THE ATTRACTOR FOR A NONLINEAR HYPERBOLIC EQUATION IN THE UNBOUNDED DOMAIN.

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Abstract. We study the long-time behavior of solutions for damped nonlinear hyperbolic equations in the unbounded domains. It is proved that under the natural assumptions these equations possess the locally compact attractors which may have the infinite Hausdorff and fractal dimension. That is why we obtain the upper and lower bounds for the Kolmogorov's entropy of these attractors.

Moreover, we study the particular cases of these equations where the attractors occurred to be finite dimensional. For such particular cases we establish that the attractors consist of finite collections of finite dimensional unstable manifolds and every solution stabilizes to one of the finite number of equilibria points.

Contents

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Introduction

In this paper the long-time behavior for the solutions of damped hyperbolic equations

$$
\begin{cases} \partial_t^2 u + \gamma \partial_t u - \Delta_x u + f(u) + \lambda_0 u = g(t);\ x \in \Omega \\ u\big|_{t=0} = u_0\,, \quad \partial_t u\big|_{t=0} = u_0'\,, \quad u\big|_{\partial\Omega} = 0 \end{cases} \tag{0.1}
$$

is studied.

Here $\Omega \subset \mathbb{R}^n$ is an unbounded domain in $\mathbb{R}^n$ with a sufficiently smooth boundary (see §1), $u = u(t,x)$ is unknown function, Δ_x is a Laplacian with respect to $x = (x_1, \cdots, x_n)$, f and g are given functions and λ_0 and γ are fixed positive constants.

It is assumed that the nonlinear term $f(u)$ satisfies the conditions

$$
\begin{cases} 1.\, f \in C^2(\mathbb{R}, \mathbb{R}), \quad f'(u) \ge -C \\ 2.\, |f'(u)| \le C(1 + |u|^{q_1}), \quad |f''(u)| \le C(1 + |u|^{q_2}) \end{cases} \tag{0.2}
$$

where $q_1 < 2/(n-2)$ if $n \ge 3$ and $q_2 \le 1$ if $n = 3$ and $q_2 = 0$ for $n > 3$.

Moreover, it is assumed also that the function f can be decomposed in a sum of two functions

$$
f(u) = f_1(u) + f_2(u) \tag{0.3}
$$

where the function f_2 is bounded together with the derivatives

$$
|f_2(u)| + |f_2'(u)| + |f_2''(u)| \le C \tag{0.4}
$$

and the function f_1 satisfies the assumption

$$
f_1(u) \cdot u \ge 0 \tag{0.5}
$$

It is well known that in many cases the longtime behavior of dynamical systems, generated by evolutionary equations of mathematical physics can be naturally described in terms of attractors of the corresponding semigroups (see [2], [16], [24]). In bounded domains the existence of the attractor is established for a large class of equations such as reaction-diffusion equations, nonlinear wave equations, 2D Navier–Stokes system, etc. Under some natural assumptions it is proved that in the autonomous case for all equations mentioned above the attractor has the finite Hausdorff and fractal dimension (see [2], [24]).

The equations of mathematical physics which can depend explicitly on t in bounded domains Ω are considered in [4], [6]. Recall that according to the construction of the uniform attractor, suggested there, one should consider not only the initial nonautonomous problem but simultaneously the whole family of problems which are obtained from the initial one by all positive shifts along the t-axis and their closure in the appropriate topology. For instance, in order to construct the attractor for the equation (0.1) where $g = g(t)$ depends explicitly on t one should consider the family of equations of the type (0.1) with all right-hand sides belonging to the hull $\mathcal{H}^+(g)$ of the initial right-hand side g which can be defined in the following way

$$
\mathcal{H}^+(g) := \left[T_h g, h \ge 0\right]_{L^2_{loc}(\mathbb{R}_+ \times \Omega)}, \quad (T_h g)(t) := g(t+h) \tag{0.6}
$$

(see §5 and §6). As usual, we restrict ourselves to consider only such right-hand sides g for which the hull (0.6) is compact in $L^2_{loc}(\mathbb{R}_+ \times \Omega)$.

Moreover, if this hull is in a certain sense "infinite dimensional" (for example the right-hand side is almost-periodic by t with the infinite number of independent frequences) then the uniform attractor of the corresponding equation naturally has infinite Hausdorff and fractal dimension.

Thus, in contrast to the autonomous case in the nonautonomous one the fractal dimension is not a convenient quantitative characteristic of the "size" of attractors and consequently the problem of finding another characteristics arises.

One of possible approaches to handle this problem, which is suggested in [6], is to estimate Kolmogorov's ε-entropy of the attractor. Recall, that by definition Kolmogorov's ε-entropy $\mathbb{H}_\varepsilon(\mathcal{A}, \Phi)$ of the attractor $\mathcal{A}$ is the logarithm from the minimal number $N_\varepsilon(\mathcal{A}, \Phi)$ of ε-balls in the appropriate phase space Φ which cover the attractor:

$$\mathbb{H}_\varepsilon(\mathcal{A}, \Phi) = \ln N_\varepsilon(\mathcal{A}, \Phi) \tag{0.7}$$

Note that since $\mathcal{A}$ is compact then (0.7) is well defined and finite for every $\varepsilon > 0$.

