THE REGULAR ATTRACTOR FOR A NONLINEAR ELLIPTIC SYSTEM IN A CYLINDRICAL DOMAIN

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Introduction

In the half-cylinder $\Omega_+ = \mathbb{R}_+ \times \omega$, where ω is a bounded domain in $\mathbb{R}^n$ with a sufficiently smooth boundary, we consider the following quasilinear second-order elliptic boundary value problem:

$$\begin{cases} a(\partial_t^2 u + \Delta_x u) - \gamma \partial_t u - f(u) = g(t), \\ u|_{t=0} = u_0, \quad u|_{\partial\omega} = 0. \end{cases} \tag{0.1}$$

Here $u = u(t,x) = (u^1, \cdots, u^k)$ is the unknown vector function, $g(t) = g(t,x)$, $f(u)$ are given vector functions, $(t,x) \in \Omega_+$, Δ_x denotes the Laplacian with respect to the variables $x = (x_1, \cdots, x_n)$, and $\gamma = \gamma^*$, $a = a^*$ are positive self-adjoint matrices ($\gamma, a \in L(\mathbb{R}^k, \mathbb{R}^k)$),

$$0 < a_- \operatorname{Id} \le a \le a_+ \operatorname{Id}, \ \ 0 < \gamma_- \operatorname{Id} \le \gamma \le \gamma_+ \operatorname{Id}. \tag{0.2}$$

We suppose that the nonlinear function $f(u)$ satisfies the conditions

$$\begin{cases} 1. \quad f \in C^1(\mathbb{R}^k, \mathbb{R}^k), \\ 2. \quad f(u) \cdot u \ge -C, \ \forall\, u \in \mathbb{R}^k, \\ 3. \quad f'(u) \ge -K, \ K \ge 0, \ \forall u \in \mathbb{R}^k. \end{cases} \tag{0.3}$$

Here and below $u \cdot v$ denotes the scalar product in $\mathbb{R}^k$.

The right-hand side g is supposed to be in the space $L_p(\Omega_T)$ for every $T \ge 0$ ($\Omega_T \equiv [T, T+1] \times \omega$) and some $p > \max\{2, \frac{n+1}{2}\}$ and to have a finite norm

$$|g|_b = \sup_{T \ge 0} \|g, \Omega_T\|_{0,p} < \infty. \tag{0.4}$$

Here and in the following $\|v, \Omega_T\|_{l,p} \equiv \|v\|_{W_{l,p}(\Omega_T)}$, and the symbol $W_{l,p}$ denotes the Sobolev space of distributions whose derivatives up to order l belong to the space L_p (see [9]).

A function u satisfying (0.1) in the sense of distributions is said to be a solution of the problem (0.1) if $u \in W_{2,p}(\Omega_T)$ for every $T \ge 0$ and the following norm is finite:

$$\|u\|_b = \sup_{T \in \mathbb{R}_+} \|u, \Omega_T\|_{2,p} < \infty. \tag{0.5}$$

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The initial value u_0 are assumed to be in the trace space V_0 on $\{t=0\}$ of functions from $W_{2,p}(\Omega_0)$. It is known (see [10]) that

$$V_0 = \operatorname{Tr}\big|_{t=0}\big(W_{2,p}(\Omega_+)\cap\{u|_{\partial\omega}=0\}\big) = W_{2-1/p,p}(\omega)\cap \overset{0}{W}_{1,p}(\omega). \tag{0.6}$$

Equations of type (0.1) can appear for instance when studying the equilibrium points and travelling wave solutions of certain evolutionary equations (see Remark 5.3).

The problem (0.1) was studied under different assumptions on the nonlinear part f and the right-hand side g in [3], [4], [6], [13], [14], [15], [18], [19], [21].

In this paper we mainly consider the case where the elliptic boundary value problem (0.1) has a unique solution for every $u_0\in V_0$. To prove the uniqueness we assume that there exists a number $\lambda_0>0$ such that the operator family

$$L_K(\lambda) = a\lambda^2 - \gamma\lambda + a\Delta_x + K\ : \overset{0}{W}_{1,2}(\omega)\to W_{-1,2}(\omega) \tag{0.7}$$

satisfies the condition

$$-(L_K(\lambda_0)v,v)>0 \text{ for every } v\in \overset{0}{W}_{1,2}(\omega),\ v\neq 0. \tag{0.8}$$

It is proved in Lemma 1.1 that the condition (0.8) holds if

$$\gamma^2 > 4a_+(K-a_-\mu_1)\,\mathrm{Id}\,.$$

Here we denote by $\mu_1>0$ the first Dirichlet eigenvalue of $-\Delta_x$ in ω.

The trajectory attractor for the problem (0.1) without the uniqueness assumptions was constructed in [4].

The main task of this paper is to study the behaviour of the solutions of equation (0.1) as $t\to\infty$. The following theorem proved in Sections 2 and 3 is of fundamental significance in that connection.

Theorem 1. *1. Assume that the above assumptions are satisfied. Then the problem* (0.1) *has a unique solution u satisfying* (0.5) *and the following estimate holds:*

$$\|u,\Omega_T\|_{2,p}\le Q(\|u_0\|_{V_0})e^{-\alpha T}+Q(|g|_b). \tag{0.9}$$

Here the constant $\alpha>0$ and the monotone function $Q:\mathbb{R}_+\to\mathbb{R}_+$ are independent of the initial data u_0.

2. Let $S_t: V_0\to V_0$ be the solving operator for the problem (0.1) ($S_tu(0)=u(t)$). *Then $S_t\in C^1(V_0,V_0)$ for every $t\ge 0$ and its Fréchet derivative $D_{u_0}S_t(u_0)$ satisfies the estimate*

$$\|D_{u_0}S_t(u_0)\|_{L(V_0,V_0)}\le Q(\|u_0\|_{V_0})e^{(\lambda_0-\alpha)t}. \tag{0.10}$$

Moreover, we prove in Section 3 that the problem (0.1) is equivalent to the following evolution problem in the space V_0:

$$\begin{cases} \partial_t u + (-\Delta_x)^{1/2}u = F(t,u)+G(t), \\ u\big|_{t=0}=u_0,\ \ u\in C_b(\mathbb{R}_+,V_0)\cap C_b^1(\mathbb{R}_+,V_0'). \end{cases} \tag{0.11}$$

Here $V_0'=(-\Delta_x)^{1/2}V_0$ and, for G and the nonlinear operator F, we have respectively $G\in C(\mathbb{R}_+,V_0')$ and $F\in C(\mathbb{R}_+\times V_0,V_0)$ (see Theorem 3.3).

Sections 4 and 5 of this paper are devoted to a more detailed study of the "autonomous" case

$$g(t) \equiv g \in L_p(\omega). \tag{0.12}$$

In this case the operators $\{S_t, t \geq 0\}$ generate a semigroup in the space V_0. The equilibrium points of this semigroup are investigated in Section 4. In this section we give necessary and sufficient conditions for the hyperbolicity of the equilibrium z_0 of S_t, prove that the instability index ind_{z_0} for every equilibrium z_0 is finite and obtain formulae for its calculation. Moreover, we prove there that the unstable set $\mathcal{M}^+(z_0)$ for the hyperbolic equilibrium z_0 (see Definition 4.3) possesses the structure of a C^1-manifold diffeomorphic to $\mathbb{R}^d$, $d = \mathrm{ind}_{z_0}$.

We construct the attractor $\mathcal{A}$ for the semigroup $\{S_t, t \geq 0\}$ in Section 5 and prove that it is compact in the space $W_{2,p}(\omega)$. Moreover, in the potential case

$$f = -\nabla P, \ \ P \in C^2(\mathbb{R}^k, \mathbb{R}^k), \tag{0.13}$$

the following theorem is proved.

Theorem 2. *Assume that the above assumptions are satisfied. Let all of the equilibria $\mathcal{R} = \{z_1, \cdots, z_N\}$ for the semigroup S_t be hyperbolic. Then the attractor $\mathcal{A}$ is regular (see Theorem 5.3 and its corollaries),*

$$\mathcal{A} = \cup_{i=1}^N \mathcal{M}^+(z_i), \tag{0.14}$$

and exponential (see Theorem 5.4 and its corollary).

Moreover,

$$\dim \mathcal{M}^+(z_i) = \mathrm{ind}_{z_i} = \#\{\lambda \in \sigma(a\Delta_x - f'(z_i)), \ \lambda > 0\}$$

For evolutionary equations possessing a global Lyapunov function the representation (0.14) for the attractor was obtained in [2], [17].

For the proof of Theorems 1 and 2 we need a number of auxiliary results, which are discussed in §1.

§1 The linear elliptic equation in a half-cylinder

This section deals with the following elliptic system in a half-cylinder $\Omega_+ = \mathbb{R}_+ \times \omega$ ($\omega \subset\subset \mathbb{R}^n$):

$$\begin{cases} a(\partial_t^2 u + \Delta_x u) - \gamma \partial_t u - q(t,x)u = g(t), \\ u\big|_{t=0} = u_0, \ \ u\big|_{\partial\omega} = 0. \end{cases} \tag{1.1}$$

Here u, g are vector functions, $q \in L_\infty(\Omega_+, \mathcal{L}(\mathbb{R}^k, \mathbb{R}^k))$, a, γ are self-adjoint matrices, and $a > 0$ is positive. Let us fix a constant $K \in \mathbb{R}$ so that

$$q(t,x) \geq -K \,\mathrm{Id} \ \text{ for almost all } (t,x) \in \Omega_+. \tag{1.2}$$

We also assume that the operator family (0.7) (with K defined by (1.2)) satisfies the condition (0.8) for some $\lambda_0 \geq 0$ and that for the right-hand side g the following norm is finite:

$$|g|_{\lambda_0} \equiv \sup_{T \in \mathbb{R}_+} e^{-\lambda_0 T} \|g, \Omega_T\|_{0,p} < \infty. \tag{1.3}$$

A function u satisfying (1.1) is said to be a solution of the problem (1.1) if we have $u \in H_{2,p}(\Omega_T)$ for every $T \geq 0$ and

$$\|u\|_{\lambda_0} = \sup_{T \in \mathbb{R}_+} e^{-\lambda_0 T} \|u, \Omega_T\|_{2,p} < \infty. \tag{1.4}$$

We choose the initial condition u_0 to belong to the space V_0 (see (0.6)).

Lemma 1. *Condition* (0.8) *holds if* $\gamma \geq 0$ *and*

$$\gamma^2 > 4(K - \mu_1 a_-)a_+ \,\mathrm{Id}\,. \tag{1.5}$$

Here by $\mu_1 > 0$ *we denote the first Dirichlet eigenvalue of* $-\Delta_x$ *in the domain* ω.

Proof. It follows from (1.5) that there exists a number $\gamma_0 \geq 0$, such that

$$\gamma \geq \gamma_0 \,\mathrm{Id},\ \ \gamma_0^2 > 4(K - \mu_1 a_-)a_+.$$

Thus $a_+\lambda_0^2 - \gamma_0\lambda_0 - \mu_1 a_- + K < 0$ for $\lambda_0 = \gamma_0/2a_+$ and, therefore, for every $v \in H^0_{1,2}(\omega)$, $v \neq 0$,

$$\begin{aligned}(L_K(\lambda_0)v, v) = (av, v)\lambda_0^2 - (\gamma v, v)\lambda_0 - (a\nabla v, \nabla v) + (Kv, v) \leq \\ \leq (a_+\lambda_0^2 - \gamma_0\lambda_0 - \mu_1 a_- + K)\|v\|^2_{0,2} < 0.\end{aligned}$$

Here and below $(a\nabla v, \nabla v) = a\nabla v \cdot \nabla v \equiv \sum_i a\partial_{x_i} v \cdot \partial_{x_i} v$. Lemma 1 is proved.

Lemma 2. *Condition* (0.8) *is equivalent to the following one:*

$$-(L_K(\lambda_0)v, v) \geq \varepsilon\|v\|^2_{1,2} \text{ for some } \varepsilon > 0. \tag{1.6}$$

Proof. Let us suppose that (1.6) is false. Then there exist $v_i \in W^0_{1,2}(\omega)$, $\|v_i\|_{1,2} = 1$, $i \in \mathbb{N}$, such that

$$(L_K(\lambda_0)v_i, v_i) = -\varepsilon_i,\ \ \varepsilon_i \geq 0,\ \ \varepsilon_i \to 0 \text{ as } i \to \infty. \tag{1.7}$$

Without loss of generality we may assume that $v_i \rightharpoonup v_0 \in W^0_{1,2}(\omega)$ in the space $W^0_{1,2}$ and $v_i \to v_0$ in the space L_2. Taking the limit $i \to \infty$ in (1.7) and using $(a\nabla v_0, \nabla v_0) \leq \liminf_{i\to\infty}(a\nabla v_i, \nabla v_i)$, we find that $(L_K(\lambda_0)v_0, v_0) \geq 0$. Hence in view of condition (0.8) $v_0 = 0$. Rewriting formula (1.7) in the form

$$((a + \varepsilon_i \,\mathrm{Id})\nabla v_i, \nabla v_i) = (av_i, v_i)\lambda_0^2 - (\gamma v_i, v_i)\lambda_0 + (Kv_i, v_i),$$

we then obtain that $\|v_i\|_{1,2} \to 0$ for $i \to \infty$ which contradicts our assumptions on the sequence v_i. Lemma 2 is proved.

The main result of this section is stated in the following theorem:

Theorem 1. *Let the above assumptions be satisfied. Then the problem* (1.1) *has a unique solution* $u(t)$. *Moreover, following estimate holds:*

$$\|u, \Omega_T\|^p_{2,p} \leq Ce^{p\lambda_0 T}\left(\|u_0\|^p_{V_0}e^{-\alpha T} + \int_{\mathbb{R}_+} e^{-\alpha|T-s| - p\lambda_0 s}\|g(s)\|^p_{0,p}\,ds\right) \tag{1.8}$$

for some $\alpha > 0$.

To prove this theorem we need a number of auxillary results.

