|u_0\|_{V_0}^{p-1}+|g|_b)<\infty \tag{1.30}$$

Indeed the estimate (1.31) follows immediately from the estimate (1.25)

§2 The solution existence.

In this Section we prove the solvability of the problem (0.1). For the fisrt we solve the following auxiliary problem in finite cylinder

$$\left\{\begin{array}{l} a(\partial_t^2 u+\Delta_x u)+\gamma\partial_t u-f(u)=g(t)\\ u\big|_{t=0}=u_0\ \ ;\ \ u\big|_{t=M}=u_1\end{array}\right. \tag{2.1}$$

Here $u_0, u_1 \in V_0$ and $u \in H_{2,\mathbb{Q}}(\Omega_{0,M})$.

We'll get solution u of the main problem (0.1) as a limit of solutions u_M of the corresponding auxiliary problems (2.1) when $M \to \infty$.

Theorem 2.1. *Let u – be the solution of the problem* (2.1). *Then the following estimate is valid uniformly with respect to $M \to \infty$*

$$\begin{aligned} &\|u,\Omega_T\|_{2,\mathbb{Q}} \le \\ &\le C(1+\chi(1-T)\|u_0\|_{V_0}^{p-1}+\chi(T-M+1)\|u_1\|_{V_0}^{p-1}+\|g,\Omega_{T-1,T+2}\cap\Omega_{0,M}\|_{0,2}) \end{aligned} \tag{2.2}$$

The proof of this estimate is the same as the proof of estimate (1.25) given in the previuos Section for the case of semibounded cylinder.

Theorem 2.2. *The problem* (2.1) *has at least one solution for any $u_0, u_1 \in V_0$.*

Proof. Let us introduce the space

$$W=\left\{w\in H_{2,\mathbb{Q}}(\Omega_{0,M})\ :\ w\big|_{t=0}=w\big|_{t=M}=0\right\} \tag{2.3}$$

and rewrite the problem (2.1) as an equation in the space W. For the first we rewrite our problem with respect to new function $w = u - v$, where $w \in W$, $v \in H_{2,\mathbb{Q}}(\Omega_{0,M})$.

$$\left\{\begin{array}{l} \partial_t^2 w+\Delta_x w=a^{-1}(-\gamma\partial_t w+f(v+w)+g_1(t))\\ w\big|_{t=0}=w\big|_{t=M}=0\end{array}\right. \tag{2.4}$$

Here $g_1 = -a(\partial_t^2 v + \Delta_x v) - \gamma\partial_t v + g$.

Let's denote by A the inverse operator for the Laplacian with respect to variables $(t, x) \in \Omega_{0,M}$ and appropriate boundary conditions ($w\big|_{t=0} = 0$, $w\big|_{t=M} = 0$, $\partial_n w\big|_{x\in\partial\omega} = 0$). Then due to results of Appendix 1

$$A\ : L_2(\Omega_{0,M}) \to W \tag{2.5}$$

Aplying the operator A to both sides of equation (2.4) we obtain

$$w+F(w)=h\equiv -A(\partial_t^2 v+\Delta_x v) \tag{2.6}$$

where

$$F(w)=-Aa^{-1}(-\gamma\partial_t w+f(v+w)+g-\gamma\partial_t v)$$

We'll use Leray–Shauder principle in the following form (see [10])

Leray–Shauder principle. *Let D be a bounded open set of B-space W and let $F : \overline{D} \to W$ be a compact continious operator. Let also the point $h \in D$ be such that*

$$w + sF(w) \neq h \text{ forall } w \in \partial D \text{ and } s \in [0,1] \tag{2.7}$$

Then the equation

$$w + F(w) = h$$

has at least one solution in D.

Let D – be a ball of sufficiently large radius in W and Let us suppose that

$$w_s + sF(w_s) = h \text{ for some } s \in [0,1] \text{ and } w_s \in \partial D \tag{2.8}$$

Equation (2.8) can be written in the following form

$$\begin{cases} a(\partial_t^2 u_s + \Delta_x u_s) + s\gamma\partial_t u_s - sf(u_s) = sg(t) \\ u_s\big|_{t=0} = u_0 \ ; \ u_s\big|_{t=M} = u_1 \end{cases} \tag{2.9}$$

Here $u_s = w_s + v$.

Equation (2.9) has the view (2.1). It is not difficult to see that the estimate (2.2) is uniform with respect to $s \in [0,1]$. Hence

$$||w_s||_W \leq K$$

for all solutions of (2.9) uniform with respect to $s \in [0,1]$. So condition (2.7) is valid if the radius of D is greater then K.

Let's prove the compactness of operator F. It is sufficient to prove the compactness only for the nonlinear part $Aa^{-1}f(w+v)$ of this operator. To do this let's decompose previous nonlinear operator in the composition of three continious operators $A \circ F_2 \circ F_1$, and one of them is compact ($F_1 : W \to L^{2(p-1)}$ – embedding operator, wich is compact because $2(p-1) < q$ (see Theorem A.7) and $F_2 w = a^{-1}f(v+w)$). As known, operator F_2 is continious from $L^{2(p-1)}$ to L^2 (due to conditions (0.2) and Krasnoselskiy theorem (see [11])). Hence operator F is compact and according to Leray–Shauder principle the problem (2.1) has at least one solution. □

Theorem 2.3. *The problem* (0.1) *has at least one solution* $u \in H_{\mathbb{Q},b}(\Omega_+)$

Proof. Let's consider a sequence u_M of solutions of auxiliary problems (2.1) with $M = 1, 2, \dots$ and $u_1\big|_{t=M} = 0$. It follows from Theorem 2.1 that

$$||u_M, \Omega_{0,N}||_{2,\mathbb{Q}} \leq C(u_0, N, g)$$

uniform with respect to $M \geq N$ (for every fixed N). So using Cantor diagonalization process one can extract from u_M a subsequence (which will be denoted for simplicity as u_M again) with the following property

$$u_m\big|_{\Omega_{0,N}} \rightharpoonup u\big|_{\Omega_{0,N}} \text{ in the space } H^2_{\mathbb{Q}}(\Omega_{0,N})$$

for some $u \in H_{\mathbb{Q},b}(\Omega_+)$. Let's prove that u is a solution of (0.1). It is sufficient to prove that for every $\Phi \in C_0^\infty(\Omega_+)$ the following equality is valid

$$-\langle a\partial_t u, \partial_t \Phi\rangle - \langle a\nabla_x u, \nabla_x \Phi\rangle + \langle \gamma\partial_t u, \Phi\rangle - \langle f(u), \Phi\rangle = \langle g, \Phi\rangle \tag{2.10}$$

It follows from the definition of u_M that

$$-\langle a\partial_t u_M, \partial_t \Phi\rangle - \langle a\nabla_x u_M, \nabla_x \Phi\rangle + \langle \gamma\partial_t u_M, \Phi\rangle - \langle f(u_M), \Phi\rangle = \langle g, \Phi\rangle \tag{2.11}$$

for sufficiently large M. Taking a limit $M \to \infty$ in the equation (2.11) we obtain (2.10). Indeed the only nontrivial problem is to prove that

$$\langle f(u_M), \Phi\rangle \to \langle f(u), \Phi\rangle \tag{2.12}$$

Let's suppose that $\operatorname{supp}\Phi \subset \Omega_{0,N}$. It follows from the conditions (0.2) and Theorem A.7 that embedding $H^2_{\mathbb{Q}} \subset L^{2(p-1)}$ is compact. Hence $u_M \to u$ in $L^{2(p-1)}(\Omega_{0,N})$ and $f(u_M) \to f(u)$ in $L^2(\Omega_{0,N})$. Theorem 2.3 is proved.

§3 Trajectory attractor of nonlinear elliptic system.

In this Section we construct the trajectory attractor for the problem (0.1). Recall shortly the main concepts and definitions of the abstract theory of trajectory attractors for dynamical systems (see [6], [7] for complete exposition).

Definition 3.1. *The right–hand side g of the problem* (0.1) *is said to be translation compact in the space*

$$\Xi^+ = \left[L^2_{\mathrm{loc}}(\mathbb{R}_+, L_2(\omega))\right]^k$$

if it's hull

$$\mathcal{H}^+(g) = [T_s g, s \geq 0]_{\Xi^+} \ , \ (T_s g)(t) = g(t+s)$$

is a compact set in Ξ^+. Here $[\cdot]_{\Xi^+}$ means the closure in the space Ξ^+ .

The right–hand side g of the problem (0.1) *is said to be weak translation compact in the space Ξ^+ if it's weak hull*

$$\mathcal{H}^+_w(g) = [T_s g, s \geq 0]_{(\Xi^+)^w}$$

is a compact set in $(\Xi^+)^w$. Here and below $(\Xi^+)^w$ means the space Ξ^+ endowed by a weak topology.

Remark 3.2. *It is diffcult to prove (see* [20]*) that if the function g is translation–compact (in a strong topology) then*

$$\mathcal{H}^+(g) = \mathcal{H}^+_w(g) \tag{3.1}$$

Remark 3.3. *It is evidant that t–periodic or quasi–periodic (or almost periodic by Bochner in $L^2(\omega)$) function g is translation compact in the space Ξ^+ (in strong topology). So the concept of a translation–compact function is a some generalization for a concept of an almost–periodic function.*

Remark 3.4. *It follows immediately from the hull's definition that*

$$T_s\mathcal{H}^+(g) \subset \mathcal{H}^+(g) \text{ and } T_s\mathcal{H}^+_w(g) \subset \mathcal{H}^+_w(g) \text{ for } t \geq 0 \tag{3.2}$$

i.e the semigroup of shifts $\{T_s,\ s \geq 0\}$ *acts in the spaces* $\mathcal{H}^+(g)$ *and* $\mathcal{H}^+_w(g)$.

Now we formulate the nessesary and sufficient conditions of translation compactness and weak translation compactness in the space Ξ^+.

Theorem 3.5 [8].
1. A function g *is weak translation compact in* Ξ^+ *if and only if it is bounded with respect to* $t \to \infty$ *i.e* $|g|_b < \infty$.
2. A function g is translation compact in Ξ^+ *if and only if the following conditions is valid*
a) For every fixed $t > 0$ *the set* $\{\int_s^{t+s} g(z)\,dz\,,\ s \in \mathbb{R}_+\}$ *is precompact in the space* $\left[L^2(\omega)\right]^k$.
b) There exists the function $\beta(s)$, $s \geq 0$, $\beta(s) \to 0$ *when* $s \to +0$, *such that*

$$\int_t^{t+1} \|g(z) - g(z+l)\|_{L_2(\omega)}\,dz \leq \beta(|l|)\ ,\ \forall t \in \mathbb{R}_+\ ;\ t+l \in \mathbb{R}_+ \tag{3.3}$$

Remark 3.6. *Condition* (3.3) *is valid for example if*

$$\|T_s g, [0,1] \times \omega\|_{\delta,2} \leq C\ ,\ \forall s \in \mathbb{R}_+$$

for some $\delta > 0$.

To construct the trajectory attractor for the problem (0.1) we consider (together with the equation (0.1)) family of problems of view (0.1) obtained by all positive shifts of the initial problem (0.1) and their limits in the appropriate topology

$$\begin{cases} a(\partial_t^2 u + \Delta_x u) + \gamma\partial_t u - f(u) = \sigma(t) \\ \sigma \in \Sigma \end{cases} \tag{3.4}$$

here we take $\Sigma = \mathcal{H}^+(g)$ if g is translation–compact in a strong topology and else we take $\Sigma = \mathcal{H}^+_w(g)$.

Definition 3.7. *For every function* σ *from* Σ *we define* K^+_σ *of as a set of all solutions for the equation* (3.4) *with a fixed* $\sigma \in \Sigma$ *and with an arbitrary* $u_0 \in V_0$.
We denote by K^+_Σ *the following set:*

$$K^+_\Sigma = \cup_{\sigma\in\Sigma} K^+_\sigma$$

It follows from (3.2) that a semigroup $\{T_s, s \geq 0\}$ of all nonnegative shifts along the t–axis $((T_s v)(t) \equiv v(t+s))$ acts in the space K^+_Σ i.e.

$$T_s K^+_\Sigma \subset K^+_\Sigma \text{ for } s \geq 0 \tag{3.5}$$

We endowed the set K^+_Σ by the toplogy induced from the embedding $K^+_\Sigma \subset \Theta^+_0$ if $\Sigma = \mathcal{H}^+(g)$ (the strong topology case) and from the embedding $K^+_\Sigma \subset (\Theta^+_0)^w$ if $\Sigma = \mathcal{H}^+_w(g)$ (the weak topology case) (see Apendix 1 for Θ^+_0 definition).

Definition 3.8. *The (global) atrractor of the semigroup $\{T_s, s \geq 0\}$ acting in topological space K_Σ^+ is called the trajectory attractor of the family* (3.4) *i.e. a set $\mathcal{A}_\Sigma \subset K_\Sigma^+$ is the trajectory attractor of* (3.4) *if the following conditions are valid*

(1) *$\mathcal{A}_\Sigma$ – is a compact set in K_Σ^+*
(2) *$\mathcal{A}_\Sigma$ is strictly invariant with respect to the semigroup T_s action, i.e.*

$$T_s \mathcal{A}_\Sigma = A_\Sigma$$

(3) *$\mathcal{A}_\Sigma$ is an attracting set for the semigroup $\{T_s,\ s \geq 0\}$, i.e. for every neighbourhood $\mathcal{O}(\mathcal{A}_\Sigma)$ in K_Σ^+ topology there exist such number $S_\mathcal{O}$ that*

$$T_s \mathcal{K}_\Sigma^+ \subset \mathcal{O}(\mathcal{K}) \text{ for every } s \geq S_\mathcal{O}. \tag{3.6}$$

Remark 3.9. *Usually one requare that the attracting property be valid only for bounded (in some sence) subsets of K_Σ^+ but due to the estimate* (1.26) *the set $T_1 K_\Sigma^+$ is bounded (as in Θ_0^+ so in $\mathcal{F}_0^+$). Hence the attracting property* (3.6) *is automatically valid for all subsets of K_Σ^+ with the same constant $S_\mathcal{O}$ (see also* [22]*).*

Theorem 3.10 [8]**.** *Let the following conditions be valid:*

1) There exists a compact attracting set $P \subset K_\Sigma^+$ for the semigroup $\{T_s, s \geq 0\}$.