The estimates for the ε-entropy of the attractors of nonautonomous reaction-diffusion equations in *bounded* domains has been obtained in [6]. The autonomous reaction-diffusion equations in $\mathbb{R}^n$ has been considered in [7] and [27]. The entropy of attractors for autonomous and nonautonomous RDE in the unbounded domains has been studied in [11] and [27].

Recall that for unbounded domains Ω the behavior of solutions for (0.1) becomes much more complicated. In this case even the problem of finding the appropriate phase space for (0.1) becomes nontrivial. For instance, the reaction diffusion systems in unbounded domains have been studied in weighted Sobolev spaces $W^{l,p}_\phi(\Omega)$ with $\phi(x) = \phi_\alpha(x) = (1 + |x|^2)^{\alpha/2}$ in [1], [3], [9]. The case of general weights ϕ is considered in [10].

In this paper we assume that the solution $u(t, x)$ is bounded with respect to $|x| \to \infty$. To be more precise, we introduce the spaces $W^{l,p}_b(\Omega)$ by the following expression

$$W^{l,p}_b(\Omega) := \{v : \|v\|_{W^{l,p}_b} = \sup_{x_0 \in \Omega} \|v\|_{W^{l,p}(\Omega \cap B^1_{x_0})} < \infty\} \tag{0.8}$$

(here and below we denote by $B^R_{x_0}$ the R-ball in $\mathbb{R}^k$ centered in x_0) and require that for every fixed $t \geq 0$ the solution $\xi_u(t) := (u(t), \partial_t u(t))$ belongs to the space

$$E_b(\Omega) := \left(W^{1,2}_b(\Omega) \cap \{v\big|_{\partial\Omega} = 0\}\right) \times L^2_b(\Omega) \tag{0.9}$$

The behavior of solutions for reaction-diffusion equations and systems in the unbounded domains in the spaces (0.8) has been studied in [7], [8], [11], [13], [21], [22], [28].

The attractor for the autonomous hyperbolic equation (0.1) in $\mathbb{R}^n$ under the assumptions which are similar to (0.2)–(0.5) has been constructed in [14].

Note that under the above assumptions (as in the case of parabolic equations) the attractor $\mathcal{A}$ of the equation (0.1) may have (and has in general) infinite Hausdorff and fractal dimension even in the autonomous case (see Section 11). Thus, in contrast to the case of bounded domains where the infinite dimensional attractor can appear only in the nonautonomous case and only due to the "infinite dimensional" external time-depended forces, in the case where Ω is unbounded the infinite dimensionality appears even in the autonomous case and has consequently the internal nature.

In this paper we give a systematical study of Kolmogorov's ε-entropy of attractors for autonomous and nonautonomous hyperbolic equations of the type (0.1) in unbounded domains Ω.

It is known (see Remark 4.2) that in general the attractor $\mathcal{A}$ of the problem (0.1) is not compact in the uniform topology of the space (0.8) but only in a local topology of the space $E_{loc}(\Omega)$, that is why (following to [28]) we consider the entropy of restrictions $\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}$ and study the dependence of $\mathbb{H}_\varepsilon(\mathcal{A}\big|_{\Omega\cap B^R_{x_0}})$ on three parameters ε, R and x_0.

It is proved in Section 9 that the entropy of the uniform attractor of the nonautonomous equation (3.1) possesses the estimate

$$
\begin{aligned}
(0.10)\quad \mathbb{H}_\varepsilon(\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}, E_b(\Omega\cap B^R_{x_0})) &\le C\,\mathrm{vol}_{x_0,\Omega}(R+K\ln\frac{1}{\varepsilon})\ln\frac{1}{\varepsilon}+\\
&\mathbb{H}_{\varepsilon/L}\left(\mathcal{H}^+(g)\big|_{[0,K\ln\frac{1}{\varepsilon}]\times\Omega\cap B^{R+K\ln\frac{1}{\varepsilon}}_{x_0}}, L^2_b([0,K\ln\frac{1}{\varepsilon}]\times\Omega\cap B^{R+K\ln\frac{1}{\varepsilon}}_{x_0})\right)
\end{aligned}
$$

where $\mathrm{vol}_{x_0,\Omega}(r)=\mathrm{vol}(\Omega\cap B^r_{x_0})$, $\mathrm{vol}(\cdot)$ means the n-dimensional volume, and constants C,K,L are independent of R, ε and x_0. Particularly, for autonomous equations in $\Omega=\mathbb{R}^n$, the estimate (0.10) implies that

$$
(0.11)\qquad \mathbb{H}_\varepsilon(\mathcal{A}\big|_{B^R_{x_0}}, E_b(\Omega\cap B^R_{x_0}))\le C(R+K\ln\frac{1}{\varepsilon})^n\ln\frac{1}{\varepsilon}
$$

We verify also the sharpness of the estimate (0.11). To this end we consider in Section 11 a special class of equations of the form (0.1) which contains for instance the equations

$$
(0.12)\qquad \partial_t^2u+\gamma\partial_t u-\Delta_x u-\alpha u+u|u|^p=0,\quad \gamma,\alpha>0,\quad x\in\mathbb{R}^n
$$

with $p<1+2/(n-2)$.