Lemma 3. *Let u be a solution of the problem* (1.1). *Then the following estimate holds:*

$$\|u,\Omega_T\|_{1,2}^2 \le Ce^{2\lambda_0 T}\left(\|u_0\|_{V_0}^2 e^{-\alpha T} + \int_{\mathbb{R}_+} e^{-\alpha|T-s|-2\lambda_0 s}\|g(s)\|_{0,2}^2\, ds\right). \qquad (1.9)$$

Proof. Consider a function $v \in W_{2,p}(\Omega_+)$ such that

$$v\big|_{t=0} = u_0,\ \ v\big|_{\partial\omega} = 0,\ \ \operatorname{supp} v \subset \Omega_0,\ \ \|v,\Omega_+\|_{2,p} \le C\|u_0\|_{V_0}, \qquad (1.10)$$

where the constant C is independent of u_0 (it exists by the definition of the space V_0). Rewrite equation (1.1) for the new unknown $\theta(t) = u(t)e^{-\lambda_0 t} - v(t)$:

$$\begin{cases} a\partial_t^2\theta - (\gamma - 2a\lambda_0)\partial_t\theta + [a\lambda_0^2 - \gamma\lambda_0 + a\Delta_x - q(t,x)]\theta = g_1(t), \\ \theta\big|_{t=0} = 0,\ \ \theta\big|_{\partial\omega} = 0. \end{cases} \qquad (1.11)$$

Here

$$g_1(t) = g(t)e^{-\lambda_0 t} - a\partial_t^2 v + (\gamma - 2a\lambda_0)\partial_t v - [a\lambda_0^2 - \gamma\lambda_0 + a\Delta_x - q(t,x)]v.$$

Multiplying equation (1.11) by $\theta(t)e^{-\alpha|T-t|} = \theta(t)\varphi_T(t)$ in $\mathbb{R}^k$, where α is a sufficiently small positive number which will be fixed below, and afterwards integrating over Ω_+, we obtain

$$\begin{aligned} \left\langle a\partial_t^2\theta, \theta(t)\varphi_T(t)\right\rangle - \langle(\gamma - 2a\lambda_0)\partial_t\theta, \theta(t)\varphi_T(t)\rangle + \\ + \left\langle[a\lambda_0^2 - \gamma\lambda_0 + a\Delta_x - q(t,x)]\theta, \theta(t)\varphi_T(t)\right\rangle = \\ = \langle g_1(t), \theta(t)\varphi_T(t)\rangle. \end{aligned} \qquad (1.12)$$

Here $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L_2(\Omega_+)$. It follows from (1.4) and from the definition of θ that the function $\theta(t)$ remains bounded as $t \to +\infty$, i.e.,

$$\|\theta\|_b = \sup_{T\ge 0}\|u,\Omega_T\|_{2,p} < \infty.$$

Hence all the integrals in (1.12) are well-defined.

Let us now separately estimate each term on the left-hand side of (1.12). Using the positiveness of a, the estimate

$$|\partial_t\varphi_T(t)| \le \alpha\varphi_T(t), \qquad (1.13)$$

and the Cauchy-Schwartz inequality we obtain after integrating by parts

$$\begin{aligned} -\left\langle a\partial_t^2\theta, \theta(t)\varphi_T(t)\right\rangle = \langle a\partial_t\theta\cdot\partial_t\theta, \varphi_T(t)\rangle + \langle a\partial_t\theta\cdot\theta, \partial_t\varphi_T(t)\rangle \ge \\ \ge C\left\langle|\partial_t\theta|^2, \varphi_T(t)\right\rangle - {}_1\alpha^2\left\langle|\theta|^2, \varphi_T(t)\right\rangle. \end{aligned} \qquad (1.14)$$

Analogously, since a and γ are self-adjoint we can infer that

$$\begin{aligned} |\langle(\gamma - 2a\lambda_0)\partial_t\theta, \theta(t)\varphi_T(t)\rangle| = 1/2|\langle\partial_t[(\gamma - 2a\lambda_0)\theta\cdot\theta], \varphi_T(t)\rangle| = \\ = 1/2| - \langle[(\gamma - 2a\lambda_0)\theta\cdot\theta], \partial_t\varphi_T(t)\rangle| \le C_2\alpha\left\langle|\theta|^2, \varphi_T(t)\right\rangle. \end{aligned} \qquad (1.15)$$

It follows from estimate (1.6) that

$$-\left\langle [a\lambda_0^2-\gamma\lambda_0+a\Delta_x-q(t,x)]\theta,\theta(t)\varphi_T(t)\right\rangle \geq \\ \geq \varepsilon\left\langle |\nabla\theta|^2,\varphi_T(t)\right\rangle \geq \frac{\varepsilon}{2}\left\langle |\nabla\theta|^2,\varphi_T(t)\right\rangle+\frac{\varepsilon}{2}\mu_1\left\langle |\theta|^2,\varphi_T(t)\right\rangle. \quad (1.16)$$

Here $\mu_1>0$ is the first eigenvalue of the Laplacian in ω.

Inserting these estimates into (1.6), (1.12) and choosing $\alpha>0$ small enough so that

$$C_2\alpha+C_1\alpha^2\leq\frac{\varepsilon}{2}\mu_1$$

we find

$$\left\langle |\partial_t\theta|^2,\varphi_T(t)\right\rangle+\frac{\varepsilon}{2}\left\langle |\nabla\theta|^2,\varphi_T(t)\right\rangle\leq C\left\langle |g_1|^2,\varphi_T(t)\right\rangle. \quad (1.17)$$

It follows from the definition of $\varphi_T(t)$ that $\varphi_T(t)\geq e^{-\alpha}$ for $t\in[T,T+1]$. Hence

$$\|\theta,\Omega_T\|_{1,2}^2\leq e^{\alpha}\frac{2}{\varepsilon}(\left\langle |\partial_t\theta|^2,\varphi_T(t)\right\rangle+\frac{\varepsilon}{2}\left\langle |\nabla\theta|^2,\varphi_T(t)\right\rangle)\leq C\left\langle |g_1|^2,\varphi_T(t)\right\rangle. \quad (1.18)$$

Finally, using (1.10) and the definition of g_1 we conclude

$$\left\langle |g_1|^2,\varphi_T(t)\right\rangle\leq 2\left\langle |g|^2,e^{-2\lambda_0 t}\varphi_T(t)\right\rangle+Ce^{-\alpha T}\|u_0\|_{V_0}^2.$$

Inserting this estimate into the right-hand side of (1.18) and replacing $\theta(t)$ by $u(t)e^{-\lambda_0 t}-v(t)$, we obtain the estimate (1.9). Lemma 3 is proved.

Lemma 4. *Let u be a solution of the equation* (1.1). *Then for every $1<r\leq p$ and every $\nu>0$ the following estimate holds:*

$$\|u,\Omega_T\|_{2,r}\leq C_\nu(\|u,\Omega_{T-\nu,T+1+\nu}\|_{1,r}+ \\ +\|g,\Omega_{T-\nu,T+1+\nu}\|_{0,r}+\chi(\nu-T)\|u_0\|_{V_0}). \quad (1.19)$$

Here $\Omega_{T_1,T_2}=[T_1,T_2]\times\omega$ when $T_2>T_1\geq 0$ and $\Omega_{T_1,T_2}=[0,T_2]\times\omega$ when $T_1<0\leq T_2$, $\chi(z)$ is the Heaviside function which equals zero for $z<0$ and equals one for $z\geq 0$, and the constant C_ν is independent of T.

Proof. Let $\psi_T(t)\in C_0^\infty(\mathbb{R}_+)$ be a cut-off function, $\psi_T(t)=1$ for $t\in[T,T+1]$ and $\psi_T(t)=0$ for $t\notin[T-\nu,T+1+\nu]$. Consider the function $w=\psi_T(t)u(t)$. It follows from equation (1.1) that

$$\begin{cases} \partial_t^2 w+\Delta_x w=h_u(t), \ \ w\big|_{\partial\omega}=0, \\ w\big|_{t=T+1+\nu}=0, \ \ w\big|_{t=\max\{T-\nu,0\}}=\psi_T(0)u_0. \end{cases} \quad (1.20)$$

Here

$$h_u=\psi_T''u+2\psi_T'\partial_t u+a^{-1}\psi_T\left(\gamma\partial_t u+q(x,t)u+g(t)\right). \quad (1.21)$$

Formula (1.21) yields

$$\|h_u,\Omega_{T-\nu,T+1+\nu}\|_{0,r}\leq C_\nu(\|u,\Omega_{T-\nu,T+1+\nu}\|_{1,r}+\|g,\Omega_{T-\nu,T+1+\nu}\|_{0,r}). \quad (1.22)$$

Using the $(H_{2,r},L_r)$-regularity theorem for solutions of the Laplace equation (see [10]) and taking into account that $\psi_T(0)=0$ for $T\geq\nu$, we obtain

$$\|u,\Omega_T\|_{2,r}\leq\|w,\Omega_{T-\nu,T+1+\nu}\|_{2,r}\leq \\ \leq C_1(\|h_u,\Omega_{T-\nu,T+1+\nu}\|_{0,r}+\chi(\nu-T)\|u_0\|_{V_0}). \quad (1.23)$$

The estimate (1.19) is an immediate consequence of the inequalities (1.22), (1.23). Lemma 4 is proved.

Lemma 5. *Let u be a solution of the problem* (1.1). *Then*

$$||u,\Omega_T||_{2,p} \le C(||u,\Omega_{T-1,T+2}||_{1,2} + ||g,\Omega_{T-1,T+2}||_{0,p} + \chi(1-T)||u_0||_{V_0}) \quad (1.24)$$

Proof. According to S. L. Sobolev's embedding theorem (see [10])

$$\|u,\Omega_{T-\nu,T+1+\nu}\|_{1,l(r)} \le C\|u,\Omega_{T-\nu,T+1+\nu}\|_{2,r}. \quad (1.25)$$

Here

$$l(r) = \begin{cases} r(n+1)/(n+1-r) & \text{if } r < n+1, \\ \infty & \text{if } r > n+1. \end{cases} \quad (1.26)$$

It follows from the estimates (1.19) and (1.25) that

$$\|u,\Omega_T\|_{1,m(r)} \le C(\|u,\Omega_{T-\nu,T+1+\nu}\|_{1,r} + \\ + \|g,\Omega_{T-\nu,T+1+\nu}\|_{0,r} + \chi(\nu-T)\|u_0\|_{V_0}). \quad (1.27)$$

where $m(r) = \min\{l(r),p\}$. Let us define a sequence of the exponents r_i in the following way: $r_0 = 2$, $r_{N+1} = m(r_N)$. Formula (1.26) implies that $r_N = p$ for $N \ge N_0 = \left[\frac{\ln p/2}{\ln(1+2/(n+1))}\right] + 1$, where $[z]$ denotes the integer part of the real z.

Iterating formula (1.27) N_0 times with the first exponent being $r = r_0 = 2$, we get

$$\|u,\Omega_T\|_{1,p} \le C(\|u,\Omega_{T-N_0\nu,T+1+N_0\nu}\|_{1,2} + \\ + \|g,\Omega_{T-N_0\nu,T+1+N_0\nu}\|_{0,p} + \chi(N_0\nu-T)\|u_0\|_{V_0}). \quad (1.28)$$

Inserting this estimate into (1.19) with $\nu = 1/(N_0+1)$, we eventually obtain the estimate (1.24). Lemma 5 is proved.

The inequalities (1.9) and (1.24) imply the estimate (1.8).

An immediate corollary of this estimate is the uniqueness of the solution for the problem (1.1). Thus for completing the proof of Theorem 1 it remains to prove the existence of a solution for the problem (1.1). To construct this solution we need the analogue of Theorem 1 for the following auxiliary problem of the form (1.1) in the finite cylinder $\Omega_{0,M}$, $M \in \mathbb{N}$,

$$\begin{cases} a(\partial_t^2 u + \Delta_x u) - \gamma\partial_t u - q(t,x)u = g(t), & u\big|_{\partial\omega} = 0, \\ u\big|_{t=0} = u_0, \quad u\big|_{t=M} = u_1. \end{cases} \quad (1.29)$$

Lemma 6. *The problem* (1.29) *has the unique solution u for all u_0, $u_1 \in V_0$. Moreover, the following estimate holds uniformly with respect to $M \in \mathbb{N}$:*

$$\|u,\Omega_T\|_{2,p}^p \le Ce^{p\lambda_0 T}\Big(\|u_0\|_{V_0}^p e^{-\alpha T} + \|u_1\|_{V_0}^p e^{-\alpha(M-T)-p\lambda_0 M} + \\ + \int_0^M e^{-\alpha|T-s|-p\lambda_0 s}\|g(s)\|_{0,p}^p\, ds\Big). \quad (1.30)$$

Proof. The proof of the a priori estimate (1.30) for the case of the finite cylinder $\Omega_{0,M}$ is completely analogous to the proof of the estimate (1.8) for the case of the semi-finite cylinder Ω_+ derived above (Lemmas 3 to 5). The existence of solutions for the problem (1.29) can be shown using the a priori estimate (1.30) and the Leray-Schauder principle (see, for instance, [4]). Lemma 6 is proved.

End of the proof of Theorem 1. Consider the sequence u_M, $M \in \mathbb{N}$, of solutions for the problems (1.29) in the cylinders $\Omega_{0,M}$ with boundary conditions $u_M\big|_{t=0} = u_0$ and $u_M\big|_{t=M} = 0$. Due to the uniformity of the estimate (1.30), the sequence u_M ($M \geq T$) is bounded in the space $H_{2,p}(\Omega_T)$ for every $T \geq 0$. Using the reflexivity of the space $H_{2,p}$ and Cantor's diagonal procedure we can extract a subsequence u_{M_k} from u_M that converges weakly to some function u in the space $H_{2,p}(\Omega_T)$ for every $T \geq 0$. It is not difficult to check (see [4]) that u is a solution for the problem (1.1). Theorem 1 is proved.

Below we also need lower estimates on the norms of solutions for the problem (1.1).

Theorem 2. *Let the function u belong to $W_{2,2}(\Omega_T)$ for every $T \geq 0$ and let u satisfy equation* (1.1) *with $g \equiv 0$ (the condition* (1.4) *is not assumed to be valid). Then the following estimate holds:*

$$\|u(t)\|_{1,2}^2 + \|\partial_t u(t)\|_{0,2}^2 \geq C(\|u(0)\|_{1,2}^2 + \|\partial_t u(0)\|_{0,2}^2)e^{-\beta t^2 + C_u t}, \tag{1.31}$$

where the constant $C_u \in \mathbb{R}$ depends on the function u and the constants β, $C > 0$ depend only on the coefficients of the equation (1.1).