2) The set K_Σ^+ is closed in the space Θ_0^+ (or sequentially closed in the space $(\Theta_0^+)^w$ in the weak topology case).

Then the family (3.6) *possesses a trajectory attractor $\mathbb{A} = \mathcal{A}_\Sigma$ in K_Σ^+.*

Definition 3.11. *The trajectory attractor $\mathbb{A}^w$ for the family* (3.6) *with $\Sigma = \mathcal{H}_w^+(g)$ (weak topology case) is said to be a weak trajectory attractor of the initial problem* (0.1)*.*

Analogously the trajectory attractor $\mathbb{A} = \mathbb{A}^s$ for the family (3.6) *with $\Sigma = \mathcal{H}^+(g)$ (strong topology case) is said to be a (srong) trajectory attractor of the initial problem* (0.1)*.*

Theorem 3.12.

1. Let the condition (0.3) *be valid. Then the problem* (0.1) *possesses a weak trajectory attractor $\mathbb{A}^w$.*

2. Let the right–hand side g be translation–compact in Ξ^+ (with the strong topology). Then the problem (0.1) *possesses a strong trajectory attractor $\mathbb{A} = \mathbb{A}^s$.*

Proof. Let us verify the conditions of previous Theorem.

Lemma 3.13. *The set K_Σ^+ is (sequentially) closed in the space $(\Theta_0^+)^w$.*

Proof. Let $u_n \in K_{\sigma_n}^+$, $u_n \to u$ in $(\Theta_0^+)^w$. Due to the compactness of Σ in $(\Xi^+)^w$ we may suppose without loss of generality that $\sigma_n \rightharpoonup \sigma \quad \Xi^+$. It is nessesary to prove that $u \in K_\sigma^+$. By definition the functions $u_n(t)$ are bounded solutions of the following problems

$$\begin{cases} a(\partial_t^2 u_n + \Delta_x u_n) + \gamma \partial_t u_n - f(u_n) = \sigma_n(t) \\ u_n|_{t=0} = u_n^0 \ ; \ u_n \in V_0 \end{cases} \tag{3.7}$$

Taking a limit $n \to \infty$ in (3.4) we obtain now (as in the proof of Theorem 2.3) that $u \in K_\sigma^+$. □

So the second condition of Theorem 3.10 is valid. Let us verify the first one. It follows from the estimate (1.25) that the set

$$P = B_R \cap K_\Sigma^+ \ , \ s > 0$$

where B_R is sufficiently large ball in the space F_0^+ is an attracting (and even absorbing) set for the semigroup $\{T_s, \ s \geq 0\}$. Let us consider firstly the weak topology case. Then as known the set B_R is a compact and metrizable subset of $(\Theta_0^+)^w$ Indeed the ball B_R is bounded in Θ_0^+ and Θ_0 is reflexive separable F–space hence B_R is semicompact and metrizable in a weak topology and metrizable. Hence due to the convexivity B_R is a metrizable compact in a weak topology. It follows from Lemma 1 now that the set P also compact in a weak topology.

Let us suppose now that the right–hand side g of the problem (0.1) is translation–compact in strong topology.

Lemma 3.14. *Let the previous condition be valid. Then for every $s \geq 0$ the set $T_s P$ is compact in the space Θ^+.*

Proof. Without loss of generality we can suppose that $s = 1$. Let $u_n \in B_R \cap K_{\sigma_n}^+$ be an arbitrary sequence. Due to the compactness of the set P in a weak topology we can extract from $\{u_n\}$ a subsequence (for simplisity we denote it also by $\{u_n\}$) weakly convergent to some $u \in K_\sigma^+$ and due to the compactness of the hull Σ in Ξ^+ (in a strong topology) we can suppose that $\sigma_n \to \sigma$ in Ξ^+.

To prove Lemma it is sufficient to prove that $u_n \to u$ in $[H_{2,\mathbb{Q}}(\Omega_T)]^k$ for an arbitrary $T \geq 1$.

By definition the functions u_n satisfy the equations (3.7) Let's multiply (3.7) by the cut–off function $\phi \in C_0^\infty(\mathbb{R})$, such that $\phi \equiv 1$ if $t \in [T, T+1]$ and $\phi \equiv 0$ if $t \notin [T-1, T+2]$. We obtain

$$\begin{cases} \partial_t^2(\phi u_n) + \Delta(\phi u_n) = a^{-1}(\phi\sigma_n + \phi f(u_n) - \gamma\phi\partial_t u_n) + \\ \qquad\qquad + 2\partial_t\phi\partial_t u_n + (\partial_t^2\phi)u_n = h_n(t) \\ \phi u_n|_{t=T-1} = 0 \ ; \ \phi u_n|_{t=T+2} = 0 \ ; \ \partial_n(\phi u_n)|_{x\in\partial\omega} = 0 \end{cases} \tag{3.8}$$

Arguing as in the proof of Theorem 2.3 we prove that $f(u_n) \to f(u)$ and $\partial_t u_n \to \partial_t u$ in Ξ^+. So $h_n \to h$ in the space $[L_2(\Omega_{T-1,T+2})]^k$ and by definition (see Appendix 1)

$$\phi u_n \to \phi u \text{ in } \mathcal{G} \text{ hence, } u_n \to u \text{ in } [H_{2,\mathbb{Q}}(\Omega_T)]^k$$

□

So all assumptions of Theorem 3.10 is valid. Theorem 3.12 is proved.

Corollary 3.15. *Let the right–hand side g be strongly translation compact in Ξ^+. Then*

$$\text{dist}_{2,\mathbb{Q}}(\Pi_{t_1,t_2} T_s K_\Sigma^+ \ , \ \Pi_{t_1,t_2}\mathbb{A}) \to 0 \ \textit{when} \ s \to +\infty$$

where

$$\text{dist}_{2,\mathbb{Q}}(M, N) = \sup_{x\in M} \inf_{y\in N} \|x - y, \Omega_{t_1,t_2}\|_{2,\mathbb{Q}}$$

Here by Π_{t_1,t_2} we denote the restriction operator to the segment $t \in [t_1, t_2]$.

Indeed, the assertions of this Corollary follows immediately from the attrctor's definition.

Corollary 3.16. *Let the right–hand side g be weakly translation compact in Ξ^+. Then*

$$\mathrm{dist}_{H^{3/2+\varepsilon,2}(\Omega_{t_1,t_2})}\left(\Pi_{t_1,t_2}T_sK_\Sigma^+\,,\,\Pi_{t_1,t_2}\mathbb{A}\right)\to 0 \ \textit{when}\ s\to+\infty$$

and

$$\mathrm{dist}_{L^q(\Omega_{t_1,t_2})}\left(\Pi_{t_1,t_2}T_sK_\Sigma^+\,,\,\Pi_{t_1,t_2}\mathbb{A}\right)\to 0 \ \textit{when}\ s\to+\infty$$

Where $\varepsilon>0$ is a sufficiently small positive number and $q<2\frac{n+1}{n-3}$.

Indeed, the assertions of this Corollary follows from the compactness of embeddings $H^2_{\mathbb{Q}}\subset H^{3/2+\varepsilon,2}$ and $H^2_{\mathbb{Q}}\subset L^q$ which have been proved in Appendix 1.

Now we investigate the structure of trajectory attractor $\mathbb{A}$ constructed in the previous Theorem.

Definition 3.17. *Let $\omega(\Sigma)$ be the ω–limit set (attractor) of the semigroup $\{T_s,\ s\geq 0\}$ acting in the compact space Σ. As known (see [2]) it is not empty and could be represented in the following form*

$$\omega(\Sigma)=\cap_{t\geq 0}\left[\cup_{s\geq t}\Sigma\right]_\Sigma$$

Here $[\cdot]_\Sigma$ means the closure in the space Σ.

Definition 3.18. *A function $\xi(t)$, $t\in\mathbb{R}$ is said to be a complete symbol of* (3.4) *in the space $\omega(\Sigma)$ if*

$$\Pi_+\xi_s(\cdot)\in\omega(\Sigma)\ ,\ \forall s\in\mathbb{R}$$

Here $\xi_s(t)=\xi(t+s)$, and Π_+ is the restriction operator to the semiaxis $t\geq 0$.

The set of all complete symbols of (3.4) we denote by $Z(\Sigma)$.

Lemma 3.19 [8]**.** *For every $\sigma\in\omega(\Sigma)$ there exists a complete symbol $\xi\in Z(\Sigma)$ such that $\Pi_+\xi=\sigma$.*

Definition 3.20. *Let $\xi\in Z(\Sigma)$ be a complete symbol of* (3.4)*. Let us denote by K_ξ – the set of all (bounded) solutions of the equation* (3.4) *in the whole axis $t\in\mathbb{R}$, in which we replace $\sigma(t)$ by $\xi(t)$.*

Theorem 3.21 [8]**.** *The attractor $\mathbb{A}$ has the following structure:*

$$\mathbb{A}=\Pi_+\cup_{\xi\in Z(\Sigma)}K_\xi \tag{3.9}$$

Corollary 3.22 [20]**.** *Let the right–hand side g of the equation* (0.1) *be strongly translation–compact in Ξ^+. Then the weak trajectory attractor of the problem* (0.1) *concides with it's strong attractor*

$$\mathbb{A}^s=\mathbb{A}^w$$

§4 Stabilization of solutions when $t \to \infty$

In this Section we ivestigate the long-time solutions behaviour in the case when right-hand side $g(t)$ of (0.1) can be represnted in the following form

$$g(t,x) = g_+(x) + g_1(t,x) \tag{4.1}$$

when $g_+ \in L^2(\omega)$ doesn't depend on t and g_1 satisfies the following condition

$$T_s g_1 \to 0 \quad \text{, when } s \to +\infty \tag{4.2}$$

in the space Ξ^+ or in the space or $(\Xi^+)^w$. It is not difficult to check that in the first case the function g is strong translation compact in Ξ^+ and in the second case it is weak translation compact.

Theorem 4.1. *Let the condition* (4.2) *be valid. Then the equation* (0.1) *with the right–hand side* (4.1) *possesses a strong (weak) trejectory attractor* $\mathbb{A} = \mathbb{A}_g$ *which consides with the attractor of the limit autonomous equation*

$$a(\partial_t^2 u + \Delta_x u) - \gamma \partial_t u - f(u) = g_+ \tag{4.3}$$

i.e

$$\mathbb{A}_g = \mathbb{A}_{g_+} \tag{4.4}$$

Proof. The attractor existance follows immediatly from Theorem 3.12. Let us check the equality (4.4).

It is easy to reduce from the condition (4.2) that

$$Z(g) \equiv Z(\Sigma) = w(\Sigma) = g_+$$

Here Σ is the strong (weak) hull of the right–hand side g in the space Ξ^+ (see Section 3). Hence formula (4.4) is valid due to Theorem 3.21. Theorem 4.1 is proved.

Let us suppose now that the nonlinear term $f(u)$ in the left–hand side of the equation (0.1) is gradient-like

$$f(u) = -\nabla F(u), \quad F \in C(\mathbb{R}^k, \mathbb{R}) \tag{4.5}$$

For every $u \in H_{\mathbb{Q},b}(\Omega_+)$ we introduce the function $\mathcal{F}_u(t)$ by the following formula

$$\mathcal{F}_u(t) = \frac{1}{2}(a\partial_t u(t), \partial_t u(t)) - \frac{1}{2}(a\nabla_x u(t), \nabla_x u(t)) + (F(u(t)), 1) - (g_+, u(t)) \tag{4.6}$$

where $(\cdot,\cdot)$ denotes the scalar product in the cross-section space $L^2(\omega)$.

Theorem 4.2.
1. The function $\mathcal{F}_u$ is well-defined for every $u \in H_{\mathbb{Q},b}(\Omega_+)$ and belonges to the space $H^{1,1}_{\mathrm{b}}(\mathbb{R}_+)$.
2. Let us suppose that u is a solution of the problem (0.1). Then

$$\frac{d\mathcal{F}_u(t)}{dt} = -(\gamma\partial_t u(t), \partial_t u(t)) + (g_1(t), \partial_t u(t)) \tag{4.7}$$

Proof. Let us suppose that $u \in H_{\mathbb{Q},b}(\Omega_+)$. Then due to the embedding (A.20) the first, the second and the fourth term in the right-hand side of (4.6) are well–posed. It remains to check the third term. It follows from (A.16) and (4.5) that

$$|F(u)| \le C(1 + |u|^p) \tag{4.8}$$

Then due to the embedding (A.16) and Krasnoselskij Theorem

$$(F(u(t)), 1) \in C_b(\mathbb{R}_+)$$

Hence the definition of $\mathcal{F}_u(t)$ is correct.
Let us calculate it's derivative. It is not difficult to obtain using the ordinary methods of distributions theory that $\mathcal{F}_u \in H^{1,1}_b(\mathbb{R}_+)$ and

$$\frac{d}{dt}\mathcal{F}_u(t) = (\partial_t u, a(\partial_t^2 u + \Delta_x u) - f(u) - g_+) \tag{4.9}$$

Hence the first part of Theorem 4.2 is proved.
Let us suppose now that u is a solution of the problem (0.1). Then (4.5) follows immediately from the formula (4.9). Theorem 4.2 is proved.