It is proved that the entropy of the attractor of (0.12) possesses the following estimate

$$
(0.13)\qquad \mathbb{H}_\varepsilon(\mathcal{A}\big|_{B^R_{x_0}}, E_b(B^R_{x_0}))\ge C_1R^n\ln\frac{1}{\varepsilon}
$$

for $R\ge R_0$ and $\varepsilon<\varepsilon_0$ and consequently the estimate (0.11) is sharp if $R\sim\ln\frac{1}{\varepsilon}$ or $R\gg\ln\frac{1}{\varepsilon}$. For the case where $R\ll\ln\frac{1}{\varepsilon}$ (particularly for $R=1$) we obtain that for every $\delta>0$ there exists $C_\delta>0$ such that

$$
(0.14)\qquad \mathbb{H}_\varepsilon(\mathcal{A}\big|_{B^1_{x_0}}, E_b(B^1_{x_0}))\ge C_\delta\left(\ln\frac{1}{\varepsilon}\right)^{n+1-\delta}
$$

and consequently the estimate (0.11) is sharp (in a some sense) for every R and ε.

It is worth to emphasize that the estimates (0.10)–(0.11) and (0.13)–(0.14) are very similar to the ones obtained in [28] for parabolic equations and consequently these estimates seem to have a universal nature. From the other side if Ω is bounded and g is independent of t then (0.10) implies that

$$\mathbb{H}_\varepsilon(\mathcal{A}, E_b(\Omega)) \le C \ln \frac{1}{\varepsilon}$$

which reflects the well-known heuristic principle that the equations of mathematical physics in bounded domains have finite dimensional attractors. Thus, the estimates mentioned above may be considered as a natural generalization of this principle to the case of unbounded domains.

Note that although the attractor of the equation (0.1) is not compact in general in $E_b(\Omega)$ but there is a number of particular cases where this equation has nontrivial compact in $E_b(\Omega)$ attractors. One of these particular cases is considered in the paper. Namely, we assume that the function f_2 in (0.4) equals zero identically ($f_2 \equiv 0$) and the right-hand side $g(t)$ (with compact hull $\mathcal{H}^+(g)$) satisfies the following assumption

$$\int_T^{T+1} \|g(t), \Omega \cap B^1_{x_0}\|^2_{0,2}\, dt \to 0 \tag{0.15}$$

when $|x_0| \to 0$ uniformly with respect to $T \in \mathbb{R}_+$.

It is proved in Section 10 that under the above assumptions the entropy of globally compact (i.e. compact in $E_b(\Omega)$) attractor $\mathcal{A}$ possesses the following estimate which essentially improves the estimate (0.10):

$$\mathbb{H}_\varepsilon\left(\mathcal{A}, E_b(\Omega)\right) \le C \ln \frac{1}{\varepsilon} + \mathbb{H}_{\varepsilon/L}\left(\mathcal{H}^+(g), L^2_b([0, K \ln \frac{1}{\varepsilon}] \times \Omega)\right) \tag{0.16}$$

Particularly, this estimate implies that in the autonomous case (or in the case where $g(t)$ is quasiperiodic with respect to t) the attractor $\mathcal{A}$ has a finite fractal dimension in $E_b(\Omega)$.

The finite dimensional attractor for reaction-diffusion equations in $\mathbb{R}^n$ with quasiperiodic right-hand sides satisfying (0.15) has been obtained in [12].

Moreover, we give a detailed study of the autonomous finite dimensional attractor in the above particular case of the equation (0.1).

It is proved in Section 12 that there is an equilibria point u_0 of the equation (0.1) such that for every $(u(t), \partial_t u(t)) \in \mathcal{A}$

$$\|\partial_t u(t), \Omega \cap B^1_{x_0}\|^2_{0,2} + \|u(t) - u_0, \Omega \cap B^1_{x_0}\|^2_{1,2} \le C e^{-\delta |x_0|} \tag{0.17}$$

with C and $\delta > 0$ depending only on the equation. Using this spatial asymptotic of the attractor we construct the Liapunov function on it and prove that for generic g's the attractor is regular, i.e. that it consists of a finite collection of finite dimensional unstable manifolds:

$$\mathcal{A}_g = \cup_{i=1}^N \mathcal{M}^+(u_i) \tag{0.18}$$

where u_i are the equilibria points of (0.1) and $\mathcal{M}^+(u_i)$ are the C^1-submanifolds in $E_b(\Omega)$.

Note also that the results of the paper has been announced in [29] for the case $\Omega = \mathbb{R}^n$.

The paper is organized as follows.

The definition of functional spaces and a number of a priori estimates for linear and nonlinear equations of the type (0.1) are given in Part 1. Moreover the problems of the solution's existence and it's uniqueness are analyzed there.

The locally and globally compact attractors for (0.1) are constructed in Part 2.

Upper and lower bounds of the ε-entropy of the attractors constructed above are given in Part 3.

The spatial asymptotic for the solutions belonging to the attractor are obtained in Part 4. Using this asymptotic we derive stabilization of solutions of the equation (0.1) when $t \to \infty$ and prove the attractor's regularity.

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Part 1. A priori estimates,existence of solutions, uniqueness.

In this part we derive a number of a priori estimates for the equations of the type (0.1), prove the existence of solutions and their uniqueness.