The proof of this theorem under a bit stronger assumptions on the function u was given in [1]. For the reader's convenience, below we give the scheme of this proof adopted to our case.

Let us introduce the functions

$$p(t) = \partial_t u(t) + (-\Delta_x)^{1/2} u(t), \;\; r(t) = \partial_t u(t) - (-\Delta_x)^{1/2} u(t). \tag{1.32}$$

Then $u \in W_{2,2}(\Omega_T)$ for every $T > 0$ implies that for every $M > 0$

$$p,\, r \in H_{1,2}([0,M], L_2(\omega) \cap L_2([0,M], H^0_{1,2}(\omega)) \subset C([0,M], L_2(\omega)). \tag{1.33}$$

It can be easily derived from equation (1.1) that

$$\begin{cases} \partial_t p - (-\Delta_x)^{1/2} p = a^{-1}\gamma\, \frac{p+r}{2} + a^{-1} q (-\Delta_x)^{-1/2}\, \frac{p-r}{2}, \\ \partial_t r + (-\Delta_x)^{1/2} r = a^{-1}\gamma\, \frac{p+r}{2} + a^{-1} q (-\Delta_x)^{-1/2}\, \frac{p-r}{2} \end{cases}$$

and, introducing the notation $\xi_u(t) = \begin{pmatrix} p(t) \\ r(t) \end{pmatrix}$,

$$\partial_t \xi_u(t) - \mathcal{B}\xi_u(t) = \mathcal{T}(t)\xi_u(t), \;\; \mathcal{B} = \begin{pmatrix} -(-\Delta_x)^{1/2} & 0 \\ 0 & (-\Delta_x)^{1/2} \end{pmatrix},$$

where the operator $\mathcal{T}(t)$ satisfies the following estimate uniformly with respect to $t \in [0, \infty]$:

$$\|\mathcal{T}(t)\xi_u\|_{0,2} \leq C\|\xi_u\|_{0,2}. \tag{1.34}$$

Let $y(t) = \|\xi_u(t)\|_{0,2}^2$. Suppose also that $y(0) > 0$ (otherwise there is nothing to prove). Then it follows from (1.33) that $y(t) > 0$ when $t \in [0, M]$ for some $M > 0$. Let us define the function

$$l(t) = \ln y(t) - \int_0^t F(s)\, ds, \;\; F(t) = \frac{2(\mathcal{T}(t)\xi_u(t), \xi_u(t))}{y(t)}, \tag{1.35}$$

where $t \in [0, M]$.

Lemma 7. *Let $u(t)$ satisfy equation* (1.1) *and let $l(t)$ be defined by* (1.35). *Then $l \in AC([0,M])$, $l' \in AC([0,M])$, and the following estimate holds:*

$$l''(t) + 4C^2 \geq 0, \tag{1.36}$$

where the constant $C > 0$ is the same as in (1.34).

Proof. The estimate (1.36) is proved in [1]. It follows immediately from (1.33) that $l(t)$ is absolutely continuous. The absolute continuity of its derivative $l'(t)$ can be easily obtained using (1.33) and the equality

$$l'(t) = \frac{2(\mathcal{B}\xi_u(t), \xi_u(t))}{y(t)}$$

which can be verified directly. Lemma 7 is proved.

End of the proof of Theorem 2. Integrating the estimate (1.36) twice, we obtain

$$y(t) \geq y(0)e^{-2C^2t^2 + l'(0)t + \int_0^t F(s)\,ds}. \tag{1.37}$$

It follows from inequality (1.34) that $\int_0^t F(s)\,ds \geq -4Ct$. Thus

$$\|p(t)\|^2_{0,2} + \|r(t)\|^2_{0,2} \geq (\|p(0)\|^2_{0,2} + \|r(0)\|^2_{0,2})e^{-2C^2t^2 - 4Ct + l'(0)t}.$$

Theorem 2 is proved.

Corollary 1. *Let $u_1(t)$, $u_2(t)$ be two solutions of equation* (1.1). *If $u_1(T) = u_2(T)$ for some $T \geq 0$, then $u_1 \equiv u_2$.*

Indeed, according to Theorem 1, $u(t) = u_1(t) - u_2(t) = 0$ for every $t \geq T$, hence $\partial_t u(T) = 0$. Now estimate (1.31) implies that $u(0) = 0$, and, therefore, $u(t) = 0$ for every $t \in \mathbb{R}_+$.

§2 The nonlinear system. A priori estimates. The existence of solutions.

In this section we obtain the a priori estimate (0.9) for the solutions of the nonlinear problem (0.1) and afterwards we prove based on this estimate that the problem (0.1) has at least one solution for every $u_0 \in V_0$. We suppose that $a > 0$, γ are self-adjoint matrices ($a, \gamma \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^k)$), and that the nonlinear term $f \in C(\mathbb{R}^k, \mathbb{R}^k)$ satisfies the second condition from (0.3).

In contrast to the previous section we consider here only solutions of the problem (0.1) which are *bounded* with respect to $t \to \infty$, i.e., solutions for which the norm (0.5) is finite. It is also assumed that g satisfies (0.4) and $u_0 \in V_0$.

The main result of this section is stated in the following theorem.

Theorem 1. *Let the above assumptions be satisfied. Let u be a solution of the problem* (0.1). *Then the estimate* (0.9) *holds.*

To prove this theorem we need a number of the auxiliary results.

Lemma 1. *Let u be a solution of* (0.1)*. Then*

$$\|u(T)\|_{0,2}^2 \le C\left(\|u(0)\|_{0,2}^2 e^{-\alpha T} + 1 + |g|_b^2\right) \tag{2.1}$$

for some positive constant α.

The proof of this estimate is also decomposed into several steps. It the first step we obtain a rough estimate of the form (2.1) which will be improved afterwards.

Proposition 1. *Let u be the solution of the problem* (0.1)*. Then*

$$\|u(T)\|_{0,2}^2 \le C e^{mT}\left(1 + |g|_b^2 + \|u_0\|_{0,2}^2\right) \tag{2.2}$$

for some (*positive*) *constant m.*

Proof. Let us multiply the equation (0.1) by $ue^{-\varepsilon t}\chi_\tau(t)$ in $\mathbb{R}^k$, where $\chi_\tau(t)=\chi(t-\tau)$ ($\chi(z)$ is the Heaviside function), and $\varepsilon > 0$ is a sufficiently small positive number which will be fixed below, and integrate the obtained equality over Ω_+,

$$\left\langle a\partial_t^2 u, u\chi_\tau e^{-\varepsilon t}\right\rangle - \left\langle \gamma\partial_t u, u\chi_\tau e^{-\varepsilon t}\right\rangle - \left\langle a\nabla_x u, \nabla_x u\chi_\tau e^{-\varepsilon t}\right\rangle - \\ - \left\langle f(u), u\chi_\tau e^{-\varepsilon t}\right\rangle = \left\langle g, u\chi_\tau e^{-\varepsilon t}\right\rangle. \tag{2.3}$$

We estimate the first and the second term of (2.3) using the integrating by part (see (1.14), (1.15)),

$$- \left\langle a\partial_t^2 u, u\chi_\tau e^{-\varepsilon t}\right\rangle = \left\langle a\partial_t u, \partial_t u\chi_\tau e^{-\varepsilon t}\right\rangle - \varepsilon\left\langle a\partial_t u, u\chi_\tau e^{-\varepsilon t}\right\rangle + \\ + (a\partial_t u(\tau), u(\tau))e^{-\varepsilon\tau} \ge C\left\langle |\partial_t u|^2, \chi_\tau e^{-\varepsilon t}\right\rangle - \\ - C_1\varepsilon^2\left\langle |u|^2, \chi_\tau e^{-\varepsilon t}\right\rangle + y'(\tau)e^{-\varepsilon\tau}. \tag{2.4}$$

Here and in the following $y(\tau) = \frac{1}{2}(au(\tau), u(\tau))$.

$$- \left\langle \gamma\partial_t u, u\chi_\tau e^{-\varepsilon t}\right\rangle = -\frac{1}{2}\varepsilon\left\langle \gamma u, u\chi_\tau e^{-\varepsilon t}\right\rangle + \\ + \frac{1}{2}(\gamma u(\tau), u(\tau))e^{-\varepsilon\tau} \le C_2\varepsilon\left\langle |u|^2, \chi_\tau e^{-\varepsilon t}\right\rangle + C_3 y(\tau)e^{-\varepsilon\tau}. \tag{2.5}$$

The second condition of (0.3) implies that

$$- \left\langle f(u), u\chi_\tau e^{-\varepsilon t}\right\rangle \le \left\langle C, \chi_\tau e^{-\varepsilon t}\right\rangle \le C_1. \tag{2.6}$$

By Friedrichs' inequality,

$$\left\langle a\nabla_x u, \nabla_x u\chi_\tau e^{-\varepsilon t}\right\rangle \ge 2C_4\left\langle |u|^2, \chi_\tau e^{-\varepsilon t}\right\rangle.$$

Applying Hölder's inequality to the last term in (2.3), we obtain

$$|\left\langle g, u\chi_\tau e^{-\varepsilon t}\right\rangle| \le C_4\left\langle |u|^2, u\chi_\tau e^{-\varepsilon t}\right\rangle + C_5\left\langle |g|^2, \chi_\tau e^{-\varepsilon t}\right\rangle \le \\ \le C_4\left\langle |u|^2, u\chi_\tau e^{-\varepsilon t}\right\rangle + C_6|g|_b^2. \tag{2.7}$$

Inserting these estimates into (2.3) and choosing $\varepsilon > 0$ small enough so that $C_1\varepsilon^2 + C_2\varepsilon \le C_3$, we find after a short calculation

$$\left(y'(\tau) - C_3 y(\tau)\right)e^{-\varepsilon\tau} \le C(1 + |g|_b^2). \tag{2.8}$$

Gronwall's inequality applied to estimate (2.8) yields (2.1). Proposition 1 is proved.

Proposition 2. *Let u be a solution of* (0.1). *Then the following estimate holds for $T \geq 1$:*

$$\|u, \Omega_T\|^2_{1,2} \leq C \left(\|u, \Omega_0\|^2_{0,2} e^{-\alpha T} + 1 + |g|^2_b \right). \tag{2.9}$$

Proof. Let us introduce a cut-off function $\psi(t) \in C_0^\infty(\mathbb{R}_+)$ such that $\psi(t) \geq 0$, $\psi(t) = 0$ for $t \leq 0$ and $\psi(t) = 1$ for $t \geq 1$. Multiplying equation (0.1) by $u(t)\psi(t)e^{-\alpha|T-t|} = u\psi\phi_T$ in $\mathbb{R}^k$ and integrating over Ω_+, we obtain

$$\left\langle a\partial_t^2 u, u\psi\phi_T \right\rangle - \langle a\nabla u, \nabla(u\psi\phi_T)\rangle - \langle \gamma\partial_t u, u\psi\phi_T\rangle - \\ - \langle f(u), u\psi\phi_T\rangle = \langle g, u\psi\phi_T\rangle. \tag{2.10}$$

Let us estimate every term in (2.10) separately. Integrating by part the first term and using $\operatorname{supp}\psi' \subset [0,1]$,, we obtain analogously to (1.14)

$$- \left\langle a\partial_t^2 u, u\psi\phi_T \right\rangle = \langle a\partial_t u \cdot \partial_t u, \psi\phi_T\rangle + \frac{1}{2}\langle \partial_t[au\cdot u], \psi'\phi_T\rangle + \langle a\partial_t u, u\psi\phi_T'\rangle = \\ = \langle a\partial_t u \cdot \partial_t u, \psi\phi_T\rangle + \langle a\partial_t u, u\psi\phi_T'\rangle - \frac{1}{2}\langle au\cdot u, (\psi'\phi_T)'\rangle \geq \\ \geq C\left\langle |\partial_t u|^2, \psi\phi_T\right\rangle - C_1\alpha^2\left\langle |u|^2, \psi\phi_T\right\rangle - C_2\|u,\Omega_0\|^2_{0,2}e^{-\alpha T}. \tag{2.11}$$

Analogously, since a is self-adjoint we can infer that

$$\langle \gamma\partial_t u, u\psi\phi_T\rangle = -\frac{1}{2}\langle \gamma u\cdot u, \psi\phi_T'\rangle - -\frac{1}{2}\langle \gamma u\cdot u, \psi'\phi_T\rangle \geq \\ \geq -C(\alpha\left\langle |u|^2, \psi\phi_T\right\rangle + \|u,\Omega_0\|^2_{0,2}e^{-\alpha T}). \tag{2.12}$$

According to the second condition of (0.3),

$$\langle f(u), u\psi\phi_T\rangle = \langle f(u)\cdot u, \psi\phi_T\rangle \geq -\langle C, \psi\phi_T\rangle \geq -C_1. \tag{2.13}$$

Applying the Cauchy-Schwartz inequality to the right-hand side of (2.10), we derive

$$|\langle g, u\psi\phi_T\rangle| \leq \mu\left\langle |u|^2, \psi\phi_T\right\rangle + C_\mu\left\langle |g|^2, \psi\phi_T\right\rangle. \tag{2.14}$$

Inserting the estimates (2.11) to (2.14) into the equation (2.10), we obtain after a short calculation

$$\left\langle |\partial_t u|^2 + |\nabla u|^2, \psi\phi_T\right\rangle \leq C(\|u,\Omega_0\|^2_{0,2}e^{-\alpha T} + 1 + \left\langle |g|^2, \phi_T\right\rangle). \tag{2.15}$$

The estimate (2.15) implies (2.9) as in the proof of Lemma 1.3. Proposition 2 is proved.

Proof of Lemma 1. Inserting the estimate (2.2) into the estimate (2.9) and using the evident inequality

$$\|u(T)\|_{0,2} \leq C\|u, \Omega_T\|_{1,2},$$

we obtain estimate (2.1) for the case $T \geq 1$. For $T \leq 1$, the inequality (2.1) is an immediate consequence of estimate (2.2). Lemma 1 is proved.