Theorem 4.3. *Let the conditions of previous Theorem be valid. Let us suppose also that the matrix γ in the left-hand side of (0.1) is sign-defined*

$$\gamma + \gamma^* > 0 \text{ or } \gamma + \gamma^* < 0$$

and function $g_1(t) = g_1(t, x)$ from (4.1) satisfies at least one of the following conditions

$$\begin{cases} 1. & \int_0^\infty \|g_1(t)\|_{0,2}\, dt < \infty \\ 2. & \partial_t g_1 \in L_1^{\mathrm{loc}}(\mathbb{R}_+, L_2(\omega)) \text{ and } \int_0^\infty \|\partial_t g_1(t)\|_{0,2}\, dt < \infty \\ 3. & \sum_{N=0}^\infty \|G_1, \Omega_N\|_{0,2} < \infty \text{ for some } G_1 \text{ such that } \partial_t G_1 = g_1 \end{cases} \tag{4.10}$$

Then every solution u of the problem (0.1) possesses the finite dissipative integral

$$\int_0^\infty \|\partial_t u(t)\|^2_{0,2}\, dt < \infty \tag{4.11}$$

Proof. Let us integrate (4.6) over $t \in [0, T]$

$$\int_0^T (\gamma\partial_t u, \partial_t u)dt = \mathcal{F}_u(0) - \mathcal{F}_u(T) + \int_0^T (g_1, \partial_t u)\, dt$$

It follows now from the sign-definess of matrix γ that

$$\int_0^T \|\partial_t u(t)\|_{0,2}^2\, dt \le C|\mathcal{F}_u(T) - \mathcal{F}_u(0)| + C\Big|\int_0^T (g_1, \partial_t u)\, dt\Big| \tag{4.12}$$

Theorem 4.2 implies that function $\mathcal{F}_u(T)$ is bounded with respect to $T \to \infty$ hence it sufficient to obtain the boundness of the integral in the right-hand side of (4.12).

Let the first condition of (4.10) be valid. Then

$$\begin{aligned} \Big|\int_0^T (g_1, \partial_t u)\, dt\Big| &\le \int_0^T \|g_1(t)\|_{0,2}\|\partial_t u(t)\|_{0,2}\, dt \le \\ &\le \sup_{t\in[0,T]} \|\partial_t u(t)\|_{0,2} \int_0^T \|g_1(t)\|_{0,2}\, dt \le \|u\|_b \int_0^\infty \|g_1(t)\|_{0,2}\, dt \end{aligned} \tag{4.13}$$

So $|\int_0^T (g_1, \partial_t u)\, dt|$ is bounded with respect to $T \to \infty$.

Let the second condition of (4.10) be valid. Then applying the partial integration formula we obtain

$$\Big|\int_0^T (g_1, \partial_t u)\, dt\Big| \le |(g_1(T), u(T))| + |(g_1(0), u(0))| + \Big|\int_0^T (\partial_t g_1(t), u(t))\, dt\Big| \tag{4.14}$$

The integral in the right-hand side of (4.14) estimates in the same way as in (4.13). To estimate the first to terms in the previous formula it is sufficient to prove that under above assumptions $g_1 \in C_b(\mathbb{R}_+, L^2(\omega))$. Let us consider an arbitrary segment $[N, N+1] \subset \mathbb{R}_+$ and let $[T, t]$ be in this segment. Then

$$\begin{aligned} \|g_1(T)\|_{0,2} &\le \|g_1(t)\|_{0,2} + \|g_1(T) - g_1(t)\|_{0,2} \le \\ &\le \|g_1(t)\|_{0,2} + \int_t^T \|\partial_t g_1(t)\|_{0,2} dt \le \|g_1(t)\|_{0,2} + \int_0^\infty \|\partial_t g_1(t)\|_{0,2}\, dt \end{aligned} \tag{4.15}$$

Let us integrate the inequality (4.15) over $t \in [N, N+1]$

$$\|g_1(T)\|_{0,2} \le C\|g_1, \Omega_N\|_{0,2} + \int_0^\infty \|\partial_t g_1(t)\|_{0,2} dt \le |g_1|_b + \|\partial_t g_1\|_{L_1(\mathbb{R}_+, L^2(\omega))}$$

But the constant N was choosen arbitraryly hence $g_1 \in C_b(\mathbb{R}_+, L^2(\omega))$.

Let the third condition of (4.10) be valid. Then applying the partial integration formula again we obtain

$$\Big|\int_0^T (g_1, \partial_t u)\, dt\Big| \le |(G_1(T), \partial_t u(T))| + |(G_1(0), \partial_t u(0))| + \Big|\int_0^T (G_1(t), \partial_t^2 u(t))\, dt\Big|$$

The first two terms in the right-hand side can be estimated as in the previous case. Let us estimate the integral

$$\begin{aligned} \Big|\int_0^T (G_1(t), \partial_t^2 u(t))\, dt\Big| &\le \int_0^T \|G_1(t)\|_{0,2}\|\partial_t^2 u(t)\|_{0,2} dt \le \\ &\le \sum_{N=0}^{[T]} \|G_1, \Omega_N\|_{0,2}\|\partial_t^2 u, \Omega_N\|_{0,2} \le C\|u\|_b \sum_{N=0}^{\infty} \|G_1, \Omega_N\|_{0,2} \end{aligned}$$

Theorem 4.3 is proved.

Theorem 4.4. *Let the all asumptions of previous Theorem be valid. Let us suppose also that the limit problem in the cross section* ω

$$\left\{ \begin{array}{l} a\Delta_x v_+ - f(v_+(x)) = g_+(x) \\ \partial_n v_+\big|_{x\in\partial\omega} = 0 \end{array} \right. \tag{4.16}$$

has the finite number of solutions

$$v_+ \in \mathcal{V}_+ = \{v_+^1(x), \cdots, v_+^l(x)\} \tag{4.17}$$

Then for every solution u *of the problem* (0.1) *there exists an equlibria* $v_+^N(x) \in \mathcal{V}_+$ *such that*

$$(T_s u)(t,x) \to v_+^N(x) \text{ in the space } \Theta^+ \text{ , when } s \to +\infty \tag{4.18}$$

Here by Θ^+ *we denote the space* Θ_0^+ *if* g *is strong translation compact in* Ξ *and* $\Theta^+ = (\Theta_0^+)^w$ *if* g *is weak translation compact.*

Remark. As known (see for instance [2]) there exists an open dence set in $L^2(\omega)$ such that the set $\mathcal{V}_+$ is finite for every g_+ from this set.

Proof. Let u be a solution of the problem (0.1). Let us consider the ω-limit set $\omega(u)$ of the point $u \in \Theta^+$ under the $\{T_s, s \geq 0\}$ semigroup action. Recall that $u_+ \in \omega(u)$ if and only if there exists the sequence $\{s_j, j \in \mathbb{N}\}$, $s_j \to \infty$ such that

$$T_{s_j} u \to u_+ \text{ in the space } \Theta^+ \tag{4.19}$$

Theorem 4.1 implies that $\{T_s, s \geq 0\}$ possesses an atractor $\mathbb{A}$ in $K_\Sigma^+ \subset \Theta^+$ hence (see [2]) $w(u)$ is nonempty *connected* compact set in Θ^+. Let u_+ be in $\omega(u)$. It means that there exists a sequence $s_j \in \mathbb{R}_+$ such that for every $T \in \mathbb{R}_+$

$$T_{s_j} u \rightharpoonup u_+ \text{ in the space } H^2_{\mathbb{Q}}(\Omega_T), \text{ when } s_j \to \infty$$

Particulary

$$\|T_{s_j}\partial_t u - \partial_t u_+, \Omega_T\|_{0,2} \to 0 \text{ , when } s_j \to \infty$$

But it follows from the dissipative integral (4.11) existance that

$$\|T_{s_j}\partial_t u, \Omega_T\|_{0,2} = \|\partial_t u, T_{s_j}\Omega_T\|_{0,2} \to 0 \text{ , when } s_j \to \infty$$

Hence $\|\partial_t u_+, \Omega_T\|_{0,2} = 0$ and $u_+(t,x) \equiv u_+(x)$.

It follows now from the condition (4.2) and from Lemma 3.3 that $u_+(x)$ is a solution of the limit problem (4.16). So

$$w(u) \subset \mathcal{V}_+ \tag{4.20}$$

But the set $w(u)$ must be connected and the set $\mathcal{V}_+$ is descrete hence

$$w(u) = \{v_+^N\} \text{ for some } N \in \{1, \cdots, l\} \tag{4.21}$$

The attracting property for $\{T_s, s \geq 0\}$ (see §3) implies immediately now that (4.18) is valid. Theorem 4.4 proved

Corollary 4.5. *Arguing as in the prove of Corollary 3.16 we obtain as in the case of strong translation compactness of* g *as in the weak one condition* (4.20) *implies that*

$$\begin{cases} \lim_{t\to+\infty} \|u(t,\cdot) - v_+^N(\cdot)\|_{0,p_0} \to 0 \ , \ when\ t \to \infty \\ \lim_{t\to+\infty} \|\partial_t u(t,\cdot)\|_{\varepsilon,2} \to 0 \end{cases} \tag{4.22}$$

where the exponent p_0 *is given in Corollary A.1.*

Corollary 4.6. *Let the function* g_+ *satisfied the conditions of Theorem 4.4. Then any solution* $u(t)$, $t \in \mathbb{R}$ *of the equation* (4.4) *in the whole cylinder* $\Omega = \mathbb{R}\times\omega$ *is a heteroclinic orbit i.e. there exist two different equilibria* w_u^+ *and* w_u^- *from the set* $\mathcal{V}_+$ *such that*

$$T_s u \to w_u^+ \ when\ s \to +\infty \ and\ T_s u \to w_u^- \ when\ s \to -\infty \tag{4.23}$$

Indeed due to the estimate (1.25) (see Remark 1.7) any solution of the problem (4.4) is bounded as with respect to $t \to \infty$ so with respect to $t \to -\infty$. So the convergence (4.23) follows now from Theorem 4.4. Hence it remains to prove that $w_u^+ \neq w_u^-$. Integrating the formula (4.7) with $g_1 \equiv 0$ over $\mathbb{R}$ we obtain that

$$\mathcal{F}_u(+\infty) - \mathcal{F}_u(-\infty) = \mathcal{F}_{w^+} - \mathcal{F}_{w^-} = \int_{\mathbb{R}} (\gamma\partial_t u, \partial_t u)\, dt \neq 0 \tag{4.24}$$

Thus $w^+ \neq w^-$.

Let us give now some examples of the pertrubation term $g_1(t,x)$ satisfying the conditions of previous Theorem.

Example 4.7. Let

$$g_1(t,x) = \varphi(t) g_0(x) \tag{4.25}$$

where $g_0 \in L^2(\omega)$ and

$$\varphi(t) = \frac{|\sin(t^2)|}{1+t^2} \tag{4.26}$$

Then it is not difficult to check that this function satisfies the first condition of (4.10) and condition (4.2) is valid for the strong topology choice.

Example 4.8. Let the function g_1 have the form (4.25) with the following function $\varphi(t)$

$$\varphi(t) = \frac{t}{1+t^2} \tag{4.27}$$

Then it is not difficult to check that this function satisfies the second condition of (4.10) and condition (4.2) is valid for the strong topology choice.

Example 4.9. Let the function g_1 have the form (4.25) with the following function $\varphi(t)$

$$\varphi(t) = \sin(t^3) \tag{4.28}$$

Then as known $T_s\varphi \to 0$ when $s \to \infty$ in a weak topology of the space $L^2([T, T+1])$ for every $T \in \mathbb{R}_+$ hence g_1 satisfies condition (4.2) with the weak topology choice. Let us check that this function satisfies the third condition of (4.10). Let G_1 be the following function

$$G_1(t,x) = \Phi(t)g_0(x) \ , \text{ where } \Phi(t) = -\int_t^\infty \sin(s^3)\, ds$$

We must check that

$$\sum_{N=0}^{\infty} \|\Phi, [N, N+1]\|_{0,2} < \infty \tag{4.29}$$

In order to do it we represent Φ in the following equivalent form

$$\Phi(t) = \frac{1}{3}\cos(t^3)t^{-2} - \frac{2}{9}\int_{t^3}^{\infty} \frac{\cos v}{v^{5/3}}\, dv$$

It follows immediately from this representation that

$$\Phi(t) = \mathcal{O}(t^{-2}) \ , \text{ when } t \to \infty$$

Hence

$$\|\Phi, [N, N+1]\|_{0,2} = \mathcal{O}(N^{-\frac{3}{2}})$$

and so (4.29) is valid.

Part 2. Asymptotics in the three-dimensional case

In the second part, we describe the asymptotics of solutions to the linear system (0.??) in case the half-cylinder $\Omega_+ = \mathbb{R}_+ \times \omega$ is three-dimensional and conclude from this depiction the existence of the trajectory attractor for the singular part of the solutions to the nonlinear elliptic system (0.1).

5. Elliptic regularity for the Neumann problem for the Laplace operator

Before discussing elliptic regularity for the Neumann problem for the Laplace operator on the half-cylinder $\Omega_+ = \mathbb{R}_+ \times \omega$, we discuss elliptic regularity for the Neumann problem for the Laplace operator on the infinite cone $\Gamma \subset \mathbb{R}^2$ and the infinite wedge $\mathbb{R} \times \Gamma \subset \mathbb{R}^3$. Since in this section we shall make use of the Fourier transformation, in contrast to the rest of the paper functions appearing are complex-valued. Since all differential operators considered have real-valued coefficients, the conclusions are easily specified to the real-valued case. When speaking about a solution to the Neumann problem, we always mean a variational solution that is in particular in H^1. Further, subscripts b, loc in the notation of Sobolev spaces on a cylinder or half-cylinder have the same meaning as before, while subscript N indicates the subspace of functions satisfying the homogeneous Neumann boundary condition, where it makes sense, i.e., on $\partial\Gamma$, $\mathbb{R} \times \partial\Gamma$, $\partial\omega$, $\mathbb{R} \times \partial\omega$, and $\mathbb{R}_+ \times \partial\omega$, respectively. We shall also employ notation with subscript Q to designate the space of all variational solutions to the corresponding Neumann problem with right-hand side in L^2. We always have $H^2_Q \supseteq H^2_N$. If Γ or at least one of the conical points of ω has an obtuse angle, then $H^2_Q \neq H^2_N$, otherwise $H^2_Q = H^2_N$.