Section 1 contains the definitions of the functional spaces which are necessary to deal with the equations (0.1) in unbounded domains and a number of simple properties of these spaces which will be essentially used throughout of the paper.

The linear equation of the view (0.1) is analyzed in Section 2.

In Section 3 arguing in the spirit of [14] we establish the existence of solutions for the nonlinear equation (0.1) and their uniqueness and derive the estimates which will be used in the next Part in order to prove the attractor's existence.

§1 Functional spaces

In this Section we introduce several classes of Sobolev spaces in unbounded domains and recall shortly some of their properties which will be essentially used below. For a detailed study of these spaces see [10], [28].

Definition 1.1. *A function $\phi \in L^\infty_{loc}(\mathbb{R}^n)$ is called a weight function with the rate of growth $\mu \geq 0$ if the condition*

$$\phi(x+y) \leq C_\phi e^{\mu|x|}\phi(y), \;\; \phi(x) > 0 \tag{1.1}$$

is satisfied for every $x, y \in \mathbb{R}^n$.

Remark 1.1. *It is not difficult to deduce from* (1.1) *that*

$$\phi(x+y) \geq C_\phi^{-1} e^{-\mu|x|}\phi(y) \tag{1.2}$$

is also satisfied for every $x, y \in \mathbb{R}^n$.

The following example of weight functions are of fundamental significance for our purposes:

$$\phi_{\{\varepsilon\},x_0}(x) = e^{-\varepsilon|x-x_0|}, \; \varepsilon \in \mathbb{R}, \; x_0 \in \mathbb{R}^n$$

(Evidently this weight has the rate of growth $|\varepsilon|$.)

Definition 1.2. *Let $\Omega \subset \mathbb{R}^n$ be some (unbounded) domain in $\mathbb{R}^n$ and let ϕ be a weight function with the rate of growth μ. Define the space*

$$L^p_\phi(\Omega) = \left\{ u \in D'(\Omega) \; : \; \|u, \Omega, \|_{\phi,0,p} \equiv \int_\Omega \phi(x)|u(x)|^p \, dx < \infty \right\}$$

Analogously the weighted Sobolev space $W^{l,p}_\phi(\Omega)$, $l \in \mathbb{N}$ is defined as the space of distributions whose derivatives up to the order l inclusively belong to $L^p_\phi(\Omega)$.

For the simplicity of notations we will right throughout of the paper $W^{s,p}_{\{\varepsilon\}}$ instead of $W^{s,p}_{e^{-\varepsilon|x|}}$.

We define also another class of weighted Sobolev spaces

$$W^{l,p}_{b,\phi}(\Omega) = \left\{ u \in D'(\Omega) \; : \|u, \Omega\|^p_{b,\phi,l,p} = \sup_{x_0 \in \mathbb{R}^n} \phi(x_0) \|u, \Omega \cap B^1_{x_0}\|^p_{l,p} < \infty \right\}$$

Here and below we denote by $B^R_{x_0}$ the ball in $\mathbb{R}^n$ of radius R, centered in x_0, and $\|u, V\|_{l,p}$ means $\|u\|_{W^{l,p}(V)}$.

We will write $W^{l,p}_b$ instead of $W^{l,p}_{b,1}$.

Proposition 1.1.

1. Let $u \in L^p_\phi(\Omega)$, where ϕ is a weight function with the rate of growth μ. Then for any $1 \le q \le \infty$ the following estimate is valid

$$\left(\int_\Omega \phi(x_0)^q \left(\int_\Omega e^{-\varepsilon|x-x_0|} |u(x)|^p \, dx \right)^q dx_0 \right)^{1/q} \le C \int_\Omega \phi(x)|u(x)|^p \, dx \tag{1.3}$$

for every $\varepsilon > \mu$, where the constant C depends only on ε, μ and C_ϕ from (1.1) *(and independent of Ω).*

2. Let $u \in L^\infty_\phi(\Omega)$. Then the following analogue of the estimate (1.3) *is valid*

$$\sup_{x_0 \in \Omega} \left\{ \phi(x_0) \sup_{x \in \Omega} \{ e^{-\varepsilon|x-x_0|} |u(x)| \} \right\} \le C \sup_{x \in \Omega} \{ \phi(x)|u(x)| \} \tag{1.4}$$

The proof of this Proposition can be found in [10] or [28].

For the more detailed study of functional spaces defined above we need some regularity assumptions on the domain $\Omega \subset \mathbb{R}^n$ which are assumed to be valid throughout of the paper.

We suppose that there exists a positive number $R_0 > 0$ such that for every point $x_0 \in \Omega$ there exists a smooth domain $V_{x_0} \subset \Omega$ such that

$$B^{R_0}_{x_0} \cap \Omega \subset V_{x_0} \subset B^{R_0+1}_{x_0} \cap \Omega \tag{1.5}$$

Moreover it is assumed also that there exists a diffeomorphizm $\theta_{x_0} : B^1_0 \to V_{x_0}$ such that $\theta_{x_0}(x) = x_0 + p_{x_0}(x)$ and

$$\|p_{x_0}\|_{C^N} + \|p^{-1}_{x_0}\|_{C^N} \le K \tag{1.6}$$

where the constant K is assumed to be independent of $x_0 \in \Omega$ and N is large enough. For simplicity we suppose below that (1.5) and (1.6) hold for $R_0 = 2$.