Lemma 2. *Let u be a solution of the problem* (0.1)*. Then*

$$\|u(T)\|_{0,\infty} \le C(\|u_0\|_{V_0} e^{-\alpha T} + 1 + |g|_b). \tag{2.16}$$

Proof. Consider the function $w(t,x) = au(t,x)\cdot u(t,x)$. It is readily seen that this function satisfies the equation

$$\begin{cases} \partial_t^2 w(t) + \Delta_x w(t) = h_u(t), \\ w\big|_{t=0} = au_0 \cdot u_0, \end{cases} \tag{2.17}$$

where

$$h_u(t) = 2(a\partial_t u(t)\cdot\partial_t u(t) + a\nabla u(t)\cdot\nabla u(t) + \gamma\partial_t u(t).u(t) + \\ + f(u(t))\cdot u(t) + g(t)\cdot u(t)) \ge -C_1(1 + |g(t)|\cdot|u(t)| + |u(t)|^2) \equiv h(t). \tag{2.18}$$

Let us consider the auxiliary problem

$$\begin{cases} \partial_t^2 v(t) + \Delta_x v(t) = h(t), \\ v\big|_{t=0} = au_0 \cdot u_0. \end{cases} \tag{2.19}$$

According to Sobolev's embedding theorem $H_{2,p} \subset C$ for $p > (n+1)/2$. Hence, (0.5) implies that $h \in L_p(\Omega_T)$ for every $T \ge 0$ and $|h|_b < \infty$.

It follows from Theorem 1.1 with $\gamma = 0$, $q = 0$, $K = 0$, and $\lambda_0 = 0$ that

$$\|v, \Omega_T\|_{2,p}^p \le C(\|u_0\|_{V_0}^{2p} e^{\alpha T} + \int_{\mathbb{R}_+} e^{-\alpha|T-s|}\|h(s)\|_{0,p}^p\, ds). \tag{2.20}$$

Applying the comparison principle for bounded solutions of the problem (2.19) (see [4]) we obtain that

$$w(t,x) \le v(t,x) \text{ almost everywhere in } \Omega_+. \tag{2.21}$$

By our assumptions $p > \frac{n+1}{2}$, hence (2.20), (2.21), and the fact $w \ge 0$ imply that

$$\|u(T)\|_{0,\infty}^{2p} \le C\left(\|u_0\|_{V_0}^{2p} e^{-\alpha T} + \int_{\mathbb{R}_+} e^{-\alpha|T-s|}\|h(s)\|_{0,p}^p\, ds\right). \tag{2.22}$$

Applying Hölder's inequality to the function h, we obtain

$$\|h(s)\|_{0,p}^p \le C(1 + (|g(s)|^p|u(s)|^p, 1) + \|u(s)\|_{0,2p}^{2p}) \le \\ \le C_1(1 + \|g(s)\|_{0,p}^p\|u(s)\|_{0,\infty}^p + \|u(s)\|_{0,2p}^{2p}). \tag{2.23}$$

Let us estimate the last term on the right-hand side of (2.23) in the following way:

$$\|u(s)\|_{0,2p}^{2p} \le \|u(s)\|_{0,\infty}^{2p(1-\theta)}\|u(s)\|_{0,2}^{2p\theta} \le C_\mu\|u(s)\|_{0,2}^{2p} + \mu\|u(s)\|_{0,\infty}^{2p}, \tag{2.24}$$

where $\theta = 1/p \in (0,1)$. The estimate (2.24) holds for every $\mu > 0$.

Analogously,

$$\int_{\mathbb{R}_+} e^{-\alpha|T-s|}\|g(s)\|_{0,p}^p\|u(s)\|_{0,\infty}^p\, ds \le |g|_b^p \sup_{s\ge 0}\{e^{-\alpha|T-s|/4}\|u(s)\|_{0,\infty}^p\} \le \\ \le C_\mu|g|_b^{2p} + \mu\sup_{s\ge 0}\{e^{-\alpha|T-s|/2}\|u(s)\|_{0,\infty}^{2p}\}. \tag{2.25}$$

Inserting the estimates (2.23), (2.24), and (2.25) into the inequality (2.22) and using the L_2-norm estimate for $u(s)$ obtained in Lemma 1, we obtain after some simple calculation

$$\|u(T)\|_{0,\infty}^{2p} \le C_\mu(1 + |g|_b^{2p} + \|u_0\|_{V_0}^{2p} e^{-\alpha T}) + \mu\sup_{s\ge 0}\{e^{-\alpha|T-s|/2}\|u(s)\|_{0,\infty}^{2p}\}. \tag{2.26}$$

To complete the proof of Lemma 2 were need the following proposition.

Proposition 3. *Let a function* $z \in C_b(\mathbb{R}_+)$ *satisfy the inequality*

$$z(t) \le C_1 + C_0 e^{-\alpha t} + \mu \sup_{s\ge 0}\{e^{-\beta|t-s|}z(s)\} \tag{2.27}$$

for some $\alpha \ge \beta > 0$ *and* $\mu < 1/2$. *Then*

$$z(t) \le 2(C_0 e^{-\beta t} + C_1). \tag{2.28}$$

Proof. Multiplying the inequality (2.27) by $e^{-\beta|l-t|}$, $l \in \mathbb{R}_+$ and taking $\sup_{t\ge 0}$ of both sides of the obtained inequality we have

$$\sup_{t\ge 0} e^{-\beta|l-t|}z(t) \le C_0 \sup_{t\ge 0} e^{-\beta|l-t|}e^{-\alpha t} + C_1 \sup_{t\ge 0} e^{-\beta|l-t|} + \\ + \mu \sup_{t\ge 0}\sup_{s\ge 0}\{e^{-\beta(|t-s|+|t-l|)}z(s)\}. \tag{2.29}$$

A simple calculation reveals that $\sup_{t\ge 0} e^{-\beta|l-t|-\alpha t} = e^{-\beta l}$. Changing the order of supremums in the last term on the right-hand side of (2.29), we get

$$\sup_{t\ge 0}\sup_{s\ge 0}\{e^{-\beta(|t-s|+|t-l|)}z(s)\} = \sup_{s\ge 0}\{\sup_{t\ge 0} e^{-\beta(|t-s|+|t-l|)}\}z(s) = \\ = \sup_{s\ge 0} e^{-\beta|l-s|}z(s) = \sup_{t\ge 0}\{e^{-\beta|l-t|}z(t)\}.$$

Inserting these formulae into (2.29) and using $\mu \le \frac{1}{2}$, we conclude that

$$\sup_{t\ge 0} e^{-\beta|l-t|}z(t) \le 2(C_0 e^{-\beta l} + C_1).$$

This estimate together with (2.27) implies (2.28). Proposition 3 is proved.

End of the proof of Lemma 2. Apply Proposition 3 to the inequality (2.26) and $z(t) = \|u(t)\|_{0,\infty}^{2p}$. The estimate (2.16) is an immediate consequence of (2.28). Lemma 2 is proved.

Lemma 3. *Let* u *be a solution of the problem* (0.1). *Then*

$$\|f(u), \Omega_T\|_{0,\infty} \le Q(\|u_0\|_{V_0})e^{-\alpha T} + Q(|g|_b) \tag{2.30}$$

for some monotone function Q.

The proof of (2.30) which is based on the estimate (2.16) is given in [4].

End of the proof of Theorem 1. Let us rewrite the equation (0.1) as a linear equation,

$$a\partial_t^2 u - \gamma\partial_t u + a\Delta_x u = g(t) + f(u(t)) = h(t). \tag{2.31}$$

Equation (2.31) has the form (1.1) with $q(t,x) \equiv 0$. Moreover, the family (0.7) with $K = 0$ evidently satisfies (0.8) for $\lambda_0 = 0$. Applying the estimate (1.8) with $\lambda_0 = 0$ to the equation (2.31), we obtain the estimate

$$\|u, \Omega_T\|_{2,p}^p \le C\|u_0\|_{V_0}^p e^{-\alpha T} + \int_{\mathbb{R}_+} e^{-\alpha|T-s|}\|h(s)\|_{0,p}^p\, ds. \tag{2.32}$$

Inserting the estimate (2.30) into (2.32), after some simple calculation we obtain (0.9). Theorem 1 is proved.

To prove the existence of solutions of the problem (0.1) we need, as in Section 1, the following auxiliary problems of the form (0.1) in the finite cylinder $\Omega_{0,M}$, $M \in \mathbb{N}$,

$$\begin{cases} a(\partial_t^2 u + \Delta_x u) - \gamma\partial_t u - f(u) = g(t), \ u\big|_{\partial\omega} = 0, \\ u\big|_{t=0} = u_0, \ u\big|_{t=M} = u_1. \end{cases} \tag{2.33}$$

Theorem 2. *Let u_0, $u_1 \in V_0$ and let u be a solution of the problem* (2.33)*. Then the following estimate holds:*

$$\|u, \Omega_T\|_{2,p} \le Q(\|u_0\|_{V_0}^2 + \|u_1\|_{V_0}^2)(e^{-\alpha T} + e^{-\alpha(M-T)}) + Q(\|g\|_b), \tag{2.34}$$

where Q is a monotone function independent of $M \ge 2$.

Proof. The proof of (2.34) is analogous to that one of the estimate (0.9) given above (Lemmas 1 to 3). That's why we give below only the scheme of the proof of Theorem 2.

The analogue of the estimate (2.1) for the problem (2.33) is the following one:

Lemma 4. *Let u be the solution of the problem* (2.33)*. Then for $T \in [0, M]$*

$$\|u(T)\|_{0,2}^2 \le C\left((\|u(0)\|_{0,2}^2 + \|u(M)\|_{0,2}^2)(e^{-\alpha T} + e^{-\alpha(M-T)}) + 1 + |g|_b^2\right). \tag{2.35}$$

To prove this lemma we need the following propositions.

Proposition 4. *Let u be the solution of the problem* (2.33)*. Then the estimate*

$$\|u(T)\|_{0,2}^2 \le C(\|u(0)\|_{0,2}^2 + \|u(M)\|_{0,2}^2)\min\{e^{mT}, e^{m(M-T)}\} \tag{2.36}$$

holds uniformly with respect to $M \in \mathbb{N}$. Here m is a (*positive*) *number.*

Proof. Multiplying equation (2.33) by $ue^{-\varepsilon t}$ in $\mathbb{R}^k$ and integrating over $[\tau, s] \times \omega$, $[\tau, s] \subset [0, M]$ we obtain analogously to (2.8) the following estimate which holds uniformly with respect to $M \in \mathbb{N}$,

$$(y'(\tau) - C_1 y(\tau))\, e^{-\varepsilon\tau} - (y'(s) + C_2 y(s))\, e^{-\varepsilon s} \le C(1 + |g|_b^2). \tag{2.37}$$

Without loss of generality we may assume that $C_1 > 0$ and $C_2 > 0$.

Multiplying the inequality (2.37) by $e^{(\varepsilon + C_2)s}$ and integrating over $s \in [\tau, M]$ we find after some simple calculation

$$(y'(\tau) - C_1 y(\tau))\, e^{-\varepsilon\tau} \le$$
$$\le C(1 + |g|_b^2) + (C_2 + \varepsilon) y(M) e^{C_2 M}\left(e^{(C_2+\varepsilon)M} - e^{(C_2+\varepsilon)\tau}\right)^{-1}. \tag{2.38}$$

Without loss of generality we may also assume that $\tau \le M - 1$. Then

$$e^{C_2 M}\left(e^{(C_2+\varepsilon)M} - e^{(C_2+\varepsilon)\tau}\right)^{-1} \le \left(1 - e^{-C_2 - \varepsilon}\right)^{-1} \le C.$$

Hence (2.38) implies

$$(y'(\tau) - C_1 y(\tau))\, e^{-\varepsilon\tau} \le C(1 + |g|_b^2 + y(M)). \tag{2.39}$$

Using Hölder's inequality we derive from (2.39) that

$$y(t) \le C(1 + |g|_b^2 + |y(0)| + |y(M)|)e^{mt} \text{ for } t \le M - 1 \tag{2.40}$$

for some $m > 0$. Applying (2.40) to the problem (2.39) with t replaced by $M - t$ we obtain the estimate

$$y(t) \le C(1 + |g|_b^2 + |y(0)| + |y(M)|)e^{m(M-t)} \text{ for } t \ge 1. \tag{2.41}$$

The estimates (2.40) and (2.41) imply (2.36). Proposition 4 is proved.

Proposition 5. *Let u be the solution of the problem* (2.33). *Then the following estimate holds for $T \in [1, M-2]$:*

$$\|u, \Omega_T\|_{1,2}^2 \le C\left(\|u, \Omega_0\|_{0,2}^2 e^{-\alpha T} + \|u, \Omega_{M-1}\|_{0,2}^2 e^{-\alpha(M-T)} + 1 + |g|_b^2\right). \quad (2.42)$$

The proof of Proposition 5 is completely analogous to that one Proposition 2. Only instead of multiplying by $u(t)\psi(t)\phi_T(t)$ one should multiply equation (2.33) by $u(t)\psi(t)\psi(M-t)\phi_T(t)$.

The estimates (2.36) and (2.42) together imply the estimate (2.35) (see the proof of Lemma 1). Lemma 4 is proved.

Lemma 5. *Let u be the solution of the problem* (2.33). *Then*

$$\|u(T)\|_{0,\infty}^2 \le C(\|u_0\|_{V_0}^2 + \|u_1\|_{V_0}^2)(e^{-\alpha T} + e^{-\alpha(M-T)}) + 1 + |g|_b^2). \quad (2.43)$$

The proof of Lemma 5 is analogous to that one of Lemma 2. Only for estimating the C-norm of solutions of the auxiliary equation (2.19) one should use the estimate (1.30) instead of the estimate (1.8).

As proved in [4], the estimate (2.43) and continuity of f imply the estimate

$$\|f(u), \Omega_T\|_{0,\infty} \le Q(\|u_0\|_{V_0}^2 + \|u_1\|_{V_0}^2)(e^{-\alpha T} + e^{-\alpha(M-T)}) + Q(|g|_b) \quad (2.44)$$

for some monotone function Q.

Now rewriting equation (2.23) in the form (2.31) and applying the estimate (1.30) with $\lambda = 0$ we obtain the estimate (2.34). Theorem 2 is proved.