Henceforth let $\Gamma \subset \mathbb{R}^2$ be an open cone with opening α. Since it turns out that for $\alpha < \pi$ we have H^2-regularity, we shall suppose that $\alpha > \pi$. Further let $y = (y_1, y_2)$ be euclidian coordinates in $\mathbb{R}^2$, while (r, θ) denote polar coordinates. We assume that $\Gamma = \{(r, \theta); 0 < \theta < \alpha\}$. We fix a cut-off function $\psi \in C_0^\infty(\overline{\Gamma})$, depending only on the radial coordinate r, such that $\psi(r) = 1$ in a neighbourhood of 0 and ψ is supported sufficiently close to 0. The model cone Γ arises through flatting out the boundary of ω near a conical point of $\partial\omega$, i.e, through introducing suitable local coordinates. To deal with such a situation, on Γ we shall consider the operator $1 - \Delta_y - M(y, \partial_y)$, where $M(y, \partial_y) = \sum_{|\gamma| \le 2} b_\gamma(y) \partial_y^\gamma$ is a second-order partial differential operator with coefficients from $C^\infty(\overline{\Gamma})$ subject to the following conditions:

(a) $\|b_\gamma\|_{L^\infty(\Gamma)} \le \delta$ for $\gamma \in \mathbb{N}^2$, $|\gamma| \le 2$;
(b) $b_\gamma(0) = 0$ for $\gamma \in \mathbb{N}^2$, $|\gamma| = 2$;
(c) $\|\partial_r b_\gamma\|_{L^\infty(\operatorname{supp}\psi)} \le \delta$ for $\gamma \in \mathbb{N}^2$, $|\gamma| = 2$

for a certain $\delta > 0$ sufficiently small.

The proof of the following lemma shows that $H_Q^2(\Gamma)$ defined as the space of solutions v to

$$(1 - \Delta_y - M(y, \partial_y))v = g \text{ in } \Gamma, \ \partial_n v|_{\partial\Gamma} = 0 \tag{5.1}$$

with right-hand side $g \in L^2(\Gamma)$ is actually independent of the operator $M(y, \partial_y)$ satisfying (a)–(c) provided that $\delta > 0$ is small enough. For the case $M(y, \partial_y) \equiv 0$ it is known that

$$H_Q^2(\Gamma) = H_N^2(\Gamma) \oplus \operatorname{span}\{S\}, \ \ S(y) = \psi(r) r^{\pi/\alpha} \cos(\pi\theta/\alpha), \tag{5.2}$$

see [9], [13]. Notice that $S \in H^{1+\pi/\alpha-\varepsilon}(\Gamma)$ for any $\varepsilon > 0$, but $S \notin H^{1+\pi/\alpha}(\Gamma)$.

Lemma 5.1. *For $\delta > 0$ sufficiently small, the differential operator*

$$1 - \Delta_y - M(y, \partial_y) \colon H_Q^2(\Gamma) \to L^2(\Gamma) \tag{5.3}$$

realizes an isomorphism, where $H_Q^2(\Gamma)$ is the space given in (5.2)*. Moreover, if $v \in H_Q^2(\Gamma)$ and $(1 - \Delta - M(y, \partial_y))v = g$, then v is the unique solution to the problem* (5.1)*.*

Proof. It is known that $1 - \Delta$ is an isomorphism from $H_Q^2(\mathbb{R} \times \Gamma)$ onto $L^2(\mathbb{R} \times \Gamma)$. Furthermore, it is readily seen that $M(y, \partial_y)$ maps $H_Q^2(\Gamma)$ into $L^2(\Gamma)$, where

$$\big\|1 - \Delta - M(y, \partial_y)\big\|_{H_Q^2 \to L^2} \le C(\delta)$$

with some constant $C(\delta) > 0$, and $C(\delta) \to 0$ as $\delta \to 0$. Now choose $\delta > 0$ so small that

$$\big\|M(y, \partial_y)\big\|_{H_Q^2(\Gamma) \to L^2(\Gamma)} < \big\|(1 - \Delta)^{-1}\big\|_{L^2(\Gamma) \to H_Q^2(\Gamma)},$$

where $(1 - \Delta)^{-1}$ stands for the inverse to $1 - \Delta \colon H_Q^2(\Gamma) \to L^2(\Gamma)$. This shows that the differential expression $1 - \Delta - M(y, \partial_y)$ in (5.3) realizes an isomorphism.

From Theorem A.3 (and its corresponding version for model cones) we infer that solutions to the problem (5.1) belong to $H^{3/2+\varepsilon}(\Gamma)$ for a certain $\varepsilon > 0$. Thus in defining the space of variational solutions to (5.1) we may replace the quadratic form by the differential expression yielding the coincidence of the spaces $H^2_Q(\Gamma)$ for different $M(y,\partial_y)$. □

Remark. (a) The same proof yields that $H^2_Q(\Gamma) = H^2_N(\Gamma)$ when $\alpha < \pi$. In subsequent discussions we again assume that $\alpha > \pi$.

(b) From (5.2) it follows that each $v \in H^2_Q(\Gamma)$ can uniquely be written in the form

$$v = v_0 + dS, \tag{5.4}$$

where $v_0 \in H^2(\Gamma)$, $d \in \mathbb{C}$. Hence an equivalent norm on $H^2_Q(\Gamma)$ is given by $\left\{\|v_0\|^2_{H^2(\Gamma)} + |d|^2\right\}^{1/2}$. Moreover, for v being a solution to (5.1) we get the estimate

$$\left\{\|v_0\|^2_{H^2(\Gamma)} + |d|^2\right\}^{1/2} \le C\,\|g\|_{L^2(\Gamma)}, \tag{5.5}$$

where the constant $C > 0$ is independent of the operator $M(y,\partial_y)$ as long as the requirements (a)–(c) with the same δ, $\delta > 0$ as small as in Lemma 5.1, are fulfilled.

(c) Notice further that the coefficient d in (5.4) is independent of the particular choice of the cut-off function ψ, i.e., in choosing another cut-off function possessing the same properties as ψ we obtain the same d as before.

Now we want to discuss the space $H^2_Q(\mathbb{R}\times\Gamma)$ of solutions v to the problem

$$(1 - \partial_t^2 - \Delta_y - M(y,\partial_y))v = g \ \text{ in } \mathbb{R}\times\Gamma, \ \partial_n v|_{\mathbb{R}\times\partial\Gamma} = 0 \tag{5.6}$$

with right-hand side $g \in L^2(\mathbb{R}\times\Gamma)$, where $M(y,\partial_y)$ is a second-order partial differential operator as above. Again it turns out that the space $H^2_Q(\mathbb{R}\times\Gamma)$ is independent of the operator $M(y,\partial_y)$ provided that $\delta > 0$ is small enough.

We need the following result in the cases $s = 2$, $s = 0$. For a proof, see [9], [17].

Lemma 5.2. *Let $\Gamma \subset \mathbb{R}^2$ be an open cone, $s \in \mathbb{R}$. Then an equivalent norm on $H^s(\mathbb{R}\times\Gamma)$ is given by*

$$\|u\|_{H^s(\mathbb{R}\times\Gamma)} = \left\{\int_{\mathbb{R}} \langle\tau\rangle^{2s} \left\|\kappa(\tau)^{-1}\widehat{u}(\tau)\right\|^2_{H^s(\Gamma)}\, d\tau\right\}^{1/2}, \tag{5.7}$$

where $\widehat{u}(\tau) = F_{t\to\tau}u(\tau)$, $\kappa(\tau) = \kappa_{\langle\tau\rangle}$, $\langle\tau\rangle = (1+|\tau|^2)^{1/2}$, and

$$\kappa_\lambda u(y) = \lambda u(\lambda y), \quad \lambda > 0,\, y \in \Gamma,$$

for $u \in H^s(\Gamma)$.

Notice that $\{\kappa_\lambda\}_{\lambda>0}$ is a strongly continuous group on $H^s(\Gamma)$. It consists of isometries when $s = 0$.

Lemma 5.3. *Let $\Gamma \subset \mathbb{R}^2$ be an open cone as above. Then we have*

(5.8)
$$H^2_Q(\mathbb{R}\times\Gamma) = H^2_N(\mathbb{R}\times\Gamma)\oplus\Big\{F^{-1}_{\tau\to t}\{\langle\tau\rangle\psi(r\langle\tau\rangle)(r\langle\tau\rangle)^{\pi/\alpha}\cos(\pi\theta/\alpha)\widehat{d}(\tau)\};\, d\in H^2(\mathbb{R})\Big\}.$$

Proof. Let v be solution to (5.6) with right-hand side $g \in L^2(\mathbb{R}\times\Gamma)$. Upon applying the Fourier transformation $F_{t\to\tau}$ and afterwards the group action $\kappa(\tau)^{-1}$ we obtain the equation

(5.9)
$$\big(1-\Delta-M_\tau(y,\partial_y)\big)\kappa(\tau)^{-1}\widehat{v}(\tau) = \langle\tau\rangle^{-2}\kappa(\tau)^{-1}\widehat{g}(\tau) \text{ in } \Gamma,\;\; \partial_n\big(\kappa(\tau)^{-1}\widehat{v}(\tau)\big)|_{\partial\Gamma} = 0$$

with parameter $\tau \in \mathbb{R}$, where $M_\tau(y,\partial_y) = \langle\tau\rangle^{-2}M(\langle\tau\rangle^{-1}y, \langle\tau\rangle\partial_y)$. Now it is easily seen that the operator $M_\tau(y,\partial_y) = \sum_{|\gamma|\le 2}\langle\tau\rangle^{-2+|\gamma|}b_\gamma(\langle\tau\rangle^{-1}y)\partial_y^\gamma$ satisfies the set of requirements (a)–(c) with the same $\delta>0$ as $M(y,\partial_y)$.

Hence we conclude from Eq. (5.9) together with (5.1), (5.4) that

$$\kappa(\tau)^{-1}\widehat{v}(\tau) = \kappa(\tau)^{-1}\widehat{v}_0(\tau) + \widehat{d}(\tau)S(y),\;\; S(y) = \psi(y)r^{\pi/\alpha}\cos(\pi\theta/\alpha). \tag{5.10}$$

Moreover, from (5.5) we derive the estimate

$$\|\kappa(\tau)^{-1}\widehat{v}_0(\tau)\|^2_{H^2(\Gamma)} + |\widehat{d}(\tau)|^2 \le C^2\,\langle\tau\rangle^{-4}\|\kappa(\tau)^{-1}\widehat{g}(\tau)\|^2_{L^2(\Gamma)},$$

i.e.,

$$\int_{\mathbb{R}}\langle\tau\rangle^4\|\kappa(\tau)^{-1}\widehat{v}(\tau)\|^2_{H^2(\Gamma)}\,d\tau + \int_{\mathbb{R}}\langle\tau\rangle^4|\widehat{d}(\tau)|^2\,d\tau \le$$
$$\le C^2\int_{\mathbb{R}}\|\kappa(\tau)^{-1}\widehat{g}(\tau)\|^2_{L^2(\Gamma)}\,d\tau = \|g\|^2_{L^2(\mathbb{R}\times\Gamma)}$$

showing that $v_0 \in H^2(\mathbb{R}\times\Gamma)$, $d\in H^2(\mathbb{R})$ by Lemma 5.2. From (5.10) we finally get

$$v = v_0 + F^{-1}_{\tau\to t}\{\widehat{d}(\tau)(\kappa(\tau)S)(y)\} \tag{5.11}$$

yielding the decomposition (5.8) by further noting that the sum on the right-hand side of (5.8) is direct and is obviously contained in $H^2_Q(\mathbb{R}\times\Gamma)$. □

Remark. The proof of Lemma 5.3 shows that

$$\|u\|'_{H^2_Q(\mathbb{R}\times\Gamma)} = \left\{\int_{\mathbb{R}}\langle\tau\rangle^4\big\|\kappa(\tau)^{-1}\widehat{u}(\tau)\big\|^2_{H^2_Q(\Gamma)}\,d\tau\right\}^{1/2}$$

is an equivalent norm on $H^2_Q(\mathbb{R}\times\Gamma)$. Since $H^2_Q(\Gamma)$ is a cone Sobolev space of functions possessing asymptotics of a certain discrete asymptotic type near $y=0$, $H^2_Q(\mathbb{R}\times\Gamma)$ is in fact a wedge Sobolev space in the sense of B.-W. Schulze, see [15]–[17].

Next we turn our attention to the case of the cylinder $\mathbb{R}\times\omega$ and of the half-cylinder $\Omega_+ = \mathbb{R}_+\times\omega$, respectively. In the following, let ω be a bounded and polyhedral domain in $\mathbb{R}^2$. The boundary $\partial\omega$ is in particular smooth except for a finite number of conical points. For H^2-regularity holds up to conical points with an acute angle, see (a) in the remark following the proof of Lemma 5.1, only conical points of $\partial\omega$ obeying an obtuse angle have to be regarded specifically. Let $\{b_1, \dots, b_\kappa\}$ denote the set of these conical points. Let α_j be the size of the angle at b_j, $\alpha_j > \pi$. For every j, $1 \le j \le \kappa$, we choose an open cone $\Gamma_j \subset \mathbb{R}^2$, open subsets U_j, V_j in $\mathbb{R}^2$ with $U_j \ni b_j$, $V_j \ni 0$, and a diffeomorphism $\chi_j : U_j \to V_j$ such that $\chi_j(b_j) = 0$ and $\chi_j(\overline{\omega} \cap U_j) = \overline{\Gamma}_j \cap V_j$. Recall that $y = (y_1, y_2)$ are euclidian coordinates in $\mathbb{R}^2$, while (r, θ) denote polar coordinates. We assume that $\Gamma_j = \{(r, \theta);\, 0 < \theta < \alpha_j\}$. Furthermore, we suppose that the diffeomorphisms χ_j are chosen to preserve the standard euclidian structure centered at b_j up to cubic terms. Note that this assumption implies that $(\chi_j)_*\Delta = \Delta + M_j(y, \partial_y)$ close to $y = 0$, where $M_j(y, \partial_y)$ is a second-order differential operator with smooth coefficients and $M_j(0, \partial_y) = 0$. Moreover, up to translation and rotation, the faces of Γ_j can be viewed as being tangential to ω at b_j. By shrinking U_j, if necessary, we may suppose that $M_j(y, \partial_y)$ is a differential operator on Γ_j with coefficients from $C^\infty(\overline{\Gamma}_j)$ satisfying, for $\Gamma = \Gamma_j$ and $\delta > 0$ sufficiently small, the assumptions (a)–(c) previous to Lemma 5.1.