Note that in the case when Ω is bounded the conditions (1.5) and (1.6) are equivalent to the condition: the boundary $\partial\Omega$ is a smooth manifold, but for unbounded domains the only smoothness of the boundary is not sufficient to obtain the regular structure of Ω when $|x| \to \infty$ since some uniform with respect to $x_0 \in \Omega$ smoothness conditions are required. It is the most convenient for us to formulate these conditions in the form (1.5) and (1.6).

Proposition 1.2. *Let the domain* Ω *satisfy the conditions* (1.5) *and* (1.6), *the weight function – the condition* (1.1) *and let* R *be some positive number. Then the following estimates are valid*

(1.7)
$$C_2 \int_\Omega \phi(x)|u(x)|^p \, dx \leq \int_\Omega \phi(x_0) \int_{\Omega\cap B^R_{x_0}} |u(x)|^p \, dx \, dx_0 \leq C_1 \int_\Omega \phi(x)|u(x)|^p \, dx$$

Proof. Let us change the order of integration in the middle part of (1.7)

$$\int_\Omega \phi(x_0) \int_{\Omega\cap B^R_{x_0}} |u(x)|^p \, dx \, dx_0 = \int_\Omega |u(x)|^p \left(\int_\Omega \chi_{\Omega\cap B^R_x}(x_0)\phi(x_0) \, dx_0 \right) dx \tag{1.8}$$

Here $\chi_{\Omega\cap B^R_x}$ is the characteristic function of the set $\Omega \cap B^R_x$.

It follows from the inequalities (1.1) and (1.2) that

$$C_1\phi(x) \leq \inf_{x_0\in B^R_x} \phi(x_0) \leq \sup_{x_0\in B^R_x} \phi(x_0) \leq C_2\phi(x) \tag{1.9}$$

and the assumptions (1.5) and (1.6) imply that

$$0 < C_1 \leq \mathrm{vol}(\Omega \cap B^R_x) \leq C_2 \tag{1.10}$$

uniformly with respect to $x \in \Omega$.

The estimate (1.7) is an immediate corollary of the estimates (1.8)–(1.10). Proposition 1.2 is proved. ☐

Corollary 1.1. *Let* (1.5) *and* (1.6) *be valid. Then the equivalent norm in weighted Sobolev space* $W^{l,p}_\phi(\Omega)$ *is given by the following expression:*

$$\|u, \Omega\|_{\phi,l,p} = \left(\int_\Omega \phi(x_0) \|u, \Omega \cap B^R_{x_0}\|^p_{l,p} \, dx_0 \right)^{1/p} \tag{1.11}$$

Particularly, the norms (1.11) *are equivalent for different* $R \in \mathbb{R}_+$.

To study the equation (0.1) we need also weighted Sobolev spaces with fractional derivatives $s \in \mathbb{R}_+$ (not only $s \in \mathbb{Z}$). For the first we recall (see [25] for details) that if V is a bounded domain the norm in the space $W^{s,p}(V)$, $s = [s] + l$, $0 < l < 1$, $[s] \in \mathbb{Z}_+$ can be given by the following expression

$$\|u, V\|^p_{s,p} = \|u, V\|^p_{[s],p} + \sum_{|\alpha|=[s]} \int_{x\in V} \int_{y\in V} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+lp}} \, dx \, dy \tag{1.12}$$

It is not difficult to prove arguing as in Proposition 1.2 and using this representation that for any bounded domain V with a sufficiently smooth boundary

$$\|u, V\|^p_{s,p} \leq C_1 \int_{x_0\in V} \|u, V \cap B^R_{x_0}\|^p_{s,p} \, dx_0 \leq C_2 \|u, V\|^p_{s,p} \tag{1.13}$$

This justifies the following definition.

Definition 1.3. *Define the space* $W^{s,p}_{\phi}(\Omega)$ *for any* $s \in \mathbb{R}_+$ *by the norm* (1.11).

It is not difficult to check that these norms are also equivalent for different $R > 0$. Note now that the weight functions

$$\phi_{\{\varepsilon\},x_0} = e^{-\varepsilon|x-x_0|} \tag{1.14}$$

satisfy the conditions (1.1) *uniformly* with respect to $x_0 \in \mathbb{R}^n$, consequently all estimates obtained above for the arbitrary weights will be valid for the family (1.14) with constants, independent of $x_0 \in \mathbb{R}^n$. Since these estimates are of fundamental significance for us we write it explicitly in a number of corollaries formulated below.