Theorem 3. *The problem* (0.1) *has at least one solution for every $u_0 \in V_0$.*

Proof. As in the linear case (see Section 1) we first prove the existence of solutions for the problem (2.33) in the finite cylinder $\Omega_{0,M}$. The existence of solutions for (2.33) can be obtained using the a priori estimate (2.34) and the Leray-Schauder principle (see for instance [4]). The existence of solutions for the problem (0.1) can then be proved extending the limit $M \to \infty$ as in the proof of Theorem 1.1. Theorem 3 is proved.

Remark 1. The estimate (0.9) was proved in [4] under more restrictive assumptions to the nonlinear term f ($f(u) \cdot u \ge -C_1 + C_2|u|^{2+\varepsilon}$, $C_2, \varepsilon > 0$), but without the requirement $\gamma = \gamma^*$.

§3 Differentiability of solutions of the nonlinear elliptic equation with respect to the initial data u_0

In this section we prove *unique* solvability and differentiability with respect to the initial value u_0 of solutions of the problem (0.1) and verify the equivalence of the problems (0.1) and (0.11).

Below it is again assumed that a, γ are positive self-adjoint matrices, the nonlinear function f satisfies the conditions (0.3), and the operator family (0.7) satisfies the condition (0.8) for some $\lambda_0 > 0$.

Theorem 1. *Let the above assumptions be satisfied. Then for every pair of solutions* u_1 *and* u_2 *of the problem* (0.1) *the following estimate holds:*

$$\|u_1 - u_2, \Omega_T\|_{2,p} \le C\|u_1(0) - u_2(0)\|_{V_0} e^{(\lambda_0-\alpha)T}. \tag{3.1}$$

Moreover, the constant C *in this estimate depends only on* $\|u_1(0)\|_{V_0}$ *and* $\|u_2(0)\|_{V_0}$, *and the constant* $\alpha > 0$ *only on the coefficients of the family* $L_K(\lambda)$ (*see* (0.7)).

Proof. Let $v = u_2 - u_1$. Then

$$\begin{cases} a(\partial_t^2 v + \Delta_x v) - \gamma\partial_t v - q_{u_1,u_2}(t,x)v = 0, \\ v\big|_{t=0} = u_1(0) - u_2(0), \end{cases} \tag{3.2}$$

where

$$q_{u_1,u_2}(t,x) = \int_0^1 f'(u_2(t,x) - \rho(u_2(t,x) - u_1(t,x)))\, d\rho. \tag{3.3}$$

It follows from the estimate (0.9) and from Sobolev's embedding theorem $H_{2,p} \subset C$ for $p > (n+1)/2$ that

$$\|u_i, \Omega_+\|_{0,\infty} \le C\|u_i(0)\|_{V_0}, \quad i = 1, 2$$

Hence, according to (3.3) and the estimate (0.3) the following estimates hold:

$$\begin{cases} 1. \quad \|q_{u_1,u_2}(t,x), \Omega_+\|_{0,\infty} \le C_1(\|u_1(0)\|_{V_0}, \|u_2(0)\|_{V_0}), \\ 2. \quad q_{u_1,u_2}(t,x) \ge -K\,\mathrm{Id}\,. \end{cases} \tag{3.4}$$

Notice that by the definition of solutions of the problem (0.1) the functions $u_1(t)$ and $u_2(t)$ are bounded with respect to $t \to \infty$ (i.e., $\|u_i\|_b < \infty$). Consequently the function $v(t)$ is also bounded with respect $t \to \infty$. Thus

$$\|v\|_{\lambda_0} = \sup_{T \ge 0} e^{-\lambda_0 T}\|v, \Omega_T\|_{2,p} < \infty \tag{3.5}$$

and all the conditions of Theorem 1.1 hold for the equation (3.2) and for its solution v. The estimate (1.8) now implies the estimate (3.1). Theorem 1 is proved.

Corollary 1. *It follows from* (3.1) *that for every* $t \ge 0$ *the solving operator* $S_t\colon V_0 \to V_0$ *mapping the initial data* $u_0 \in V_0$ *to the solution* $u(t)$ *of the problem* (0.1) *at time* t *is correctly defined. Moreover, this operator is* (*locally*) *Lipschitz continuous with respect to* u_0, *i.e.,*

$$\|S_t(u_{01}) - S_t(u_{02})\|_{V_0} \le C\|u_{01} - u_{02}\|_{V_0} e^{(\lambda_0-\alpha)t}, \quad u_{01},\, u_{02} \in V_0. \tag{3.6}$$

Corollary 2. *Let* $V_0' = H^0_{1-1/p,p}(\omega)$. *Then the estimate* (3.1) *also implies that*

$$\|\partial_t u_1(0) - \partial_t u_2(0)\|_{V_0'} \le C\|u_1(0) - u_2(0)\|_{V_0}.$$

Consequently the locally Lipschitz operator $\Phi\colon V_0 \to V_0'$ *mapping the initial data* $u_0 \in V_0$ *to the* t*-derivative of the solution of the problem* (0.1) *at time* $t = 0$ *is correctly defined,*

$$\partial_t u\big|_{t=0} = \Phi\left(u\big|_{t=0}\right). \tag{3.7}$$

Considering the later time $t = \tau > 0$ *instead of* $t = 0$, *we obtain, analogously to* (3.7), *that*

$$\begin{cases} \partial_t u\big|_{t=\tau} = \Phi_\tau\left(u\big|_{t=\tau}\right), \\ u\big|_{t=0} = u_0 \end{cases} \tag{3.8}$$

which holds for every solution u *of the problem* (0.1) *and for every* $\tau \ge 0$ (*here we denote by* $\Phi_\tau\colon V_0 \to V_0'$ *the operator* Φ *that corresponds to the problem* (0.1) *with* $g(t)$ *replaced by* $g(t+\tau)$).

Corollary 3. *The operator $S_t\colon V_0 \to V_0$ defined in* Corollary 1 *is injective for every $t \geq 0$, i.e., $S_t\xi_1 = S_t\xi_2$ for some $t \geq 0$ implies $\xi_1 = \xi_2$.*

Indeed, this assertion follows from Corollary 1.1 applied to equation (3.2).

Theorem 2. *Let the assumptions of previous theorem be satisfied. Then the solving operator $S_t\colon V_0 \to V_0$ is Fréchet differentiable in V_0 for every fixed $t \geq 0$ and its derivative $w(t) = \mathcal{D}_{u_0}S_t(u_0)\xi$, $\xi \in V_0$, is given as the unique (according to* Theorem 1.1) *solution of the following problem:*

$$\begin{cases} a(\partial_t^2 w + \Delta_x w) - \gamma\partial_t w - f'(S_t(u_0))w = 0, \\ w\big|_{t=0} = \xi, \;\; \|w\|_{\lambda_0} < \infty. \end{cases} \tag{3.9}$$

The operator $\Phi\colon V_0 \to V_0'$ is Fréchet differentiable and its derivative is given by

$$\mathcal{D}_{u_0}\Phi(u_0)\xi = \partial_t w(0),$$

where $w(t)$ is a solution of (3.9) *with $w(0) = \xi$. Moreover, $S_t \in C(\mathbb{R}_+ \times V_0, V_0)$, $\Phi \in C^1(V_0, V_0')$, and the derivative $D_{u_0}S_t(u_0)$ is uniformly continuous with respect to $u_0 \in B$ for every bounded subset $B \subset V_0$, $t \in [T, T+1]$.*

Proof. Let $u_{01}, u_{02} \in B$, $\|B\|_{V_0} \leq M$. Denote by $u_1(t)$ and $u_2(t)$ the solutions of the problem (0.1) with initial data u_{01} and u_{02}, respectively. Let also $v(t) = u_2(t) - u_1(t)$ and $w(t)$ be a solution of the problem (3.9) with initial data $\xi = u_{02} - u_{01}$. Then the function $\theta(t) = v(t) - w(t)$ is a solution of the following problem:

$$\begin{cases} a(\partial_t^2\theta + \Delta_x\theta) - \gamma\partial_t\theta - f'(u_1)\theta = (q_{u_1,u_2} - f'(u_1))v \equiv L(t)v(t), \\ \theta\big|_{t=0} = 0, \;\; \|\theta\|_{\lambda_0} < \infty. \end{cases} \tag{3.10}$$

Recall that the function q_{u_1,u_2} on the right-hand side of the equation (3.10) is defined by (3.3).

Equation (3.10) satisfies all conditions of Theorem 1.1, hence according to the estimate (1.8),

$$\|\theta, \Omega_T\|_{2,p}^p \leq Ce^{p\lambda_0 T}\int_{\mathbb{R}_+} e^{-\alpha|T-s|-p\lambda_0 s}\|L(s)v(s)\|_{0,p}^p\, ds. \tag{3.11}$$

Estimating $\|v(s)\|_{0,p}$ using the estimate (3.6) we obtain

$$\|\theta, \Omega_T\|_{2,p}^p \leq Ce^{p\lambda_0 T}\|v(0)\|_{V_0}^p \int_{\mathbb{R}_+} e^{-\alpha|T-s|-p\alpha s}\|L(s)\|_{0,\infty}^p\, ds. \tag{3.12}$$

Thus it is necessary to prove that

$$Z(\|v(0)\|) \equiv \int_{\mathbb{R}_+} e^{-\alpha|T-s|-p\alpha s}\|L(s)\|_{0,\infty}^p\, ds \tag{3.13}$$

tends to zero as $\|v(0)\|_{V_0} \to 0$.

According to Theorem 2.1 and Sobolev's embedding theorem ($H_{2,p} \subset C$), $\|u_i\|_{0,\infty} \leq M_1 = M_1(\|B\|_{V_0})$. Hence it follows from the continuity of f' and from the definition of L that $\|L, \Omega_+\|_{0,\infty} \leq M_2 = M_2(\|B\|_{V_0})$.

Let us fix an arbitrary $\varepsilon > 0$ and choose $R(\varepsilon) > 0$ so that

$$\int_{R(\varepsilon)}^{\infty} e^{-p\alpha s} M_2^p \, ds < \frac{\varepsilon}{2}.$$

Applying the last inequality to (3.13) we obtain after standard estimates

$$Z(\|v(0)\|) \le R(\varepsilon)\|L, \Omega_{0,R(\varepsilon)}\|_{0,\infty}^p + \frac{\varepsilon}{2}. \tag{3.14}$$

f' is continuous, hence there exists $\delta_1 = \delta_1(\varepsilon)$ such that

$$|f'(\xi + \xi_1) - f'(\xi)| < \left(\frac{\varepsilon}{2R(\varepsilon)}\right)^{1/p}$$

for $|\xi| < 2M_1$, $|\xi_1| < \delta_1$. Thus

$$|L(t,x)|^p \le \int_0^1 |f'(u_1(t,x) + \rho v(t,x)) - f'(u_1(t,x))|^p d\rho \le \frac{\varepsilon}{2R(\varepsilon)}, \ \ (t,x) \in \Omega_{0,R}(\varepsilon),$$

if $\|v, \Omega_{0,R(\varepsilon)}\|_{0,\infty} < \delta_1$. Consequently, $Z(\|v(0)\|_{V_0}) < \varepsilon$ if $\|v, \Omega_{0,R(\varepsilon)}\| < \delta_1$. But it follows from the estimate (3.6) and from the embedding theorem that

$$\|v, \Omega_{0,R(\varepsilon)}\|_{0,\infty} \le Ce^{\lambda_0 R(\varepsilon)}\|v(0)\|_{V_0}.$$

Hence $Z < \varepsilon$ if $\|v(0)\|_{V_0} < \delta = \delta_1 C^{-1} e^{-\lambda_0 R(\varepsilon)}$. Thus the following formula is proved:

$$\|\theta, \Omega_T\|_{2,p} = \exp(\lambda_0 T)\overline{\overline{o}}(\|v(0)\|_{V_0}) \text{ as } \|v(0)\|_{V_0} \to 0. \tag{3.15}$$

The estimate (3.15) together with the inequality $\|\theta(t)\|_{V_0} \le C\|\theta, \Omega_t\|_{2,p}$ shows the differentiability of the operator S_t.

The differentiability of Φ is an immediate consequence of (3.15) and the estimate $\|\partial_t\theta(0)\|_{V_0'} \le C\|\theta, \Omega_0\|_{2,p}$. Notice that the right-hand side of (3.15) tends to zero as $v \to 0$ *uniformly* with respect to $u_{01}, u_{02} \in B$ and $t \in [T, T+1]$. Hence

$$\begin{aligned} &\|(D_{u_0}S_t(u_{01}) - D_{u_0}S_t(u_{02}))v(0)\|_{V_0} \le \|u_2(t) - u_1(t) - D_{u_0}S_t(u_{02})v(0)\|_{V_0} + \\ &+ \|u_1(t) - u_2(t) + D_{u_0}S_t(u_{01})v(0)\|_{V_0} \le C\|u_2 - u_1 - D_{u_0}S_t(u_{02})v(0), \Omega_T\|_{2,p} + \\ &+ C\|u_1 - u_2 + D_{u_0}S_t(u_{02})v(0), \Omega_T\|_{2,p} \le \overline{\overline{o}}(\|v(0)\|_{V_0}). \end{aligned} \tag{3.16}$$

The estimate (3.16) implies that $D_{u_0}S_t(u_0)$ is uniformly continuous with respect to $u_0 \in B$. To complete the proof of Theorem 2 it remains to prove that S_t is continuous with respect to $(t, u_0) \subset \mathbb{R}_+ \times V_0$. Due to the estimate (3.16) it is sufficient to prove only the continuity S_t with respect to t for *fixed* $u_0 \in V_0$. By definition $u(t) = S_t(u_0) \in W_{2,p}(\Omega_T)$ for every $T \ge 0$, hence according to Sobolev's embedding theorem $u \in C(\mathbb{R}_+, V_0)$. Theorem 2 is proved.

Corollary 4. *The Fréchet derivative $D_{u_0}S_t(u_0)$ satisfies the condition* (0.10).

Indeed, applying the result of Theorem 1.1 to equation (3.9) we obtain the estimate (0.10).

Remark 1. Notice that in fact we have proved in Theorem 2 that

$$\|\theta\|_{\lambda_0} = \overline{\overline{o}}(\|v(0)\|_{V_0}). \tag{3.17}$$

Thus the mapping $u_0 \mapsto S_t(u_0)$ is differentiable as a mapping from V_0 to the space $W^{\lambda_0}_{2,p}$ of distributions which have finite norm (1.2).