Further let $U_0 \subset \mathbb{R}^2$ be an open set not meeting $\{b_1, \dots, b_\kappa\}$ such that $\{U_0\} \cup \{U_j\}_{j=1}^\kappa$ forms an open covering of $\overline{\omega}$. Let $\{\phi_0\} \cup \{\phi\}_{j=1}^\kappa$ be a partition of unity subordinated to this covering, $\phi_0 + \sum_{j=1}^\kappa \phi_j = 1$ on $\overline{\omega}$, $\phi_j = 1$ in a neighbourhood of b_j for all j, $1 \le j \le \kappa$. Eventually we assume that, for $1 \le j \le \kappa$, $\psi_j = (\chi_j)_*\phi_j$ only depends on the radial variable r, i.e., $\psi_j = \psi_j(r)$.

Remark. For completeness we notice that an intrinsic interpretation of (5.4) can be given asserting that there is a short exact split sequence

$$(5.12) \qquad 0 \longrightarrow H^2_N(\omega) \longrightarrow H^2_Q(\omega) \longrightarrow \prod_{j=1}^\kappa \mathbb{C} \longrightarrow 0$$

with the surjection assigning to each function $u \in H^2_Q(\omega)$ its sequence $(d_1, \dots, d_\kappa)$ of singular coefficients. Thereby, d_j is explained as the coefficient appearing in (5.4) in front of S, for $v = (\chi_j)_*(\phi_j u)$ and $\Gamma = \Gamma_j$. To see that (5.12) is correctly defined one has to observe that the coefficient d_j is not only independent of the choice of the cut-off function ψ_j, see (c) in the remark following the proof of Lemma 5.1, but also independent of the choice of the diffeomorphism χ_j meeting all of the assumptions above. A splitting of (5.12) is obtained via (5.2) after having fixed the diffeomorphisms χ_j and the cut-off functions ψ_j. More precisely, we may write

$$u = u_0 + \sum_{j=1}^\kappa d_j\, (\chi_j)^*\big(\psi_j(r) r^{\pi/\alpha_j} \cos(\pi\theta/\alpha_j)\big)$$

for $u \in H^2_Q(\omega)$, where $u_0 \in H^2_N(\omega)$, $d_j \in \mathbb{C}$ are uniquely determined. Notice further that the coefficients d_j can be calculated using the formula

$$(5.13) \qquad d_j = \lim_{r \to 0^+} \beta_j^{-2} \Big(r^{-\pi/\alpha_j} \big((\chi_j)_*(\phi_j u)(r, \theta) - u(b_j)\big), \cos(\pi\theta/\alpha_j)\big)\Big)_{L^2(0,\alpha_j)},$$

where $\big(\cdot, \cdot\big)_{L^2(0,\alpha_j)}$ denotes the scalar product in $L^2(0, \alpha_j)$, $u(b_j)$ is the value of u at b_j, and $\beta_j = \Big\{\int_0^{\alpha_j} \big|\cos(\pi\theta/\alpha_j)\big|^2\, d\theta\Big\}^{1/2}$. Notice that $u(b_j) = (\chi_j)_*(\phi_j u)(0)$ is well-defined by Theorem A.3.

For further reference notice that an equivalent norm on $H^2_Q(\mathbb{R}\times\omega)$ is given by

$$\|u\|_{H^2_Q(\mathbb{R}\times\omega)} = \left\{ \|\phi_0 u\|^2_{H^2(\mathbb{R}\times\omega)} + \sum_{j=1}^{\kappa} \|(\chi_j)_*(\phi_j u)\|^2_{H^2_Q(\mathbb{R}\times\Gamma_j)} \right\}^{1/2}. \tag{5.14}$$

This follows from the fact that $u \in H^2_Q(\mathbb{R}\times\omega)$ if and only if $\phi_l u \in H^2_Q(\mathbb{R}\times\omega)$ for all l, $0 \le l \le \kappa$, and obviously $\phi_0 u \in H^2_Q(\mathbb{R}\times\omega)$ if and only if $\phi_0 u \in H^2(\mathbb{R}\times\omega)$, while, for $1 \le j \le \kappa$, $\phi_j u \in H^2_Q(\mathbb{R}\times\omega)$ if and only if $(\chi_j)_*(\phi_j u) \in H^2_Q(\mathbb{R}\times\Gamma_j)$.

From Lemma 5.3 and (5.14) we conclude that

$$\begin{aligned} &H^2_Q(\mathbb{R}\times\omega) = H^2_N(\mathbb{R}\times\omega) \\ &\oplus \left\{ \sum_{j=1}^{\kappa} (\chi_j)^* \big(F^{-1}_{\tau\to t}\{\langle\tau\rangle\psi_j(r\langle\tau\rangle)(r\langle\tau\rangle)^{\pi/\alpha_j}\cos(\pi\theta/\alpha_j)\widehat{d}_j(\tau)\}\big);\ d_j \in H^2(\mathbb{R}),\ 1 \le j \le \kappa \right\} \end{aligned} \tag{5.15}$$

On the analogy of (5.12) we have the following lemma.

Lemma 5.4. *For $\omega \subset \mathbb{R}^2$ being a bounded, polyhedral domain as above, there is a short exact split sequence*

$$0 \longrightarrow H^2_N(\mathbb{R}\times\omega) \longrightarrow H^2_Q(\mathbb{R}\times\omega) \xrightarrow{(\tau_1,\dots,\tau_\kappa)} \prod_{j=1}^{\kappa} H^{1-\pi/\alpha_j}(\mathbb{R}) \longrightarrow 0, \tag{5.16}$$

where the operators τ_j are given by

$$\tau_j u(t) = \lim_{r\to 0^+} \beta_j^{-2} \Big(r^{-\pi/\alpha_j}\big((\chi_j)_*(\phi_j u)(t,r,\theta) - u(t,b_j)\big), \cos(\pi\theta/\alpha_j)\Big)_{L^2(0,\alpha_j)}. \tag{5.17}$$

Moreover, a splitting of (5.16) *is given by the mapping*

$$(d_{11},\dots,d_{\kappa 1}) \mapsto \sum_{j=1}^{\kappa} (\chi_j)^* \big(F^{-1}_{\tau\to t}\{\psi_j(r\langle\tau\rangle)\widehat{d}_{j1}(\tau)\}\, r^{\pi/\alpha_j}\cos(\pi\theta/\alpha_j)\big). \tag{5.18}$$

Proof. According to (5.10) and the short exact sequence (5.12), the functions $d_j \in H^2(\mathbb{R})$ appearing in the representation of $u \in H^2_Q(\mathbb{R}\times\omega)$ as

$$\begin{aligned} u &= u_0 + \sum_{j=1}^{\kappa} (\chi_j)^* \big(F^{-1}_{\tau\to t}\{\langle\tau\rangle\psi_j(r\langle\tau\rangle)(r\langle\tau\rangle)^{\pi/\alpha_j}\cos(\pi\theta/\alpha_j)\widehat{d}_j(\tau)\}\big), \\ &= u_0 + \sum_{j=1}^{\kappa} (\chi_j)^* \big(F^{-1}_{\tau\to t}\{\psi_j(r\langle\tau\rangle)\widehat{d}_{j1}(\tau)\} r^{\pi/\alpha_j}\cos(\pi\theta/\alpha_j)\big) \end{aligned}$$

where $u_0 \in H^2_N(\mathbb{R}\times\omega)$, are uniquely determined, independently of the choice of the diffeomorphisms χ_j and the cut-off functions ψ_j. Likewise, the same is then true for the functions $d_{j1} = \langle D\rangle^{1+\pi/\alpha_j} d_j \in H^{1-\pi/\alpha_j}(\mathbb{R})$. Therefore, the surjection in (5.16) is well-defined. Moreover, it becomes clear that (5.16) is exact and a splitting of it is provided by (5.18).

Thus it remains to deal with (5.17). From (5.13), applied to $\Gamma = \Gamma_j$, $v = (\chi_j)_*(\phi_j u)$, and Eq. (5.10), in which $d = d_j$, we conclude that

$$\begin{aligned}\widehat{d}_j(\tau) &= \lim_{r\to 0^+} \beta_j^{-2}\Big(r^{-\pi/\alpha_j}\big(\langle\tau\rangle^{-1}\widehat{v}(\tau, r\langle\tau\rangle^{-1},\theta) - \langle\tau\rangle^{-1}\widehat{v}(\tau,0)\big), \cos(\pi\theta/\alpha_j)\Big)_{L^2(0,\alpha_j)}\\ &= \lim_{r\to 0^+} \beta_j^{-2}\Big((r\langle\tau\rangle)^{-\pi/\alpha_j}\langle\tau\rangle^{-1}\big(\widehat{v}(\tau,r,\theta) - \widehat{v}(\tau,0)\big), \cos(\pi\theta/\alpha_j)\Big)_{L^2(0,\alpha_j)},\end{aligned}$$

the latter line upon replacing r with $r\langle\tau\rangle$, i.e.,

$$\begin{aligned}\widehat{d}_{j1}(\tau) &= \langle\tau\rangle^{1+\pi/\alpha_j}\widehat{d}_j(\tau) = \lim_{r\to 0^+} \beta_j^{-2}\big(r^{-\pi/\alpha_j}(\widehat{v}(\tau,r,\theta) - \widehat{v}(\tau,0)), \cos(\pi\theta/\alpha_j)\big)_{L^2(0,\alpha_j)},\\ d_{j1}(t) &= \lim_{r\to 0^+} \beta_j^{-2}\Big(r^{-\pi/\alpha_j}\big((\chi_j)_*(\phi_j u)(t,r,\theta) - u(t,b_j)\big), \cos(\pi\theta/\alpha_j)\Big)_{L^2(0,\alpha_j)}.\end{aligned}$$

This proves Lemma 5.4 completely. □

Remark. (a) For the interpretation of the functions $d_{j1} \in H^{1+\pi/\alpha_j}(\mathbb{R})$, $1 \le j \le \kappa$, as coefficients in the asymptotic expansion of $u \in H^2_Q(\mathbb{R}\times\omega)$ close to the edge $\mathbb{R}\times\{b_j\}$, observe that $F^{-1}_{\tau\to t}\{\psi_j(r\tau)\widehat{d}_{j1}(\tau)\} = d_{j1}(t)$ when $r = 0$.

(b) From (5.17) we obtain in particular that taking traces on an edge is a local operation. More precisely, we have $\operatorname{supp}(\tau_j u) \subseteq \operatorname{supp}(u) \cap \big(\mathbb{R}\times\{b_j\}\big)$ for $u \in H^2_Q(\mathbb{R}\times\omega)$.

(c) It can be shown that

$$\beta_j^{-2}\big(r^{-\pi/\alpha_j}((\chi_j)_*(\phi_j u)(t,r,\theta) - u(t,b_j)), \cos(\pi\theta/\alpha_j)\big)_{L^2(0,\alpha_j)} \in H^1(\mathbb{R})$$

for $u \in H^2_Q(\mathbb{R}\times\omega)$, and convergence in (5.17) takes place in $H^{1-\pi/\alpha_j}(\mathbb{R})$.

The final goal in this section is to conclude the form of asymptotics when going over from $H^2_Q(\mathbb{R}\times\omega)$ to its factor space $H^2_Q(\mathbb{R}_+\times\omega)$. This is achieved by constructing a suitable splitting of (5.16) in terms of a continuous projection Π_2 in $H^2_Q(\mathbb{R}\times\omega)$ by means of a reformulation of the asymptotic information.

Theorem 5.5. *Let $\omega \subset \mathbb{R}^2$ be a bounded, polyhedral domain as above. Then there exists a continuous projection Π_2 in $H^2_Q(\mathbb{R}\times\omega)$ obeying the following properties:*

(a) $\ker\Pi_2 = H^2_N(\mathbb{R}\times\omega)$;
(b) $T_s\Pi_2 = \Pi_2 T_s$ *for all* $s\in\mathbb{R}$;
(c) $\operatorname{supp} u \subseteq \overline{\mathbb{R}}_-$ *implies* $\operatorname{supp}\Pi_2 u \subseteq \overline{\mathbb{R}}_-$;
(d) Π_2 *is* $(H^2_{Q,b}(\mathbb{R}\times\omega), H^2_{Q,b}(\mathbb{R}\times\omega))$*-continuous*;
(e) Π_2 *is* $(H^2_{Q,\mathrm{loc}}(\mathbb{R}\times\omega), H^2_{Q,\mathrm{loc}}(\mathbb{R}\times\omega))$*-continuous.*

In the proof of Theorem 5.5 we shall make use of the following result.