Corollary 1.2. *Let* $u \in L^p_{\{\delta\}}(\Omega)$ *for* $0 < \delta < \varepsilon$. *Then the following estimate holds uniformly with respect to* $y \in \mathbb{R}^n$

$$\left(\int_\Omega e^{-q\delta|x_0-y|}\left(\int_\Omega e^{-\varepsilon|x-x_0|}|u(x)|^p\,dx\right)^q dx_0\right)^{1/q} \le \le C_{\varepsilon,q}\int_\Omega e^{-\delta|x-y|}|u(x)|^p\,dx \tag{1.15}$$

Moreover if $u \in L^\infty_{\{\delta\}}(\Omega)$, $\delta < \varepsilon$ *then*

$$\sup_{x_0\in\Omega}\left\{e^{-\delta|x_0-y|}\sup_{x\in\Omega}\{e^{-\varepsilon|x-x_0|}|u(x)|\}\right\} \le C_{\varepsilon,\delta}\sup_{x\in\Omega}\{e^{-\delta|x-y|}|u(x)|\} \tag{1.16}$$

Corollary 1.3. *Let* $u \in W^{l,p}_{b,\phi}(\Omega)$ *and* ϕ *be a weight function with the rate of growth* $\mu < \varepsilon$. *Then*

$$C_1\|u,\Omega\|^p_{b,\phi,l,p} \le \le \sup_{x_0\in\Omega}\left\{\phi(x_0)\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|u,\Omega\cap B^1_x\|^p_{l,p}\,dx\right\} \le C_2\|u,\Omega\|^p_{b,\phi,l,p} \tag{1.17}$$

For the proof of this corollary see [28].

We will essentially use below the subspaces of $W^{l,p}_b(\Omega)$ which consist of functions decaying when $|x| \to \infty$.

Definition 1.4. *Define the space* $\dot W^{l,p}_b(\Omega)$ *by the following expression:*

$$\dot W^{l,p}_b(\Omega) := \{u \in W^{l,p}_b(\Omega) : \lim_{|x_0|\to\infty}\|u,\Omega\cap B^1_{x_0}\|_{l,p} = 0\} \tag{1.18}$$

The following Proposition gives simple compactness criteria for sets in $\dot W^{l,p}_b(\Omega)$.

Proposition 1.3. *A set* $B \in \dot W^{l,p}_b(\Omega)$ *is compact if and only if:*

1. For every $x_0 \in \Omega$ *the restriction* $B\big|_{V_{x_0}}$ *of the set* B *on* V_{x_0} *is compact in* $W^{l,p}(V_{x_0})$.

2. The set B *possesses a uniform 'tale' estimate, i.e. there exists a continuous function* $R_B(z) : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\lim_{z\to\infty} R_B(z) = 0$ *and*

$$\|u,\Omega\cap B^1_{x_0}\|_{l,p} \le R_B(|x_0|) \quad , \quad \forall u \in B \tag{1.19}$$

Indeed, the assertion of the proposition can be easily derived using the Hausdorff criteria.

A proposition formulated below will be useful in order to verify the fact that the function belong to the space $\dot W^{l,p}_b(\Omega)$.

Proposition 1.4. *Let the estimate* (1.19) *be satisfied and let the function* v *satisfy the following estimate:*

$$\|v, \Omega \cap B^1_{x_0}\|^p_{l_1,p} \le C \int_\Omega e^{-\varepsilon|x-x_0|} \|u, \Omega \cap B^1_x\|^p_{l,p}\, dx \tag{1.20}$$

for the appropriate $\varepsilon > 0$ *and* $l_1 \ge 0$. *Then* $v \in \dot W^{l_1,p}_b(\Omega)$. *Moreover there exists a function* $R_1(z)$ *which tends to zero when* $z \to \infty$ *and depends only on the function* R_B *from* (1.19) *and* $\varepsilon > 0$ *such that*

$$\|v, \Omega \cap B^1_{x_0}\|_{l_1,p} \le R_1(|x_0|) \tag{1.21}$$

Proof. Indeed, let us fix an arbitrary $\delta > 0$. Since $R_B(z) \to 0$ when $z \to \infty$ then (1.19) implies that there is $L = L(\delta) > 0$ such that

$$\|u, \Omega \cap B^1_{x_0}\|_{l,p} \le \delta \ \text{ if } |x_0| > L \tag{1.22}$$

Let us rewrite the integral in (1.20) in the following form

$$\int_\Omega e^{-\varepsilon|x-x_0|} \|u, \Omega \cap B^1_x\|^p_{l,p}\, dx = \int_{\Omega\cap B^L_0} e^{-\varepsilon|x-x_0|} \|u, \Omega \cap B^1_x\|^p_{l,p}\, dx + \\ + \int_{\Omega\backslash B^L_0} e^{-\varepsilon|x-x_0|} \|u, \Omega \cap B^1_x\|^p_{l,p}\, dx := I_1(x_0) + I_2(x_0) \tag{1.23}$$

The estimates (1.17) and (1.22) imply that

$$I_2(x_0) \le C_1 \|u, \Omega\backslash B^L_0\|_{b,l,p} \le C_1\delta \tag{1.24}$$

Since, according to (1.17) $\|u, \Omega\|_{b,l,p} \le \|R_b, \mathbb{R}\|_{0,\infty} := C_2$ then applying (1.19) again we derive that

$$I_1(x_0) \le e^{-\varepsilon(|x_0|-L)} C_3 \tag{1.25}$$

It follows from (1.25) that $I_1(|x_0|) \to 0$ when $|x_0| \to \infty$. Consequently the integral I_1 can be made arbitrary small (taking $|x_0|$ large enough). Since $\delta > 0$ is arbitrary then we have proved that $v \in \dot W^{l_1,p}_b(\Omega)$. The estimate (1.21) follows immediately from the fact that all constants used before depends only on R_B and ε and independent of the concrete choice of u. Proposition 1.4 is proved.