Now let us study the operator Φ_τ defined in Corollary 2. In order to do this we introduce the scale of Banach spaces $X_0 = V_0'$, $X_1 = V_0$, and $X_\beta = (-\Delta_x)^{-\beta/2}X_0$ for $0 \le \beta \le 1$. It is known (see for instance [10]), that

$$X_\beta = W_{1-1/p+\beta,p}(\omega) \cap \{u_0\big|_{\partial\omega} = 0\} \text{ for } \beta \ne 1/p. \tag{3.18}$$

Theorem 3. *Let the above assumptions be satisfied. Then the nonlinear operator defined in* Corollary 2 *can be represented in the following way:*

$$\Phi_\tau(v) = -(-\Delta_x)^{1/2}v + F(\tau, v) + G(\tau), \ \ v \in V_0, \ \ \tau \in \mathbb{R}_+, \tag{3.19}$$

where $G \in C_b(\mathbb{R}_+, X_1)$, and the nonlinear term F fulfils $F(\tau,\cdot) \in C(X_1, X_1) \cap C^1(X_1, X_\beta)$ for some $\beta > 0$ and every fixed $\tau \ge 0$. Moreover,

$$F \in C_b(\mathbb{R}_+, C(X_1, X_1)), \ \ \|F'_u(\tau,\xi)\|_{L(X_1,X_\beta)} \le Q(\|\xi\|_{X_1}) \tag{3.20}$$

uniformly with respect to $\tau \in \mathbb{R}_+$. Here $Q\colon \mathbb{R}_+ \to \mathbb{R}_+$ is some monotone function.

Proof. Without loss of generality we may assume that $f(0) = 0$. Let us represent the solution $u(t)$ of the problem (0.1) (with $g(t)$ replaced by $g(t+\tau)$) as a sum of two functions $u(t) = v(t) + \theta(t)$, where the function v is a solution of the problem

$$\begin{cases} a(\partial_t^2 v + \Delta_x v) - \gamma\partial_t v = g(t+\tau), \\ v|_{t=0} = u_0, \ \ \|v\|_b < \infty \end{cases} \tag{3.21}$$

and the function $\theta = u - v$ is the solution of

$$\begin{cases} a(\partial_t^2 \theta + \Delta_x \theta) - \gamma\partial_t \theta = f(S_t(u_0)), \\ \theta|_{t=0} = 0, \ \ \|\theta\|_b < \infty. \end{cases} \tag{3.22}$$

Define the operators $F_1(\tau, u_0)$ and $F_2(\tau, u_0)$ by the following formulae:

$$F_1(\tau, u_0) = \partial_t\theta\big|_{t=0}, \ \ F_2(\tau, u_0) = \partial_t v\big|_{t=0}. \tag{3.23}$$

Then $\Phi_\tau = F_1 + F_2$.

Let us first study the operator F_1. Since $f \in C^1$, $f(0) = 0$, (0.9) implies that $\|f(u), \Omega_{0,2}\|_{1,p} \le Q_1(\|u_0\|_{V_0})$ for some function Q_1 and $f(u)\big|_{\partial\omega} = 0$. Since $\theta(0) = 0$, the $(W_{3,p}, W^0_{1,p})$-regularity theorem for solutions of the Laplace equation implies analogously to the proof of Lemma 1.5 that $\theta \in W_{3,p}(\Omega_0)$ and

$$\|\theta, \Omega_0\|_{3,p} \le C\|f(u), \Omega_{0,2}\|_{1,p} \le CQ_1(\|u_0\|_{V_0}). \tag{3.24}$$

Thus $\partial_t\theta \in W_{2,p}(\Omega_0)$. Consequently, $F_1(\tau, u_0) = \partial_t\theta(0) \in X_1$. The continuity of F_1 can be proved analogously to the proof of Theorem 2. Let us check its (X_1, X_β)-differentiability.

Using Remark 1, the embedding $W_{2,p}$ into C, and the differentiability of f, it is not difficult to obtain that the mapping $u_0 \mapsto f(S_t(u_0))$ is differentiable as a mapping from X_1 to the space $L_q^{\lambda_0}(\Omega_+)$ of functions from $L_q^{\mathrm{loc}}(\Omega_+)$ which have finite norm (1.3) (with the exponent p replaced by q) for every $1 \le q \le \infty$. Now applying Theorem 1.1 to equation (3.22), we obtain that the operator Θ: $u_0 \mapsto \theta(t)$ is differentiable as a mapping from X_1 to $W_{2,q}^{\lambda_0}(\Omega_+)$. Fix some $q > p$. Then according to the embedding theorem the mapping $u_0 \mapsto \partial_t\theta(0)$ is differentiable as a mapping from X_1 to $W^0_{1-1/q,q}(\omega) \subset X_{1/p-1/q}$. Thus the operator $F_1(\tau, \cdot)$ is (X_1, X_β)-differentiable for $\beta = 1/p - 1/q > 0$.

The continuity of the derivative $D_u F_1$ can be proved as in Theorem 2.

To prove the estimate (3.20), we consider the linearized equation which corresponds to the problem (3.22) ($Q(t) = D_{u_0}F_1(\tau, u_0)\xi$)

$$\begin{cases} a(\partial_t^2 Q(t) + \Delta_x Q(t)) - \gamma\partial_t Q(t) = f'(u(t))w(t), \\ Q|_{t=0} = 0, \ \ \|Q\|_{\lambda_0} < \infty. \end{cases} \tag{3.25}$$

Here $w(t)$ is the solution of the problem (3.9) and $u(t)$ is the solution of (0.1), with $g(t)$ replaced by $g(t+\tau)$. Applying Theorem 1.1 to the equations (3.9), (3.25) and using the inequality (0.9), we obtain the estimate (3.20) for F_1.

Let us now study the operator $F_2(\tau, u_0)$. According to Theorem 1.1, there exists a unique solution v_g of the problem (3.21) with zero initial data. Let us define the function $G(\tau) = \partial_t v_g(0) \in V_0'$. Then it is not difficult to check that $G \in C_b(\mathbb{R}_+, V_0')$. Define now the function $V_{u_0}(t)$ as the unique solution of the problem

$$\begin{cases} \partial_t^2 V_{u_0} + \Delta_x V_{u_0} = 0, \\ V_{u_0}\big|_{t=0} = u_0, \ \ \|V_{u_0}\|_b < \infty. \end{cases} \tag{3.26}$$

It is easy to prove using the standard methods of the theory of analytic semigroups (see [8]) that $V_{u_0}(t) = e^{-t(-\Delta_x)^{1/2}}u_0$. Consequently, $\partial_t V_{u_0}(0) = -(-\Delta_x)^{1/2}u_0$.

Finally we consider the function $w(t) = v(t) - v_g(t) - V_{u_0}(t)$ which satisfies the equation

$$\begin{cases} a(\partial_t^2 w + \Delta_x w) - \gamma\partial_t w = \gamma\partial_t V_{u_0}(t), \\ w\big|_{t=0} = 0, \ \ \|w\|_b < \infty. \end{cases} \tag{3.27}$$

Since the right-hand side $\gamma\partial_t V_{u_0}(t)$ of equation (3.27) belongs to $W_{1,p}^{\mathrm{loc}}(\Omega_+)$, we can prove as for the operator F_1 that F_3: $u_0 \mapsto \partial_t w(0)$ is bounded linear operator from X_1 to X_1. Thus

$$F_2(\tau, u_0) = -(-\Delta_x)^{1/2}u_0 + F_3 u_0 + G(\tau).$$

Denoting now $F = F_2 + F_3$, we obtain the decomposition (3.19). Theorem 3 is proved.

We consider now the formula (3.8) as an evolutionary equation in the space V_0.

Corollary 5. *The problems* (0.1) *and* (0.11) *are equivalent.*

In fact, by the definition of the operator Φ_τ, every solution of the problem (0.1) is also a solution of the problem (3.8) or, which amounts to be the same, a solution of the problem (0.11). Consequently, we obtain the existence of solutions of the problem (0.11) from the solvability of the problem (0.1). Then to complete the proof of Corollary 5 it remains to prove that the solution of the problem (0.11) is unique.

Let $u_1(t)$ and $u_2(t)$ be two solutions of (0.11) such that $u_1(0) = u_2(0)$. Then the function $v(t) = u_1(t) - u_2(t)$ satisfies the following equation (see [8]):

$$v(t) = \int_0^t e^{(s-t)(-\Delta_x)^{1/2}} (F(s, u_1(s)) - F(s, u_2(s))\, ds. \tag{3.28}$$

Theorem 3 implies that $\|F(s, u_1(s)) - F(s, u_2(s)\|_{X_\beta} \le C\|v(s)\|_{X_1}$. It follows from the sectorial property of the operator $(-\Delta_x)^{1/2}$ (see [7], [11]) that

$$\|e^{(s-t)(-\Delta_x)^{1/2}}\|_{L(X_\beta, X_1)} \le C(t-s)^{\beta-1}. \tag{3.29}$$

Estimating the norm of the right-hand side of (3.28) using the inequality (3.29), we obtain that

$$\|v(t)\|_{X_1} \le C \int_0^t (t-s)^{\beta-1} \|v(s)\|_{X_1}\, ds. \tag{3.30}$$

Recall that $\beta - 1 > -1$. Hence we can apply Gronwall's inequality to the estimate (3.30) and obtain $\|v(t)\|_{X_1} \equiv 0$.

Corollary 6. *The problem* (3.9) *is equivalent to the following evolutionary problem in the space* V_0*:*

$$\begin{cases} \partial_t w + (-\Delta_x)^{1/2} w = F'_u(t, u(t))w, \\ w\big|_{t=0} = \xi, \quad w \in C(\mathbb{R}_+, V_0) \cap C^1(\mathbb{R}_+, V'_0). \end{cases} \tag{3.31}$$

Here the operator F *defined by* (3.19) *and* $u(t)$ *is the solution of the problem* (0.1).

The proof of this corollary is completely analogous to the proof of previous one.

§4 The nonlinear elliptic equation near an hyperbolic equilibrium

In this section we study the autonomous equation (0.1), i.e., equation (0.1) with right-hand side g independent of t,

$$g(t, x) \equiv g(x) \in L_p(\omega). \tag{4.1}$$

It is easy to verify that in this case the operators S_t: $V_0 \to V_0$, $t \ge 0$, generate a semigroup in V_0,

$$S_{t_1+t_2} = S_{t_1} S_{t_2}, \ \ t_1,\, t_2 \ge 0. \tag{4.2}$$

Let z_0 be an equilibrium of this semigroup, i.e., $S_t z_0 = z_0$ for all $t \ge 0$. Equivalently,

$$\begin{cases} a\Delta_x z_0 - f(z_0) = g, \\ z_0\big|_{\partial\omega} = 0. \end{cases} \tag{4.3}$$

Remark 1. Analogously to the proof of Theorems 2.1 and 2.2, but essentially more simple, one can prove that the problem (4.3) has at least one solution and every of its solutions $z_0 \in V_0$ belongs to the space $W_{2,p}(\omega)$.

According to Theorem 3.2, $S_t \in C^1(V_0, V_0)$ for every $t \ge 0$, hence the following definition is correct.

Definition 1. *The equilibrium* $z_0 \in V_0$ *of the semigroup* S_t *is called hyperbolic if the spectrum of the Fréchet derivative* $D_{u_0}S_t(z_0)$ *at* $t = 1$ *does not intersect the unit circle:*

$$\sigma\left(D_{u_0}S_1(z_0)\right) \cap \{\lambda \in \mathbb{C} \colon |\lambda| = 1\} = \varnothing. \tag{4.4}$$

The dimension of the kernel subspace corresponding to the part of the spectrum of $D_{u_0}S_t(z_0)$ *which lies outside of the unit ball in* $\mathbb{C}$ *is called the instability index of the hyperbolic equilibrium* z_0*:*

$$\mathrm{ind}_{z_0} = \#\{\lambda \in \sigma\left(D_{u_0}S_1(z_0)\right) : |\lambda| > 1\}. \tag{4.5}$$

Theorem 1. *Let* S_t *be the semigroup corresponding to the autonomous equation* (0.1) *and* z_0 *its equilibrium. Then*

1. The operator $D_{u_0}S_1(z_0)\colon V_0 \to V_0$ *is compact and consequently its spectrum is discrete and consists of normal eigenvalues of finite multiplicity.*

2. The equilibrium z_0 *is hyperbolic if and only if the operator family*

$$L_{z_0}(\lambda) = a\lambda^2 - \gamma\lambda + a\Delta_x - f'(z_0)\colon W_{2,p}(\omega) \cap \overset{0}{W}_{1,p}(\omega) \to L_p(\omega) \tag{4.6}$$

has no eigenvalues on the imaginary axis:

$$\sigma(L_{z_0}(\cdot)) \cap \{\mathrm{Re}\,\lambda = 0\} = \varnothing. \tag{4.7}$$

3. The instability index of the hyperbolic equilibrium z_0 *is finite and can be calculated by the following formula:*

$$\mathrm{ind}_{z_0} = \#\{\lambda \in \sigma(L_{z_0}(\cdot))\colon 0 < \mathrm{Re}\,\lambda < \lambda_0\}, \tag{4.8}$$

where the exponent λ_0 *is defined by condition* (0.8)*.*

Remark 2. As known (see for instance [10]), the spectrum of the family (4.6) is independent of $1 < p < \infty$ and consist of normal eigenvalues of finite multiplicity. Thus we may calculate the instability index ind_{z_0} under the assumption that the family (4.6) acts in a Hilbert space ($p = 2$).

Proof. According to Theorem 3.3 and Corollary 3.6, the equation (3.9), which defines the operator $D_{u_0}S_1(z_0)$, can be rewritten in the following form:

$$\begin{cases} \partial_t w = -(-\Delta_x)^{1/2} w + F'(z_0)w, \\ w\big|_{t=0} = \xi, \quad w \in C([0,1], V_0) \cap C^1([0,1], V_0'), \\ D_{u_0}S_1(z_0)\xi = w(1). \end{cases} \tag{4.9}$$

Theorem 3.3 implies that $F'(z_0) = D_{u_0}F(z_0)\colon X_1 \to X_\beta$, consequently the operator $F'(z_0)$ is compact as an operator from X_1 to X_0. Thus due to the theorem about compact perturbations of sectorial operators (see [8]), the operator $A_{z_0} = -(-\Delta_x)^{1/2} + F'(z_0)\colon X_1 \to X_0$ is also sectorial. According to the spectral mapping theorem for such operators (see [16]),

$$D_{u_0}S_1(z_0) = e^{A_{z_0}}, \quad \sigma\left(D_{u_0}S_1(z_0)\right) \setminus \{0\} = e^{\sigma(A_{z_0})}. \tag{4.10}$$

Since the operators $(-\Delta_x)^{-1/2}, (-\Delta_x)^{1/2}F'(z_0)\colon X_1 \to X_1$ are compact, it is not difficult to prove (see [11], [16]) that the operator $e^{tA_{z_0}}$ is compact for every $t > 0$. The first part of Theorem 1 is proved.