Lemma 5.6. *Let $\Gamma \subset \mathbb{R}^2$ be an open cone. Further let $\psi \in \mathcal{S}(\mathbb{R})$, $\psi_1 \in \mathcal{S}(\overline{\mathbb{R}}_+)$, $d_1 \in H^{1-\pi/\alpha}(\mathbb{R})$. Then*

$$\psi_1(r)\, F^{-1}_{\tau\to t}\big\{\big(\psi(r\langle\tau\rangle) - \psi(r\tau)\big)\,\widehat{d}_1(\tau)\big\}\, r^{\pi/\alpha}\cos(\pi\theta/\alpha) \in H^2_N(\mathbb{R}\times\Gamma). \tag{5.19}$$

Proof. Let $u(t,r) = \psi_1(r)\, F^{-1}_{\tau\to t}\big\{(\psi(r\langle\tau\rangle) - \psi(r\tau))\, \widehat{d}_1(\tau)\big\}\, r^{\pi/\alpha}\cos(\pi\theta/\alpha)$. Then we have

$$
\begin{aligned}
(5.20)\quad & \|u\|_{H^2_N(\mathbb{R}\times\Gamma)} = \\
& \left\{\int_{-\infty}^{\infty} \langle\tau\rangle^2\, \big\|\, \kappa(\tau)^{-1}\big(\psi_1(r)\big(\psi(r\langle\tau\rangle) - \psi(r\tau)\big)\widehat{d}_1(\tau)\, r^{\pi/\alpha}\cos(\pi\theta/\alpha)\big)\big\|^2_{H^2_N(\Gamma)}\, d\tau\right\}^{1/2} \\
&= \left\{\int_{-\infty}^{\infty} \langle\tau\rangle^2\, |\widehat{d}(\tau)|^2\, \big\|\psi_1(r\langle\tau\rangle^{-1})\big(\psi(r) - \psi\big(r\tau/\langle\tau\rangle\big)\big)\, r^{\pi/\alpha}\cos(\pi\theta/\alpha)\big\|^2_{H^2_N(\Gamma)}\, d\tau\right\}^{1/2} \\
&\qquad\qquad \le C\left\{\int_{-\infty}^{\infty} \langle\tau\rangle^2\, |\widehat{d}(\tau)|^2\, d\tau\right\}^{1/2},
\end{aligned}
$$

where $d = \langle D\rangle^{-1-\pi/\alpha} d_1 \in H^2(\mathbb{R})$. Thereby,

$$
\big\|\psi_1(r\langle\tau\rangle^{-1})\big(\psi(r) - \psi\big(r\tau/\langle\tau\rangle\big)\big)\, r^{\pi/\alpha}\cos(\pi\theta/\alpha)\big\|_{H^2_N(\Gamma)} \le C
$$

for a certain constant $C > 0$ independent of τ is seen from the fact that $\psi_2(r) \mapsto \psi_2(r)\, r^{\pi/\alpha}\cos(\pi\theta/\alpha)$ constitutes a bounded map from $\{\psi_2 \in \mathcal{S}(\overline{\mathbb{R}}_+); \psi_2(0) = 0\}$ into $H^2_N(\Gamma)$, while $\{\psi_1(r\langle\tau\rangle^{-1})\big(\psi(r) - \psi\big(r\tau/\langle\tau\rangle\big)\big); \tau \in \mathbb{R}\}$ for $\psi \in \mathcal{S}(\mathbb{R})$, $\psi_1 \in \mathcal{S}(\overline{\mathbb{R}}_+)$ is bounded in $\{\psi_2 \in \mathcal{S}(\overline{\mathbb{R}}_+); \psi_2(0) = 0\}$. Hence the right-hand side in (5.20) is finite proving that $u \in H^2_N(\mathbb{R}\times\Gamma)$. $\square$

Proof of Theorem 5.5. By Lemma 5.6, we are allowed to replace

$$
F^{-1}_{\tau\to t}\{\langle\tau\rangle\psi_j(r\langle\tau\rangle)(r\langle\tau\rangle)^{\pi/\alpha_j}\cos(\pi\theta/\alpha_j)\ ,\ \widehat{d}_j(\tau)\}
$$

in (5.15) by $\psi_{j1}(r) F^{-1}_{\tau\to t}\{\psi_j(r\tau)\widehat{d}_{j1}(\tau)\} r^{\pi/\alpha}\cos(\pi\theta/\alpha_j)$ i.e., we have

$$
\begin{aligned}
H^2_Q(\mathbb{R}\times\omega) &= H^2_N(\mathbb{R}\times\omega) \\
&\oplus \left\{\sum_{j=1}^{\kappa} (\chi_j)^*\big(\psi_{j1}(r) F^{-1}_{\tau\to t}\{\psi_j(r\tau)\widehat{d}_{j1}(\tau)\} r^{\pi/\alpha_j}\cos(\pi\theta/\alpha_j)\big);\right. \\
&\qquad\qquad \left. d_{j1} \in H^{1-\pi/\alpha_j}(\mathbb{R}),\ 1 \le j \le \kappa\right\},
\end{aligned}
$$

where, for each j, $1 \le j \le \kappa$, $\psi_j \in \mathcal{S}(\mathbb{R})$, $\psi_{j1} \in C^\infty_0(\overline{\mathbb{R}}_+)$, $\psi_j(0) = \psi_{j1}(0) = 1$, and ψ_{j1} is supported in V_j when considered as a function on Γ_j. If especially the ψ_j are chosen in a way such that $\operatorname{supp} F^{-1}\psi_j \subseteq \overline{\mathbb{R}}_-$ holds for all j, then

(5.21)

$$
\Pi_2 u = \sum_{j=1}^{\kappa} (\chi_j)^*\big(\psi_{j1}(r)\, F^{-1}_{\tau\to t}\{\psi_j(r\tau)(\tau_j u)\widehat{\ }(\tau)\}\, r^{\pi/\alpha_j}\cos(\pi\theta/\alpha_j)\big),\ \ u \in H^2_Q(\mathbb{R}\times\omega)
$$

is a projection in $H^2_Q(\mathbb{R}\times\omega)$ meeting all the requirements (a)–(e). That Π_2 is a projection follows from the fact that $\tau_j\Pi_2 u = \tau_j u$ holds for $u \in H^2_Q(\mathbb{R}\times\omega)$, (a), (c) are immediate, (b) is the locality of the trace operator τ_j, see (5.17), and the translation invariance of the pseudo-differential operator $d_1 \mapsto F^{-1}_{\tau\to t}(\psi_j(r\tau)\widehat{d}_1(\tau))$, where $r > 0$ is regarded as a parameter, and (e), (f) come from the observation that

$$\psi_{j1}(r)\, F^{-1}_{\tau\to t}\{\psi_j(r\tau)(\tau_j u)\hat{\ }(\tau)\}\, r^{\pi/\alpha_j}\cos(\pi\theta/\alpha_j)$$

belongs to $H^2_{Q,b}(\mathbb{R}\times\Gamma_j)$ and $H^2_{Q,\mathrm{loc}}(\mathbb{R}\times\Gamma_j)$, respectively, for u belonging to $H^2_{Q,b}(\mathbb{R}\times\Gamma_j)$ and $H^2_{Q,\mathrm{loc}}(\mathbb{R}\times\Gamma_j)$, as an easy calculation reveals. □

The following consequences of Theorem 5.5 supply the projection Π_2^+ in $H^2_{Q,b}(\mathbb{R}_+\times\omega)$ onto its closed subspace comprising the asymptotic information as well as the short exact sequences used in Section 6.

Theorem 5.7. *Let $\omega\subset\mathbb{R}^2$ be a bounded, polyhedral domain as above. Then there exists a continuous projection Π_2^+ in $H^2_{Q,b}(\mathbb{R}_+\times\omega)$ obeying the following properties:*

(a) $\ker\Pi_2^+ = H^2_{N,b}(\mathbb{R}_+\times\omega)$;
(b) $T_s\Pi_2^+ = \Pi_2^+T_s$ *for all* $s\geq 0$.

Moreover, Π_2^+ *is* $(H^2_{Q,\mathrm{loc}}(\overline{\mathbb{R}}_+\times\omega), H^2_{Q,\mathrm{loc}}(\overline{\mathbb{R}}_+\times\omega))$*-continuous.*

Proof. It follows from Theorem 5.5 (a)–(e) by continuous extension of the projection Π_2 to $H^2_{Q,b}(\mathbb{R}\times\omega)$ and its subsequent factorization to $H^2_{Q,b}(\mathbb{R}_+\times\omega)$. □

Notice that a projection Π_2^+ satisfying the requirements of Theorem 5.7 is

(5.22)
$$\Pi_2^+u = \sum_{j=1}^{\kappa}(\chi_j)^*\big(\psi_{j1}(r)\, F^{-1}_{\tau\to t}\{\psi_j(\tau r)((\tau_j u)_{\mathrm{ext}})\hat{\ }(\tau)\}\, r^{\pi/\alpha_j}\cos(\pi\theta/\alpha_j)\big),\quad u\in H^2_{Q,b}(\mathbb{R}_+\times\omega),$$

where ψ, ψ_{j1} are as in (5.21). Here $(\tau_j u)_{\mathrm{ext}}$ means an arbitrary extension of $\tau_j u\in H_b^{1-\pi/\alpha_j}(\mathbb{R}_+)$ to a function in $H_b^{1-\pi/\alpha_j}(\mathbb{R})$.

Corollary 5.8. *The short exact sequence* (5.6) *extends by continuity and factors subsequently to short split exact sequences*

$$0\longrightarrow H^2_{N,b}(\mathbb{R}_+\times\omega)\longrightarrow H^2_{Q,b}(\mathbb{R}_+\times\omega)\xrightarrow{(\tau_1,\dots,\tau_\kappa)}\prod_{j=1}^{\kappa}H_b^{1-\pi/\alpha_j}(\mathbb{R}_+)\longrightarrow 0,$$
$$0\longrightarrow H^2_{N,\mathrm{loc}}(\mathbb{R}_+\times\omega)\longrightarrow H^2_{Q,\mathrm{loc}}(\mathbb{R}_+\times\omega)\xrightarrow{(\tau_1,\dots,\tau_\kappa)}\prod_{j=1}^{\kappa}H_{\mathrm{loc}}^{1-\pi/\alpha_j}(\mathbb{R}_+)\longrightarrow 0,$$

where $(\tau_1,\dots,\tau_\kappa)$ is the vector of trace operators as before. A splitting of both is obtained from (5.22) *by replacing $\tau_j u$ with $d_{1j}\in H_b^{1-\pi/\alpha_j}(\mathbb{R}_+)$ and $H_{\mathrm{loc}}^{1-\pi/\alpha_j}(\overline{\mathbb{R}}_+)$, respectively.*

6. Regular and singular part of the trajectory attractor

In this final section we show that the trajectory attractor $\mathbb{A}$ of the problem (0.1) decomposes into a regular $\mathbb{A}_{reg}$ and a singular $\mathbb{A}_{sing}$ parts.

Let us suppose for simplicity that the right–hand side G of the problem (0.1) is strong translation compact in Ξ^+. The case of weak translation compactness could be treated analogously.

Let $K^+ = K^+_\Sigma$ be the union of all solutions for the family (3.4) see Definition 3.7. and let Π_2 be the same as in Theorem 5.7. Then one could define regular and singular parts of the union K^+ by formulas

(6.1) $$K^+_{reg} = \Pi_1 K^+\ ;\ K^+_{sing} = \Pi_2 K^+\ ,\ \text{where } \Pi_1 = Id - \Pi_2$$

Notice that by definition

$$K^+_{reg} \subset H^2_N(\Omega_+) \tag{6.2}$$

and the topology at K^+_{reg} induced by embedding $K^+_{reg} \subset \Theta^+_0$ concides with the topology induced by embedding (6.2).

It follows from Theorem 5.7 that the semigroup of positive shifts $\{T_s, s \geq 0\}$ acts as in the space K^+_{reg} so in the space K^+_{sing}, i.e.

$$T_s K^+_{reg} \subset K^+_{reg} \text{ and } T_s K^+_{sing} \subset K^+_{sing} \text{ for } s \geq 0 \tag{6.3}$$

Definition 6.1. *The attractor $\mathbb{A}_{reg}$ of the semigroup $\{T_s, s \geq 0\}$ acting in topological space K^+_{reg} is called a regular trajectory attractor for the problem* (0.1)*, see Definition 3.7.*

Analogously the attractor $\mathbb{A}_{sing}$ of the semigroup $\{T_s, s \geq 0\}$ acting in topological space K^+_{sing} is called a singular trajectory attractor for the problem (0.1)*.*

Theorem 6.2. *Let the previous assumptions be valid. Then the problem* (0.1) *possesses regular $\mathbb{A}_{reg}$ and singular $\mathbb{A}_{sing}$ trajectory attractors. Moreover*

$$\mathbb{A}_{reg} = \Pi_1 \mathbb{A} \text{ and } \mathbb{A}_{sing} = \Pi_2 \mathbb{A} \tag{6.4}$$

where $\mathbb{A}$ is a trajectory attractor for the problem (0.1)*. So*

$$\mathbb{A} = \mathbb{A}_{reg} \oplus \mathbb{A}_{sing} \tag{6.5}$$

Proof. Let us check that $\mathbb{A}_{sing} = \Pi_2 \mathbb{A}$. The assertion $\mathbb{A}_{reg} = \Pi_1 \mathbb{A}$ cuold be checked analogously.

For the first let us verify the attracting property. Let $\mathcal{O}(\Pi_2 \mathbb{A})$ be an arbitrary neighbourhood of $\Pi_2 \mathbb{A}$ in K^+_{sing} then due to Theorem 5.7 $\Pi_2^{-1} \mathcal{O}(\Pi_2 \mathbb{A})$ is some (open) neighbourhood of $\mathbb{A}$ in K^+. Hence from the attracting propery for $\mathbb{A}$ we obtain that there exists $S_{\mathcal{O}} \in \mathbb{R}_+$ such that

$$T_s K^+ \subset \Pi_2^{-1} \mathcal{O}(\Pi_2 \mathbb{A}) \text{ for } s \geq S_{\mathcal{O}} \tag{6.6}$$

Applying Π_2 to both sides of (6.6) and using the assertion (b) of Theorem 5.7 we obtain

$$T_s K^+_{sing} \subset \Pi_2 \Pi_2^{-1} (\Pi_2 \mathbb{A}) = \mathcal{O}(\Pi_2 \mathbb{A}) \text{ for } s \geq S_{\mathcal{O}}$$

Thus the attracting property for $\Pi_2 \mathbb{A}$ is valid.

For the second by the definition of $\mathbb{A}$ $\quad T_s \mathbb{A} = \mathbb{A}$ for $s \geq 0$. Applying Π_2 to both sides of this equality and using the assertion (b) of Theorem 5.7 again we obtain

$$T_s \Pi_2 \mathbb{A} = \Pi_2 \mathbb{A} \text{ for } s \geq 0$$

Thus $\Pi_2 \mathbb{A}$ is strictly invariant under $\{T_s, s \geq 0\}$ action.