Definition 1.5. In slight abuse of notations we denote by $L^p_{loc}(\mathbb{R}_+, L^p_b(\Omega))$ the F-space generated by the following sequence of seminorms

$$\|u\|^p_T := \sup_{x_0\in\Omega} \left(\int_T^{T+1} \|u(t), \Omega \cap B^1_{x_0}\|^p_{0,p}\, dt \right) \tag{1.26}$$

Moreover, we define the space $L^p_{loc}(\mathbb{R}_+, \dot L^p_b(\Omega))$ as a subspace of functions from $L^p_{loc}(\mathbb{R}_+, L^p_b(\Omega))$ which satisfies the condition

$$\lim_{|x_0|\to\infty} \left(\int_T^{T+1} \|u(t), \Omega \cap B^1_{x_0}\|^p_{0,p}\, dt \right) = 0 \tag{1.27}$$

for every fixed $T \geq 0$.

Remark 1.2. The assertions of Propositions 1.3 and 1.4 have the evident analogues for the spaces introduced in Definition 1.5, which will be essentially used in Sections 6, 8 and 12, namely:

1. A set B is compact in $L^p_{loc}(\mathbb{R}_+, \dot{L}^p_b(\Omega))$ if and only if it's x-restrictions on V_{x_0} are compact in $L^p_{loc}(\mathbb{R}_+, L^p(V_{x_0}))$ for every $x_0 \in \Omega$ and for every $T \geq 0$ there is a function $R_{B,T}(z)$ which tends to zero when $z \to \infty$ such that

$$\int_T^{T+1} \|u(t), \Omega \cap B^1_{x_0}\|^p_{0,p}\, dt \leq R_{B,T}(|x_0|), \quad \forall u \in B \tag{1.28}$$

2. Assume that the function v satisfies the estimate

$$\|v(t), \Omega \cap B^1_{x_0}\|^p_{l_1,p} \leq \int_0^t e^{-\gamma(t-s)} \int_\Omega e^{-\varepsilon|x-x_0|} \|u(s), \Omega \cap B^1_x\|^p_{l,p}\, dx\, ds \tag{1.29}$$

with the appropriate $\varepsilon, \gamma > 0$. Assume also that u satisfies (1.28) where functions $R_{B,T}(z) \equiv R_B(z)$ are independent of T. Then the function $v \in L^\infty(\mathbb{R}_+, \dot{W}^{l_1,p}_b(\Omega))$. Moreover

$$\|v(t), \Omega \cap B^1_{x_0}\|_{l_1,p} \leq R_1(|x_0|), \quad \forall t \in \mathbb{R}_+, \quad x_0 \in \Omega \tag{1.30}$$

where R_1 depends only on R_B and constants $\varepsilon, \gamma > 0$ and $\lim_{z\to\infty} R_1(z) = 0$.

The proof of these assertions are analogous to the ones given in Propositions 1.3 and 1.4.

§2 The linear equation

In this Section we derive a number of regularity results for the following linear hyperbolic equation

$$\begin{cases} \partial_t^2 v + \gamma \partial_t v - \Delta_x v + \lambda_0 v + l(t,x) v = h(t) \\ v\big|_{\partial\Omega} = 0 \\ v\big|_{t=0} = u_0 \ ; \ \ \partial_t v\big|_{t=0} = u_0' \end{cases} \tag{2.1}$$

in weighted Sobolev spaces.

It is assumed everywhere below that $\gamma, \lambda_0 > 0$ and the domain Ω satisfies the assumptions (1.5) and (1.6).

To simplify the notations we denote by $\xi_v(t)$ a pair of functions $(v(t), \partial_t v(t))$ and introduce the appropriate functional spaces for such pairs.

Definition 2.1. Let ϕ be a weight function which satisfies (1.1) and let $\kappa \geq 0$. Then

$$E^\kappa_\phi(\Omega) := \left(W^{1+\kappa,2}_\phi(\Omega) \cap \{v\big|_{\partial\Omega} = 0\}\right) \times W^{\kappa,2}_\phi(\Omega) \tag{2.2}$$

Analogously we define the spaces $E^\kappa_{loc}(\overline{\Omega})$, $E^\kappa_{b,\phi}(\Omega)$, $E^\kappa_b(\Omega)$ and $E^\kappa_{\{\varepsilon\}}(\Omega)$ (see also Definition 1.2). For simplicity we will omit the upper index κ in the case when $\kappa = 0$ (e.g. we will write $E_b(\Omega)$ instead of $E^0_b(\Omega)$). Moreover, we will write below $E^\kappa_{loc}(\Omega)$ instead of $E^\kappa_{loc}(\overline{\Omega})$

Theorem 2.1. *Let the right-hand side h of* (2.1) *belong to the space*

$$\cap_{\varepsilon>0} L^2_b(\mathbb{R}_+, L^2_{\{\varepsilon\}}(\Omega)) \tag{2.3}$$

and the function $l \in L^\infty(\mathbb{R}_+, L^p_b(\Omega))$ with $p > \max\{2, n\}$, i.e.