Let us now prove that the equality

$$\sigma\big(-(-\Delta_x)^{1/2} + F'(z_0)\big) = \sigma\left(L_{z_0}(\cdot)\right) \cap \{\operatorname{Re}\lambda < \lambda_0\} \tag{4.11}$$

holds and that the corresponding eigenvectors and their kernel subspaces coincide. Indeed, let $\lambda \in \sigma\left(-(-\Delta_x)^{1/2} + F'(z_0)\right)$ and z be the corresponding eigenvector. Then the function $w(t) = e^{\lambda t}z$ is a solution of (4.9), consequently, according to Corollary 3.6, it is the solution of the problem (3.9) with u_0 replaced by z_0. Inserting $w(t)$ into the equation (3.9) we obtain that $e^{\lambda t}L_{z_0}(\lambda)z = 0$. Thus z is an eigenvector of the family $L_{z_0}(\lambda)$. It now follows from the definition of solutions for the problem (3.9) that $\operatorname{Re}\lambda < \lambda_0$. The inverse inclusion and the assertion about the coincidence of the adjoint vectors can be proved completely analogously. Thus the equality (4.11) is proved. The second part of Theorem 1 is an immediate consequence of (4.10) and (4.11).

Let us prove the third part of the theorem. Let E_+ and E_- be the kernel subspaces of A_{z_0} corresponding to the spectral sets $\sigma_+ = \sigma(A_{z_0}) \cap \{\operatorname{Re}\lambda > 0\}$ and $\sigma_- = \sigma(A_{z_0}) \cap \{\operatorname{Re}\lambda < 0\}$. Then, as known (see [11]), $V_0 = E_+ + E_-$ and

$$\|e^{A_{z_0}t}z\| \le Ce^{-\varepsilon t}\|z\| \text{ for all } z \in E_-, \quad \|e^{A_{z_0}t}z\| \ge Ce^{\varepsilon t}\|z\| \text{ for all } z \in E_+ \tag{4.12}$$

for some $\varepsilon > 0$. Thus $\operatorname{ind}_{z_0} = \dim E_+$. Using the equality (4.11) again we obtain (4.8). Theorem 1 is proved.

In the case that the nonlinear term has a potential it is possible to obtain simpler formulae for the calculation of ind_{z_0}.

Theorem 2. *Assume that the nonlinear function f satisfies condition* (0.13)*. Then the instability index of the hyperbolic equilibrium z_0 equals the number of positive eigenvalues of the operator $a\Delta_x - f'(z_0)$ (taking into account their multiplicities):*

$$\operatorname{ind}_{z_0} = \#\{\lambda \in \sigma(a\Delta_x - f'(z_0))\colon \lambda > 0\}. \tag{4.13}$$

To prove this theorem we need the following lemma:

Lemma 1. *Let the assumptions of the previous theorem and assumption* (4.13) *be satisfied. Then the spectrum of the family* (4.6) *is real.*

Proof. According to Remark 2 it is sufficient to consider the family (4.6) in a Hilbert space ($p = 2$). Let $\lambda = \omega + i\alpha \in \sigma(L_{z_0})$. Since the spectrum L_{z_0} is discrete, there exists an $x_0 \in W^0_{1,2}(\omega)$ such that

$$L_{z_0}(\lambda)x_0 = 0. \tag{4.14}$$

Consequently, self-adjointness of $f'(z_0)$ and condition (0.8) imply that

$$\begin{aligned} 0 < -(L_{z_0}(\lambda_0)x_0, x_0) &= ([L_{z_0}(\lambda) - L_{z_0}(\lambda_0)]x_0, x_0) \\ &= (\lambda^2 - \lambda_0^2)(ax_0, x_0) - (\lambda - \lambda_0)(\gamma x_0, x_0) \\ &= (\omega^2 - \alpha^2 - \lambda_0^2)(ax_0, x_0) - (\omega - \lambda_0)(\gamma x_0, x_0) + \\ &\quad + i\{2\alpha\omega(ax_0, x_0) - \alpha(\gamma x_0, x_0)\} \end{aligned}$$

and

$$\begin{cases} \alpha[(\gamma x_0, x_0) - 2\omega(a x_0, x_0)] = 0, \\ (\omega^2 - \alpha^2 - \lambda_0^2)(a x_0, x_0) - (\omega - \lambda_0)(\gamma x_0, x_0) > 0. \end{cases} \tag{4.15}$$

Thus if $\alpha \neq 0$, then $(\gamma x_0, x_0) = 2\omega(a x_0, x_0)$. By inserting this formula into inequality (4.15), we obtain that

$$(\omega^2 + \alpha^2 + \lambda_0^2 - 2\omega\lambda_0)(a x_0, x_0) < 0$$

which contradicts the positiveness of the matrix a. Lemma 1 is proved.

Consider now the following two-parameter operator family

$$L^s(\lambda) = sL_{z_0}(\lambda) + (1-s)(\lambda^2 a_- \operatorname{Id} - \lambda\gamma_+ \operatorname{Id} + a\Delta_x - f'(z_0)), \tag{4.16}$$

where $s \in [0,1]$ and $\gamma_+ > 0$ is a positive number large enough to satisfy the condition $\gamma_+ \operatorname{Id} - \gamma > 0$.

Definition 3. *The number of eigenvalues in the interval* $(0, \lambda_0)$ (*taking into account multiplicities*) *is called the instability index of the family* (4.16) (*for a fixed* $s \in [0,1]$)*:*

$$\operatorname{ind} L^s = \#\{\sigma(L^s) \cap (0, \lambda_0)\}.$$

It is not difficult to check using condition (0.8) and the definition of $L^s(\lambda)$ that

$$-(L^s(\lambda_0)x, x) > 0, \ \text{ for all } x \in W^0_{1,2}(\omega),\ x \neq 0,\ s \in [0,1]. \tag{4.17}$$

Thus $\lambda_0 \notin \sigma(L^s)$ for $s \in [0,1]$. It is also obvious that $0 \notin \sigma(L^s)$ for $s \in [0,1]$, and it follows from Lemma 1 that that the spectrum of L^s is real for every fixed $s \in [0,1]$. Consequently, it follows from the theorem on the stability of kernel multiplicities (see [5]) that $\operatorname{ind} L^s$ is independent of $s \in [0,1]$. Thus

$$\operatorname{ind}_{z_0} = \operatorname{ind} L_{z_0} = \operatorname{ind} L^0. \tag{4.18}$$

Then it is sufficient to check the assertion of the theorem for the family

$$L^0(\lambda) = \lambda^2 a_- Id - \lambda\gamma_+ Id + a\Delta_x - f'(z_0). \tag{4.19}$$

Lemma 2. *The following equality holds:*

$$\operatorname{ind} L^0 = \#\{\lambda \in \sigma(a\Delta_x - f'(z_0))\colon \lambda > 0\}. \tag{4.20}$$

Proof. Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal system of eigenvectors of the selfadjoint operator $-a\Delta_x + f'(z_0)$ in L^2 and

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_N < 0 < \mu_{N+1} \leq \cdots$$

be the corresponding eigenvalues. Then it is not difficult to prove that the eigenvalues of the family L^0 can be calculated by the formulae

$$\lambda_i^+ = \frac{\gamma_+ + \sqrt{\gamma_+^2 + 4a_-\mu_i}}{2a_-}, \ \ \lambda_i^- = \frac{\gamma_+ - \sqrt{\gamma_+^2 + 4a_-\mu_i}}{2a_-}$$

and (4.17) for $s = 0$ is equivalent to the following conditions:

$$\gamma_+^2 + 4a_-\mu_1 > 0, \ \ \lambda_1^- < \lambda_0 < \lambda_1^+.$$

Thus the interval $(0, \lambda_0)$ contains the eigenvalues λ_i^- for $i = 1 \cdots N$ and only them. Lemma 2 is proved.

Theorem 2 is also proved.

Definition 3. *The instable set for the equilibrium z_0 of the semigroup S_t introduced in* Section 3 *is defined to be the following set:*

$$\mathcal{M}^+(z_0) = \{u_0 \in V_0\colon \exists u \in C_b(\mathbb{R}, V_0),\ u(0) = u_0, \\ (S_t u)(s) = u(t+s) \textit{ such that } \lim_{s\to-\infty} u(s) = z_0\}. \quad (4.21)$$

It follows from Definition 3 that the set $\mathcal{M}^+(z_0)$ is strictly invariant with respect to the semigroup S_t, i.e.,

$$S_t\mathcal{M}^+(z_0) = \mathcal{M}^+(z_0) \text{ for } t \geq 0. \quad (4.22)$$

Theorem 3. *Let z_0 be an hyperbolic equilibrium of the semigroup S_t and let the conditions of* Theorem 1 *be satisfied. Then the set $\mathcal{M}^+(z_0)$ possesses the structure of a C^1-manifold of dimension $d = \operatorname{ind}_{z_0}$, i.e., there exists a C^1-embedding $\pi\colon \mathbb{R}^d \to V_0$ such that $\pi(\mathbb{R}^d) = \mathcal{M}^+(z_0)$.*

For each $t \geq 0$, the restriction of S_t to the manifold $\mathcal{M}^+(z_0)$ is a C^1-diffeomorphism.

Proof. For every $\delta > 0$, we define in analogy to (4.21) the following set:

$$\mathcal{M}^+_\delta(z_0) = \{u_0 \in V_0\colon \exists u \in C(\mathbb{R}, V_0),\ u(0) = u_0, \\ (S_t u)(s) = u(t+s) \text{ such that } \lim_{s\to-\infty} u(s) = z_0,\ \|u(s)\|_{V_0} < \delta\}. \quad (4.23)$$

Using the hyperbolicity of the equilibrium z_0 and the implicit function theorem one can prove by standard arguments that for sufficiently small $\delta > 0$ the set (4.23) is a $d = \operatorname{ind}_{z_0}$-dimensional C^1-submanifold of V_0 diffeomorphic to the instable subspace E^+. Let us fix sufficiently small $\delta > 0$ and define a sequence of sets

$$\mathbb{M}^+_k = S_k(\mathcal{M}^+_\delta(z_0)) \ \ k = 0, 1, \ldots \quad (4.24)$$

Then it follows from (4.23) that

$$\mathbb{M}^+_{k-1} \subset \mathbb{M}^+_k, \ \ \mathcal{M}^+(z_0) = \cup_{k=0}^\infty \mathbb{M}^+_k. \quad (4.25)$$

As it is proved in [2], the injectivity of the operator S_k and the injectivity of its differential (see Corollaries 3.3 and 1.1) imply that, for every $k \in \mathbb{N}$, the set $\mathbb{M}_k$ is a C^1-submanifold of V_0 diffeomorphic to E_+. It follows from the representation of the set $\mathcal{M}^+(z_0)$ in (4.25) (see [12]) that the set $\mathcal{M}^+(z_0)$ possesses the structure of a C^1-manifold diffeomorphic to $\mathbb{R}^d$, where $d = \operatorname{ind}_{z_0}$. Theorem 3 is proved.

Remark 3. Notice that generally the set $\mathcal{M}^+(z_0)$ is not a C^1-submanifold of the space V_0, i.e., the topologies on $\mathcal{M}^+(z_0)$ induced by the embedding $\mathcal{M}^+(z_0) \subset V_0$ and by the structure of a C^1-manifold defined by the representation (4.25) generally do not coincide ($\pi\colon \mathbb{R}^d \to \mathcal{M}^+(z_0)$ is not a homeomorphism). However, it is proved in [2], that if the semigroup S_t possesses a global Lyapunov function, then the instable manifold $\mathcal{M}^+(z_0)$ is a C^1-submanifold of the space V_0. It will be proved in the next section that the semigroup S_t possesses a Lyapunov function if condition (0.13) holds. Thus under condition (0.13) the instable set of an hyperbolic equilibrium z_0 is a C^1-submanifold of the space V_0 of dimension $d = \operatorname{ind}_{z_0}$.

§5 The regular attractor for a nonlinear elliptic system in a cylindrical domain.

In this section we construct the attractor for the semigroup (4.2) generated by the problem (0.1) in the space V_0. Moreover, in the case where the nonlinear term f has a potential ((0.13) holds) we prove regularity for this attractor.

Let us first recall the definition of an attractor and sufficient conditions for its existence (see [2] for a detailed exposition).

Definition 1. *The set $\mathcal{A}$ is called the attractor for a semigroup S_t acting in Banach space V_0 if the following conditions hold:*

(1) *$\mathcal{A}$ is compact in V_0;*

(2) *$\mathcal{A}$ is strictly invariant, i.e., $S_t\mathcal{A} = \mathcal{A}$ for $t \geq 0$;*

(3) *$\mathcal{A}$ possesses the attracting property for bounded subsets in V_0, i.e., for every bounded $\mathcal{B} \subset V_0$*

$$\lim_{t\to\infty} \operatorname{dist}(S_t\mathcal{B}, \mathcal{A}) = 0,$$

where $\operatorname{dist}(X, Y) = \sup_{x\in X} \inf_{y\in Y} \|x - y\|_{V_0}$.