And finally the compactness for $\Pi_2 \mathbb{A}$ in K^+_{sing} is an immediate corollary of the attracor $\mathbb{A}$ compactness and from the continuity of Π_2.

Thus by definition $\Pi_2 \mathbb{A}$ is a singular trajectory attractor for the problem (0.1). Theorem 6.2 is proved.

Corollary 6.3. *Let τ_j, $1 \le j \le \kappa$, be the trace operators as given in Corollary 5.8. Then the semigroup $\{T_s; s \ge 0\}$ of positive shifts along the t–axis act in the spaces $\tau_j K^+ \subset H^{1-\pi/\alpha_j}_{\mathrm{loc}}(\overline{\mathbb{R}}_+)$ and possess the attractors $\mathbb{A}_j = \tau_j \mathbb{A}$ in them. Moreover the singular attractor $\mathbb{A}_{sing}$ possesses the futher decomposition*

$$\mathbb{A}_{sing} \simeq \oplus_{j=1}^{\kappa} \mathbb{A}_j \tag{6.7}$$

The assertion of this Corollary follows immediately from the topological isomorphizm

$$(\tau_1, \cdots, \tau_\kappa) : \Pi_2^+ H^2_{Q,b}(\mathbb{R}_+ \times \omega) \to \bigoplus_{j=1}^{\kappa} H_b^{1-\pi/\alpha_j}(\mathbb{R}_+)$$

obtained in Section 5.

Note that $\Pi_2^+ \mathbb{A}$ depends on the choice of the projection Π_2^+, while $\tau_j \mathbb{A}$ has an invariant meaning.

Finally we are concerned with the question of stabilization of asymptotics in the case when stabilization of solutions takes place, see Section 4. For that we make all assumptions of Section 4, in particular $f(u) = -\nabla F(u)$ is a gradient like, see (4.5) and the limit equation

$$a\Delta_x v_+ - f(v_+) = g_+, \quad \partial_n v_+\big|_{\partial\omega} = 0 \tag{6.8}$$

has only a finite number of solutions $v_+ = v_+^N$, $N = 1, \cdots, L$ in $H^2_{\mathbb{Q}}(\omega)$.

Let $\{d_j^N\}_{j=1}^{\kappa}$ be the sequence of singular coefficients to v_+^N (see Section 5), i.e.

$$v_+^N = v_0^N + \sum_{j=1}^{\kappa} d_j^N (\chi_j)^* (r^{\pi/\alpha_j} \psi_j(r) \cos(\pi\theta/\alpha_j)) \tag{6.9}$$

where $v_0^N = \Pi_1 v_+^N \in H^2_N(\omega)$ and $d_j^N \in \mathbb{C}^k$.

Theorem 6.4. *Let the assumptioms of Theorem 4.4 be fullfiled. Then for every solution $u(t)$ of the problem (0.1) there exists an equilibria v_+^N such that*

$$T_s u \to v_+^N \ \textit{as}\ s \to \infty\ \textit{in resp.}\ \Theta_0^+\ \textit{or}\ (\Theta_0^+)^w \tag{6.10}$$

in dependence whether the convergence in (4.2) is strong or weak. Moreover

$$T_s \Pi_1 u \to \Pi_1 v_+^N = v_0^N\ \textit{and}\ T_s \Pi_2 u \to \Pi_2 v_+^N \tag{6.11}$$

and

$$T_s \tau_j u \to d_j^N\ \textit{as}\ s \to \infty\ \textit{in resp.}\ H^{1-\pi/\alpha_j}_{\mathrm{loc}}(\overline{\mathbb{R}}_+)\ \textit{or}\ H^{1-\pi/\alpha_j}_{\mathrm{loc}}(\overline{\mathbb{R}}_+))^w \tag{6.12}$$

Proof. The assertion (6.10) follows from Theorem4.4. The rest assertions are immediate corollaries of it and of the continuity of operators Π_1, Π_2 and τ_j in appropriate spaces.

Appendix 1. Elliptic regularity.

In this Section we formulate and prove some auxilary results about the regularity of solution for a linear elliptic equation of vieu (0.1) in polyhedral domains.

Definition A.1. *Let's define $\mathcal{G}$ as the space of all functions $u \in H^{1,2}(\Omega_{T_1-1,T_2+1})$ such that u is a variational solution of the following equation*

$$\langle \partial_t u, \partial_t \Phi\rangle + \langle \nabla_x u, \nabla_x \Phi\rangle + \langle u, \Phi\rangle = \langle g, \Phi\rangle \quad , \quad \forall \Phi \in H^{1,2}(\Omega_{T_1-1,T_2+1}) \tag{A.1}$$

with the right-hand side $g \in L^2(\Omega_{T_1-1,T_2+1})$. The norm in the space $\mathcal{G}$ is

$$\|u\|^2_{\mathcal{G}} \equiv \|u, \Omega_{T_1-1,T_2+1}\|^2_{1,2} + \|g, \Omega_{T_1-1,T_2+1}\|^2_{0,2} \le C\|g, \Omega_{T_1-1,T_2+1}\|^2_{0,2} \tag{A.2}$$

(The last inequality in (A.2) *follows immediately from the unique solvability of variational problem* (A.1)*).*

We define $H^2_{\mathbb{Q}}(\Omega_{T_1,T_2})$ as the space of restrictions of functions from $\mathcal{G}$ to Ω_{T_1,T_2} with the following norm

$$\|v, \Omega_{T_1,T_2}\|_{2,\mathbb{Q}} \equiv \inf\{\|u, \Omega_{T_1-1,T_2+1}\|_{\mathcal{G}} \ : \ u \in \mathcal{G} \ ; \ u\big|_{\Omega_{T_1,T_2}} = v\} \tag{A.3}$$

Let us denote by V_0 the space of restrictions on $t = 0$ of functions from the space $H^2_{\mathbb{Q}}(\Omega_0)$ with the norm

$$\|u_0\|_{V_0} = \inf\{\|u, \Omega_0\|_{2,\mathbb{Q}} \ : \ u\big|_{t=0} = u_0\}$$

Definition A.2. *We denote by $\Theta^+_0 = \left[H^2_{\mathbb{Q},\mathrm{loc}}(\Omega_+)\right]^k$ the subspace of distribution space $D'(\Omega_+)$ with the following system of seminorms*

$$P_{[T_1,T_2]}(u) = \|u\big|_{\Omega_{T_1,T_2}}, \Omega_{T_1,T_2}\|_{2,\mathbb{Q}} < \infty \quad ; \quad [T_1, T_2] \subset [0, \infty) \tag{A.3$'$}$$

It is evidant that seminorms (A.3$'$) *generate in Θ^+ the topology of metrizable F-space (the topology of local compact convergence).*

We denote by $F^+_0 = [H_{\mathbb{Q},b}(\Omega_{T_1-1,T_2+1})]^k$ B-space of functions from Θ^+_0 which have the following norm finite

$$\|u\|_b = \sup_{T\ge 0} P_{[T,T+1]}(u)$$

Corollary A.3 (Elliptic regularity). *Let u be a (variational) solution of the following problem*

$$\begin{cases} \partial_t^2 u + \Delta_x u = g \\ u\big|_{t=T_1} = u_1 \\ u\big|_{t=T_2} = u_2 \\ \partial_n u\big|_{x\in\partial\omega} = 0 \end{cases}$$

where $u_1, u_2 \in V_0$ and $g \in L^2(\Omega_{T_1,T_2})$.

Then $u \in H^2_{\mathbb{Q}}(\Omega_{T_1,T_2})$ *and the following estimate is valid*

$$\|u,\Omega_{T_1,T_2}\|_{2,\mathbb{Q}} \le C(\|g,\Omega_{T_1,T_2}\|_{0,2} + \|u_1\|_{V_0} + \|u_2\|_{V_0})$$

Proof. By definition of V_0 there exists a function $v \in H^2_{\mathbb{Q}}(\Omega_{T_1,T_2})$ which satisfies the following conditions

$$u\big|_{t=T_1} = v\big|_{t=T_1} \text{ and } u\big|_{t=T_2} = v\big|_{t=T_2}$$

Moreover

$$\|v,\Omega_{T_1,T_2}\|_{2,\mathbb{Q}} \le C(\|u_1\|_{V_0} + \|u_2\|_{V_0})$$

Let's prove that the function $w = u - v \in H_{2,\mathbb{Q}}(\Omega_{T_1,T_2})$. This function satisfies the equation

$$\begin{cases} \partial_t^2 w + \Delta_x w = g_1 \equiv g - (\partial_t^2 v + \Delta_x v) \in L^2(\Omega_{T_1,T_2}) \\ w\big|_{t=T_1} = w\big|_{t=T_2} = \partial_n w\big|_{x\in\partial\omega} = 0 \end{cases}$$

Let's consider the cut-off function $\phi(t) \in C_0^\infty(\mathbb{R})$ such that $\phi(t) = 0$ for $t \in [T_1, T_2]$ and $\phi(t) = 1$ for $t \notin [T_1 - \varepsilon, T_2 + \varepsilon]$ where $\varepsilon < T_2 - T_1$. It is easy to check that the function

$$W(t) = \phi(t)\widehat{w}(t) \equiv \phi(t) \begin{cases} -w(2T_1 - t) & for \quad t \in (-\infty, T_1) \\ \quad w(t) & for \quad t \in [T_1, T_2] \\ -w(2T_2 - t) & for \quad t \in (T_2, \infty) \end{cases}$$

belongs to the space $\mathcal{G}$. Indeed

$$\partial_t^2 W + \Delta_x W = \phi(t)\widehat{g}_1(t) + 2\phi'(t)\partial_t\widehat{w}(t) + \phi''(t)\widehat{w}(t) \in L_2(\Omega)$$

and W satisfies the appropriate boundary conditions.

Hence according to Definition A.1 $w \in H_{2,\mathbb{Q}}(\Omega_{T_1,T_2})$. $\square$

Theorem A.4. *For all* $-\infty \le T_1 < T_2 < \infty$ *the space* $H^2_{\mathbb{Q}}(\Omega_{T_1,T_2}) \cap L^\infty(\Omega_{T_1,T_2})$ *is dense in the space* $H^2_{\mathbb{Q}}(\Omega_{T_1,T_2})$

Proof. It is sufficient to prove that $\mathcal{G} \cap L^\infty(\Omega_{T_1-1,T_2+1})$ is dense in $\mathcal{G}$. Let us consider a function $u \in \mathcal{G}$ and a function $g \in L^2(\Omega_{T_1-1,T_2+1})$ which satisfy (A.1). Let $g_n \in L^\infty(\Omega_{T_1-1,T_2+1})$ be a sequence of function with the following property

$$\lim_{n\to\infty} g_n = g \text{ in } L^2(\Omega_{T_1-1,T_2+1}) \tag{A.4}$$

Let $u_n \in \mathcal{G}$ be variational solutions of (A.1) withb right-hand sides g_n. Then, according to (A.2)

$$u_n \to u \text{ in } \mathcal{G} \tag{A.5}$$

Hence, Theorem A.4 will be proved if we prove that $u_n \in L^\infty(\Omega_{T_1-1,T_2+1})$.

To do this we shall use the Maximum principle in the following form

Lemma A.5. *Let $\Omega_{T_1-1,T_2+1} \subset \mathbb{R}^{n+1}$ – be a bounded polyhedral domain and let $u_i \in H^{1,2}(\Omega_{T_1-1,T_2+1})$, $i = 1, 2$ be variational solutions of problem* (A.1) *with right-hand sides $g_i \in H^{1,2}(\Omega_{T_1-1,T_2+1})^*$. Let the following inequallity be valid*

$$\langle g_1, \Phi\rangle \geq \langle g_2, \Phi\rangle \quad ; \quad \forall\Phi \in H^{1,2}(\Omega_{T_1-1,T_2+1}) \tag{A.6}$$

Then

$$u_1(t,x) \leq u_2(t,x) \text{ for almost all } (t,x) \in \Omega_{T_1-1,T_2+1} \tag{A.7}$$

Proof. Let us consider the function $u = u_1 - u_2$. Then

$$\langle \partial_t u, \partial_t \Phi\rangle + \langle \nabla_x u, \nabla_x \Phi\rangle + \langle u, \Phi\rangle \geq 0 \quad ; \quad \forall\Phi \in H^{1,2}(\Omega_{T_1-1,T_2+1}) \tag{A.8}$$

Let us introduce the functions $u_+(t,x) = \max\{u, 0\}$ and $u_-(t,x) = \max\{-u, 0\}$. Then $u = u_+ - u_-$. It is known (see [21]) that $u_+ \in H^{1,2}(\Omega_{T_1-1,T_2+1})$ and

$$\langle u_+, u_-\rangle = 0 \quad ; \quad \langle \nabla u_+, \nabla u_-\rangle = 0 \tag{A.9}$$

Let us replace an arbitrary function Φ in (A.4) by the function u_- and use (A.5). We obtain

$$-\langle \partial_t u_-, \partial_t u_-\rangle - \langle \nabla_x u_-, \nabla_x u_-\rangle - \langle u_-, u_-\rangle \geq 0 \tag{A.10}$$

Formula (A.10) implies that $\langle u_-, u_-\rangle = 0$ or $u_+(t,x) = 0$ for almost all $(t,x) \in \Omega_{T_1-1,T_2+1}$. Lemma A.5 is proved

Lemma A.6. *Let Ω be the same as in previous Lemma and let $u \in H^{1,2}$ be the variational solution of* (A.1). *Let us suppose also that $g \in L^\infty(\Omega_{T_1-1,T_2+1})$. Then $u \in L^\infty(\Omega_{T_1-1,T_2+1})$*

Proof. Let $-M \leq g(t,x) \leq M$ for almost all $(t,x) \in \Omega_{T_1-1,T_2+1}$. Let us consider the following two functions $u_-(t,x) = -M$ and $u_+(t,x) = M$ Then Lemma A.5 implies that $u_-(t,x) \leq u(t,x) \leq u_+(t,x)$ for almost all $(t,x) \in \Omega$. Lemma A.6 is proved. Theorem A.4 is proved.