Proposition 1 (see [2]). *Let the operator S_t: $V_0 \to V_0$ for every fixed $t \geq 0$ be continuous. Further assume that there exists a compact subset $K \subset\subset V_0$ having the attracting property for bounded subsets in V_0. Then the semigroup S_t possesses an attractor $\mathcal{A} \subset K$. Moreover, it has the following structure:*

$$\mathcal{A} = \{\xi \in V_0\colon \exists u(s),\ s \in \mathbb{R},\ \text{such that } u(0) = \xi,$$
$$\sup_{s\in\mathbb{R}} \|u(s)\|_{V_0} < \infty,\ \text{and } S_t u(s) = u(t + s),\ t \in \mathbb{R}_+,\ s \in \mathbb{R}\}. \quad (5.1)$$

Theorem 1. *Let condition* (4.1) *be satisfied. Then the semigroup S_t: $V_0 \to V_0$ generated by equation* (0.1) *possesses an attractor $\mathcal{A}$ which is compact in the space $W_{2,p}(\omega)$.*

Proof. Let us verify the conditions of Proposition 1 for the semigroup S_t generated by equation (0.1). The continuity of the operators S_t for fixed $t \geq 0$ is proved in Theorem 3.1. Let us construct the compact attracting set. The estimate (0.9) implies that the set $B = \{\xi \in V_0\colon \|\xi\|_{V_0} \leq R\}$ is an attracting set for the semigroup S_t for a sufficiently large $R = R(\|g\|_{0,p})$. Consequently, the set $B_1 = S_1 B$ also possesses the attracting property for bounded subsets in V_0. Let us prove that $B_1 \subset W_{2,p}(\omega)$. According to Remark 4.1 there exists an equilibrium $z_0 \in W_{2,p}(\omega)$ (a solution of equation (4.3)). Multiplying equation (0.1) by the cut-off function $\varphi(t)$ which equals zero for $t \notin [0, 3]$ and one for $t \in [1, 2]$ and rewriting (0.1) with respect to the new unknown function $w = \varphi(t)(u - z_0)$ we obtain

$$\begin{cases} \partial_t^2 w + \Delta_x w = 2\varphi'\partial_t u + \varphi''(u - z_0) + a^{-1}\varphi\,(f(u) - f(z_0) + \gamma\partial_t u) \equiv h_w(t), \\ w\big|_{\partial\Omega_{0,3}} = 0. \end{cases}$$

Since $f \in C^1$, we obviously have $h_w \in W^0_{1,p}(\Omega_{0,3})$ and according to (2.1) the following estimate holds:

$$\|h_w, \Omega_{0,3}\|_{1,p} \leq Q_1(\|u, \Omega_{0,3}\|_{2,p}) \leq Q_2(\|u(0)\|_{V_0}),$$

where Q_1 and Q_2 are certain monotone functions.

Applying the theorem on the $(W_{3,p}, W^0_{1,p})$-regularity for solutions of the Laplace equation we obtain that

$$\|u(1) - z_0\|_{3-1/p,p} \le C\|w, \Omega_{0,3}\|_{3,p} \le C_1\|h_w, \Omega_{0,3}\|_{1,p} \le C_1 Q_2(\|u_0\|_{V_0}).$$

Thus the set $B_1 - z_0$ is bounded in $W_{3-1/p,p}(\omega)$. Therefore, $B_1 \subset\subset W_{2,p}(\omega)$. Theorem 1 is proved.

Corollary 1. *Let z_0 be a hyperbolic equilibrium of the semigroup S_t. Then according to* (5.1),

$$\mathcal{M}^+(z_0) \subset \mathcal{A}. \tag{5.2}$$

Thus the attractor $\mathcal{A}$ contains an invariant C^1-manifold of dimension $d = \mathrm{ind}_{z_0}$.

Let us now consider the case when the semigroup S_t possesses a Lyapunov function.

Theorem 2. *Let the condition* (0.13) *hold. Then the function*

$$-\mathcal{L}(\xi) = \frac{1}{2}(a\Phi(\xi), \Phi(\xi)) - \frac{1}{2}(a\nabla\xi, \nabla\xi) - (P(\xi), 1) - (g, \xi)\colon V_0 \to R, \tag{5.3}$$

where the nonlinear operator $\Phi\colon V_0 \to V_0'$ is defined in Corollary 3.2 *and the function P is defined by* (0.13), *satisfies the equation*

$$\frac{d}{dt}\mathcal{L}(S_t\xi) = -(\gamma\Phi(S_t\xi), \Phi(S_t\xi)) \text{ for all } \xi \in V_0 \tag{5.4}$$

and, consequently, $\mathcal{L}$ is a Lyapunov function for the semigroup S_t.

Proof. Let $u(t) = S_t\xi$ be a solution of the problem (0.1). Then by the definition of Φ, $\Phi(u(t)) = \partial_t u(t)$. Consequently,

$$\mathcal{L}(u(t)) = \frac{1}{2}(a\nabla u(t), \nabla u(t)) - \frac{1}{2}(a\partial_t u(t), \partial_t u(t)) + (P(u(t)), 1) + (g, u(t)). \tag{5.5}$$

Differentiating the function (5.5) with respect to t and replacing $\partial_t^2 u(t)$ by its expression from (0.1), we obtain

$$\frac{d}{dt}\mathcal{L}(u(t)) = -(\gamma\partial_t u(t), \partial_t u(t)) \quad \mathcal{L}(u(t_2)) - \mathcal{L}(u(t_1)) = -\int_{t_1}^{t_2} (\gamma\partial_t u(t), u(t)\, dt. \tag{5.6}$$

The equality (5.4) is proved.

Since $\gamma > 0$, then it follows from (5.6) that $\mathcal{L}(S_t\xi) \le \mathcal{L}(\xi)$ and equality in this formula implies that $\partial_t u(s) \equiv 0$ for all $s \in [0, t]$, i.e., ξ is an equilibrium of S_t. Thus, $\mathcal{L}$ is a Lyapunov function for the semigroup S_t. Theorem 2 is proved.

Let us now suppose that the set of all equilibria of the semigroup S_t,

$$\mathcal{R} = \{z_i \in V_0\colon i = 1, \ldots, N,\ z_i \text{ is a solution of (4.3)}\} \tag{5.7}$$

is finite. Then the following theorem holds:

Theorem 3. *Let the conditions of the previous theorem be satisfied and let the set* (5.7) *be finite. Then for every solution* $u(t)$, $t \in \mathbb{R}$, *of equation* (0.1), $u \in C_b(\mathbb{R}, V_0)$, *there exist* $z_+ \in \mathcal{R}$ *and* $z_- \in \mathcal{R}$ *such that* $z_- \neq z_+$ *and*

$$\lim_{t\to+\infty} \|u(t) - z_+\|_{V_0} = 0, \ \lim_{t\to-\infty} \|u(t) - z_-\|_{V_0} = 0. \tag{5.8}$$

The proof of this theorem is given for instance in [4].

Corollary 2. *Let* $\mathcal{M}(z_i)$ *be the unstable set for the equilibrium* $z_i \in \mathcal{R}$ *defined by* (4.21)*. Then it follows from formula* (5.8) *and from the representation of* $\mathcal{A}$ *in the form* (5.1) *that*

$$\mathcal{A} = \cup_{i=1}^{N} \mathcal{M}^+(z_i). \tag{5.9}$$

Theorem 4. (*Regularity of elliptic attractors*)**.** *Let the semigroup* $S_t \colon V_0 \to V_0$ *generated by equation* (0.1) *possess a Lyapunov function* $\mathcal{L}$ *and let all of its equilibria* (5.7) *be hyperbolic. Then the attractor* $\mathcal{A}$ *of the problem* (0.1) *possesses the decomposition* (5.9) *and according to* Theorem 4.3 *the unstable sets* $\mathcal{M}^+(z_i)$ *are* C^1*-submanifolds in the space* V_0 *of dimension* $\kappa_i = \mathrm{ind}_{z_i}$ *diffeomorphic to* $\mathbb{R}^{\kappa_i}$.

Corollary 3. *Assume that the conditions of* Theorem 4 *are satisfied. Let the equilibria* (5.7) *be numerated such that* $\mathcal{L}(z_i) \leq \mathcal{L}(z_j)$ *if* $i \leq j$ *and suppose for simplicity that*

$$\mathcal{L}(z_1) < \mathcal{L}(z_2) < \cdots < \mathcal{L}(z_N). \tag{5.10}$$

Let

$$\mathcal{A}_k = \cup_{i=1}^{k} \mathcal{M}^+(z_i)$$

Then arguing as in [2] *we can prove that the following assertions hold:*

1. $\mathcal{A}_k$ *is stable with respect to* S_t*, i.e., for every neighborhood* $\mathcal{O}_1(\mathcal{A}_k)$ *of* $\mathcal{A}_k$ *in the space* V_0 *there exists a neighborhood* $\mathcal{O}_2(\mathcal{A}_k)$ *such that*

$$S_t\mathcal{O}_2(\mathcal{A}_k) \subset \mathcal{O}_1(\mathcal{A}_k) \ \text{for all } t \geq 0.$$

2. The boundary of $\mathcal{M}^+(z_i)$ *is strictly invariant with respect to the semigroup* S_t*, i.e.,* $S_t\partial\mathcal{M}^+(z_i) = \partial\mathcal{M}^+(z_i)$*. Moreover,* $\partial\mathcal{M}^+(z_i) \subset \mathcal{A}_{i-1}$.

3. For every compact set $K \subset \mathcal{A}_i \setminus z_i$, $\lim_{t\to\infty} \mathrm{dist}(S_tK, \mathcal{A}_{i-1}) = 0$.

Remark 1. As proved for instance in [2] the condition of hyperbolicity of all equilibria of S_t is a generic condition, i.e., the set of the right-hand sides g of the equation (0.1) which satisfy this condition is an open dense subset of the space $L_p(\omega)$. Moreover, the Leray–Schauder degree theory implies that in this case that the number of equilibria is odd: $N = 2m + 1$.

Let us study now the attracting property of the attractor $\mathcal{A}$. We will need the following definition to do so.

Definition 2. *Let the conditions of* Corollary 3 *be satisfied. A piecewise continuous function* $\hat{u}(t)$, $t \in [0, \infty]$, *is said to be a finite-dimensional composite trajectory (f.c.t) for the semigroup* S_t *in the space* V_0 *if it satisfies the following conditions:*

1. There exists $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = +\infty$*, such that*

$$\hat{u} \in C([t_i, t_i + 1), V_0), \quad i = 0, \ldots, m;$$

2. $\hat{u}(t) = S_{t-t_i}u(t_i)$ *for* $t \in [t_i, t_{i+1})$;

3. $\hat{u}(t_i) \in \mathcal{M}^+(z_{\pi(i)})$*, where* $\pi \colon \overline{1, \cdots, m+1} \to \overline{1, \cdots, N}$.

Thus a f.c.t. $\hat{u}(t)$ *is a piecewise continuous curve constructed from a finite number of pieces of continuous trajectories of the semigroup* S_t *that belong to the unstable manifolds* $\mathcal{M}^+(z_i)$.

Theorem 5. *Let the conditions of* Corollary 3 *be satisfied. Then for every* $u_0 \in V_0$ *there exists f.c.t.* $\hat{u}(t)$ *such that*

$$\|S_t u_0 - \hat{u}(t)\|_{V_0} \le Ce^{-\nu t}, \ \ t \ge 0. \tag{5.11}$$

Moreover, the constant C *can be chosen uniformly with respect to* $u_0 \in K$*, where* K *is an arbitrary compact in* V_0*, an the constant* $\nu > 0$ *depends only on the semigroup* S_t*.*

To prove this theorem we refer to the following abstract theorem proved in [2] about uniform spectral asymptotics.

Theorem 6. *Let all of the equilibria of the semigroup* $S_t \in C(\mathbb{R}_+ \times V_0, V_0)$ *acting in the Banach space* V_0 *be hyperbolic. Suppose that the semigroup* S_t *possesses the attractor* $\mathcal{A}$ *in* V_0 *and the Lyapunov function* $\mathcal{L} \in C(V_0, R)$*. Further let for every bounded subset* $B \subset V_0$ *the following condition hold:*

$$\|D_\xi S_t(\xi)\|_{L(V_0,V_0)} \le Ce^{\beta t}, \ \ \textit{for all } \xi \in B, \ C = C(B), \ \alpha = \alpha(B).$$

Moreover, assume that for every fixed t the derivative $D_\xi S_t(\xi)$ *is uniformly continuous with respect to* ξ *for* $\xi \in B$ *and every bounded set* $B \subset V_0$*. Then the assertion of* Theorem 5 *is valid for the semigroup* S_t*.*

Actually, we have already verified all conditions of Theorem 6 for the semigroup S_t generated by the problem (0.1) in Theorems 3.3, 4.1, 5.1, 5.2. Thus Theorem 5 is proved.

Remark 3. In the formulation of Theorem 6 in [2] it was also assumed that the semigroup S_t satisfies $S_t \in C^{1+\alpha}(V_0, V_0)$, but this requirement was only used in the construction of the instable manifolds $\mathcal{M}^+(z_i)$ (see [2, Remark 5.7.9]) which in our case have been constructed before in Theorem 4.3.

Corollary 4. *Let the conditions of Theorem 5 be satsfied. Then for every bounded subset* $B \subset V_0$ *the following estimate holds:*

$$\operatorname{dist}(S_t B, \mathcal{A}) \le C(B)e^{-\nu t}, \tag{5.12}$$

i.e., $\mathcal{A}$ *is an exponential attractor for the semigroup* S_t*.*

Indeed, if B is compact, then (5.12) is an immediate consequence of the estimate (5.11). But it is verified in the proof of Theorem 1 that the set $S_1 B$ is semi-compact for every bounded $B \subset V_0$. Thus the estimate (5.12) holds for every bounded subset $B \subset V_0$.

Remark 3. Equations of type (0.1) can appear in applications when studying the equilibrium points and travelling wave solutions of certain evolutionary equations. For example, the problem of finding the travelling wave solutions

$$A(t,x) = A(x_1 - \gamma t, x'), \ x = (x_1, x') \in \Omega_+, \ t \in \mathbb{R}_+, \ \gamma \in \mathbb{R}$$

for the generalized Gindsburg-Landau equation

$$\partial_t A = \mu A - (\lambda - i\beta)A|A|^2 + \Delta_x A,$$

where $A = A_1 + iA_2$, $\lambda, \beta, \mu \in \mathbb{R}$, $\lambda > 0$, is reduced to the problem of finding all bounded with respect to $t \in \mathbb{R}$ solutions of an equation of the form (0.1) in the complete cylinder $\Omega = \mathbb{R} \times \omega$, i.e., to the problem of finding the attractor of an equation of the form (0.1) in the half-cylinder Ω_+ (see (5.1)). Notice also that this equation is potential if $\beta = 0$.

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