Theorem A.7. *The following embedding is valid*

$$H^2_{\mathbb{Q}}(\Omega_{T_1,T_2}) \subset L^{q_0}(\Omega_{T_1,T_2}) \tag{A.11}$$

Here

$$q_0 \leq q = 2\frac{n+1}{n-3} \tag{A.12}$$

and if $q < q_0$ then this embedding is compact.

Moreover if $u \in H^2_{\mathbb{Q}}(\Omega_{T_1,T_2})$ then $u|u|^{\frac{q-2}{2}} \in H^{1,2}(\Omega_{T_1,T_2})$ and the following estimate is valid

$$\|u|u|^{(q-2)/2}, \Omega_{T_1,T_2}\|_{1,2} \leq C\|u, \Omega_{T_1,T_2}\|_{2,\mathbb{Q}}^{q/2}$$

Proof. Let $u \in H^2_{\mathbb{Q}}(\Omega_{T_1,T_2})$. Due to the definition A.1 it means that there exists the function $\widehat{u} \in H^{1,2}(\Omega_{T_1-1,T_2+1})$, $\widehat{u}\big|_{\Omega_{T_1,T_2}} = u$, such that

$$\text{(A.13)} \qquad \langle \partial_t \widehat{u}, \partial_t \Phi \rangle + \langle \nabla_x \widehat{u}, \nabla_x \Phi \rangle + \langle \widehat{u}, \Phi \rangle = \langle \widehat{g}, \Phi \rangle \quad , \quad \forall \Phi \in H^{1,2}(\Omega_{T_1-1,T_2+1})$$

with the right-hand side $\widehat{g} \in L^2(\Omega_{T_1-1,T_2+1})$ and

$$\|u, \Omega_{T_1,T_2}\|_{2,\mathbb{Q}} \leq C\|\widehat{g}, \Omega_{T_1-1,T_2+1}\|_{0,2}$$

Let's approximate $\widehat{g} \in L_2(\Omega_{T_1-1,T_2+1})$ by a sequence $\widehat{g}_m \to \widehat{g}$ in $L^2(\Omega_{T_1-1,T_2+1})$, $g_m \in L^\infty(\Omega_{T_1-1,T_2+1})$. Let $\widehat{u}_m$ – be a solution of variational problem (A.13) with the right-hand side $\widehat{g}$ replaced by $\widehat{g}_m$. Then due to Lemma A.6 $\widehat{u}_m \in L^\infty(\Omega_{T_1-1,T_2+1})$. Hence the function $\Phi = \widehat{u}_m|\widehat{u}_m|^{l-2}$ is in the space $H^{1,2}(\Omega_{T_1-1,T_2+1})$ where $l-2 = \frac{4}{n-3} = (q-2)/2$. Replacing in (A.13) $\widehat{u}$ by $\widehat{u}_m$ and Φ by $\widehat{u}_m|\widehat{u}_m|^{l-2}$ and arguing as in reducing the estimate (1.19) we obtain the following inequality

$$\text{(A.14)} \qquad \|\widehat{u}_m|\widehat{u}_m|^{(l-2)/2}, \Omega_{T_1-1,T_2+1}\|^2_{1,2} \leq C(1 + |\left\langle \widehat{g}_m, \widehat{u}_m|\widehat{u}_m|^{l-2}\right\rangle|)$$

It follows from Sobolev embedding theorem ($H^{1,2} \subset L^r$ for $r = \frac{2n}{n-2}$) that

$$\|\widehat{u}_m, \Omega_{T_1-1,T_2+1}\|^l_{0,q} = \|\widehat{u}_m|\widehat{u}_m|^{(l-2)/2}, \Omega_{T_1-1,T_2+1}\|^2_{0,r} \leq$$
$$\leq C\|\widehat{u}_m|\widehat{u}_m|^{(l-2)/2}, \Omega_{T_1-1,T_2+1}\|^2_{1,2}$$

Applying Holder inequality to the last term into the right–hand side of (A.14) we obtain

$$|\left\langle \widehat{g}_m, \widehat{u}_m|\widehat{u}_m|^{l-2}\right\rangle| \leq \|g, \Omega_{T_1-1,T_2+1}\|_{0,2}\|\widehat{u}_m, \Omega_{T_1-1,T_2+1}\|^{l-1}_{0,q} \leq$$
$$\leq \mu\|\widehat{u}_m, \Omega_{T_1-1,T_2+1}\|^l_{0,q} + C_\mu\|g_m, \Omega_{T_1-1,T_2+1}\|^l_{0,2}$$

for an arbitrary positive μ.

Applying these estimates to inequality (A.14) and taking sufficiently small $\mu > 0$ we get

$$\text{(A.15)} \qquad \|\widehat{u}_m, \Omega_{T_1-1,T_2+1}\|^l_{0,q} + \|\widehat{u}_m|\widehat{u}_m|^{(l-2)/2}, \Omega_{T_1-1,T_2+1}\|^2_{1,2} \leq$$
$$\leq C\|g_m, \Omega_{T_1-1,T_2+1}\|^l_{0,2}$$

We know that $\widehat{g}_m \to \widehat{g}$ in $L^2(\Omega_{T_1-1,T_2+1})$, hence the sequence $\widehat{u}_m$ is bounded in the space $L^q(\Omega_{T_1-1,T_2+1})$. Without loss of generality we can think that $\widehat{u}_m \rightharpoonup \widehat{u}$ in the space $L^q(\Omega_{T_1-1,T_2+1})$. So $\widehat{u} \in L^q(\Omega_{T_1-1,T_2+1})$ and

$$\|u, \Omega_{T_1,T_2}\|_{0,q} \leq \|\widehat{u}, \Omega_{T_1-1,T_2+1}\|_{0,q} \leq C\|\widehat{g}, \Omega_{T_1-1,T_2+1}\|_{0,2} \leq C_1\|u, \Omega_{T_1,T_2}\|_{2,\mathbb{Q}}$$

The embedding $u|u|^{l-2} \in H^{1,2}(\Omega_{T_1,T_2})$ could be proved analogously.

Let us prove the compactness of embedding (A.11) for $q_0 < q$. Indeed due to the interpolation inequality between $H^{1,2}$ and L^q

$$H^2_{\mathbb{Q}}(\Omega_{T_1,T_2}) \subset H^{\varepsilon,q_0}(\Omega_{T_1,T_2})$$

for some positive ε. It is well known that the embedding $H^{\varepsilon,q_0} \subset L^{q_0}$ is compact. Theorem A.7 is proved.

Corolary A.8. *The following embedding is valid*

$$H^2_{\mathbb{Q}}(\Omega_{T_1,T_2}) \subset C([T_1,T_2], L^{p_0}(\omega)) \tag{A.16}$$

Here $p_0 = 2l = 2 + \frac{4}{n-3}$ *– the maximum of* p *exponent in* (0.2).

Indeed it follows from the second embedding of Theorem A.7 and Sobolev embedding theorem that $u|u|^{(l-2)/2} \in C([T_1,T_2], L^2(\omega))$ if $u \in H^2_{\mathbb{Q}}(\Omega_{T_1,T_2})$. Moreover we know from the embedding $H^2_{\mathbb{Q}} \subset H^{1,2}$ that $u \in C([T_1,T_2], L^2(\omega))$ Arguing in the following as in the proof of Krasnoselski Theorem (see [11]) we obtain that $u \in C([T_1,T_2], L^{p_0}(\omega))$.

Theorem A.9. *Let* $u \in H^2_{\mathbb{Q}}(\Omega_{T_1,T_2})$. *Then* $\partial_t^2 u \in L^2(\Omega_{T_1,T_2})$, $\partial_t \nabla_x u \in L^2(\Omega_{T_1,T_2})$ *and the following estimate is valid:*

$$\|\partial_t^2 u, \Omega_{T_1,T_2}\|^2_{0,2} + \|\partial_t \nabla_x u, \Omega_{T_1,T_2}\|^2_{0,2} + \|\Delta_x u, \Omega_{T_1,T_2}\|^2_{0,2} \le C\|u, \Omega_{T_1,T_2}\|^2_{2,\mathbb{Q}} \tag{A.17}$$

Proof. By definition there exists a function $\widehat{u} \in \mathcal{G}$ such that $\widehat{u}\big|_{\Omega_{T_1,T_2}} = u$ which satisfies the equation

$$\begin{cases} \partial_t^2 \widehat{u} + \Delta_x \widehat{u} - \widehat{u} = g(x) \\ \partial_n \widehat{u}\big|_{\partial\omega} = 0 \\ \widehat{u}\big|_{t=T_1-1} = 0\ ,\ \widehat{u}\big|_{t=T_2+1} = 0 \end{cases} \tag{A.18}$$

for some function $g \in L^2(\Omega_{T_1-1,T_2+1})$ and $\|g, \Omega_{T_1-1,T_2+1}\|_{0,2} \le C\|u, \Omega_{T_1,T_2}\|_{2,\mathbb{Q}}$. We give below only formal reducing of the estimate (A.17). The rigorous proof could be obtained by using (for example) Galerkin approximations method.

Let us multiply the equation (A.18) by $\partial_t^2 \widehat{u}$ and integarte over Ω_{T_1-1,T_2+1}. We obtain after integration by part

$$\big\langle |\partial_t^2 \widehat{u}|^2, 1\big\rangle + \big\langle |\partial_t \nabla_x \widehat{u}|^2, 1\big\rangle + \big\langle |\partial_t \widehat{u}|^2, 1\big\rangle = \big\langle g, \partial_t^2 \widehat{u}\big\rangle \tag{A.19}$$

Applying Holder inequality

$$\big\langle g, \partial_t^2 \widehat{u}\big\rangle \le \frac{1}{2}\big\langle |g|^2, 1\big\rangle + \frac{1}{2}\big\langle |\partial_t^2 \widehat{u}|^2, 1\big\rangle$$

to the right–hand side of the equation (A.19) we obtain the inequality (A.17). Theorem A.9 is proved.

Corollary A.10. *It follows from the previous Theorem that*

$$u \in H^{1,2}([T_1,T_2], H^{1,2}(\omega)) \cap H^{2,2}([T_1,T_2], L^2(\omega))$$

if $u \in H^2_{\mathbb{Q}}(\Omega_{T_1,T_2})$ *hence*

$$H^2_{\mathbb{Q}}(\Omega_{T_1,T_2}) \subset C([T_1,T_2], H^{1,2}(\omega)) \cap C^1([T_1,T_2], L^2(\omega)) \tag{A.20}$$

So the functions $\|\partial_t u(t)\|_{0,2}$ *and* $\|u(t)\|_{1,2}$ *are correctly defined and continious for every* $u \in H^2_{\mathbb{Q}}$

Corollary A.11. *Let $D(A)$ be the domain of definition for the Laplace operator $Au = -\Delta_x u + u$ in $L_2(\omega)$ with Newmann boundary conditions. Then it follows from Theorem A.9 that*

$$H^2_{\mathbb{Q}}(\Omega_{T_1,T_2}) = H^{2,2}([T_1,T_2], L^2(\omega) \cap L^2([T_1,T_2], D(A)) \tag{A.21}$$

Hence due to the interpolation theory and abstract trace theorems (see [12] *and* [19]*) the space V_0 possesses the following description*

$$V_0 = D(A^{\frac{3}{4}}) \tag{A.22}$$

Let us suppose that ω has a smooth boundary $\partial\omega$ then as known (see [19] *for example) the space $D(A^{\frac{3}{4}})$ could be described explicitly*

$$V_0 = D(A^{\frac{3}{4}}) = \left\{ u_0 \in H^{\frac{3}{2},2}(\omega) \ : \ \int_\omega d^{-1}(x)|B_n(x)u_0(x)|^2\, dx < \infty \right\} \tag{A.23}$$

Here $d(x) = \inf_{y\in\partial\omega}|x-y|$ and $B_n(x) = \sum_{i=1}^n b_i(x)\partial_{x_i}$ some continious extention of the normal derivative operator from the boundary $\partial\omega$ in ω ($B_n(x)\big|_{\partial\omega} = \partial_n$).

Remark A.12. *In the case when ω – is smooth domain all results of this Section are trivial corrolaries of L_2–regularity theorem for Laplace operator (see* [19]*)*

$$H^2_{\mathbb{Q}}(\Omega_{T_1,T_2}) = \{u \in H^{2,2}(\Omega_{T_1,T_2}) \ : \ \partial_n u\big|_{\partial\omega} = 0\} \tag{A.24}$$

and Sobolev embedding theorems. But for polyhedral domains the equality (A.24) *is not valid in general (see Section 5 for example).*

Theorem A.13. *Let w be a polyhedral domain and let A be the same as in Corollary A.11. Then there exists some positive $0 < \varepsilon = \varepsilon(\omega) \le \frac{1}{2}$ such that*

$$D(A) \subset H^{\frac{3}{2}+\varepsilon,2}(\omega) \tag{A.25}$$

The proof of this Theorem is given in [9].

Corollary A.14. *Let w be polyhedral domain. Then the following embedding is valid*

$$H^2_{\mathbb{Q}}(\Omega_{T_1,T_2}) \subset H^{\frac{3}{2}+\varepsilon,2}(\Omega_{T_1,T_2}) \tag{A.26}$$

where $\varepsilon = \varepsilon(\omega)$ depends only on ω.

Indeed (A.26) follows from (A.25) and (A.21).

Corollary A.15. *Let u be in $H^2_{\mathbb{Q}}(\Omega_{T_1,T_2})$. Then it follows from* (A.26) *and Sobolev embedding theorem that*

$$\partial_n u\big|_{\partial\omega} \in H^{\varepsilon,2}([T_1,T_2]\times\partial\omega)$$

Using Green's formula (see [12]*) it is not difficult to obtain now that $\partial_n u\big|_{\partial\omega} = 0$ for every $u \in H^2_{\mathbb{Q}}(\Omega_{T_1,T_2})$. Thus the solutions u of the problem* (0.1) *from the space Θ_0^+ satisfy the homnogeneous Newmann boundary conditions in ordinary sense.*

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