$$M := \sup_{t\ge 0} \|l(t), \Omega\|_{b,0,p} < \infty \tag{2.4}$$

Then for every $\xi_v(0) = (u_0, u_0') \in \cap_{\varepsilon>0} E_{\{\varepsilon\}}(\Omega)$ the problem (2.1) *has a unique solution ξ_v in the class*

$$\cap_{\varepsilon>0} C(\mathbb{R}_+, E_{\{\varepsilon\}}(\Omega)) \tag{2.5}$$

and the following estimate is valid:

$$\begin{aligned} &\|\partial_t v(t), \Omega \cap B^1_{x_0}\|^2_{0,2} + \|v, \Omega\cap B^1_{x_0}\|^2_{1,2} \le \\ &\le Ce^{-\delta t}\left(|\partial_t v(0)|^2 + |v(0)|^2 + |\nabla_x v(0)|^2, e^{-\varepsilon|x-x_0|}\right) + \\ &\qquad + C\int_0^t e^{-\delta(t-s)}\left(|h(s)|^2, e^{-\varepsilon|x-x_0|}\right) ds \end{aligned} \tag{2.6}$$

with a sufficiently small $\varepsilon > 0$ and $\delta = \alpha - CM^2$ where $\alpha, C > 0$ some constants depending on γ and λ_0. Note that the exponent δ is positive if M is small enough. (Here and below we denote by $(\cdot,\cdot)$ the standard inner product in $L^2(\Omega)$.)

Proof. We give below only the formal proof of the estimate (2.6) which can be justified in a standard way (see e.g. [14] and [20]).

Let us introduce a function $\theta(t) = \partial_t v + \alpha v$, where $\gamma > \alpha > 0$ is a sufficiently small parameter which will be fixed below, and rewrite the equation (2.1) in the following form

$$\partial_t\theta + (\gamma-\alpha)\theta - \alpha(\gamma-\alpha)v - \Delta_x v + \lambda_0 v + lv = h(s) \tag{2.7}$$

Multiplying this equation by $\theta\phi_{x_0} := \theta e^{-\varepsilon|x-x_0|}$ (where $\varepsilon > 0$ is another small parameter) and integrating over $x \in \Omega$ we obtain after the evident transformations that

$$\begin{aligned} &1/2\partial_t\big[\left(|\theta|^2, \phi_{x_0}\right) + \lambda_0\left(|v|^2, \phi_{x_0}\right) + \left(|\nabla_x v|^2, \phi_{x_0}\right)\big] + \\ &+ (\gamma-\alpha)\left(|\theta|^2, \phi_{x_0}\right) + \lambda_0\alpha\left(|v|^2, \phi_{x_0}\right) + \alpha\left(|\nabla_x v|^2, \phi_{x_0}\right) = \\ &= -\left(\nabla_x v, \theta\nabla_x\phi_{x_0}\right) + \alpha(\gamma-\alpha)\left(v, \theta\phi_{x_0}\right) - \left(lv, \theta\phi_{x_0}\right) + \left(h, \theta\phi_{x_0}\right) \end{aligned} \tag{2.8}$$

Note that $|\nabla_x\phi_{x_0}| \le \varepsilon\phi_{x_0}$. Consequently, fixing $\varepsilon = \varepsilon(\alpha,\gamma)$ small enough we will have the estimate

$$|\left(\nabla_x v, \theta\nabla_x\phi_{x_0}\right)| \le \varepsilon\left(|\nabla_x|, |\theta|\phi_{x_0}\right) \le \frac{\gamma-\alpha}{8}\left(|\theta|^2, \phi_{x_0}\right) + \frac{\alpha}{2}\left(|\nabla_x v|^2, \phi_{x_0}\right)$$

Analogously,

$$\begin{aligned} &|\alpha(\gamma-\alpha)(v, \theta\phi_{x_0})| \le 2\alpha^2(\gamma-\alpha)(|v|^2, \phi_{x_0}) + (\gamma-\alpha)/8(|\theta|^2, \phi_{x_0}) \\ &|(lv, \theta\phi_{x_0}) \le (\gamma-\alpha)/8(|\theta|^2, \phi_{x_0}) + 2/(\gamma-\alpha)(|lv|^2, \phi_{x_0}) \\ &|(h, \theta\phi_{x_0})| \le (\gamma-\alpha)/8(|\theta|^2, \phi_{x_0}) + 2/(\gamma-\alpha)(|h|^2, \phi_{x_0}) \end{aligned}$$

Fixing now the small parameter $\alpha > 0$ in such a way that

$$\gamma - \alpha \geq \alpha, \quad \text{and} \quad 2\lambda_0 - 2\alpha(\gamma-\alpha) \geq \lambda_0$$

and inserting the obtained estimates to the equality (2.8) we derive that

$$\partial_t F_{x_0}(\theta, v) + \alpha F_{x_0}(\theta, v) \leq C\left(|h(t)|^2, \phi_{x_0}\right) + 4/(\gamma-\alpha)\left(|lv|^2, \phi_{x_0}\right) \tag{2.9}$$

where

$$F_{x_0}(\theta, v) := \left(|\theta|^2, \phi_{x_0}\right) + \lambda_0\left(|v|^2, \phi_{x_0}\right) + \left(|\nabla_x v|^2, \phi_{x_0}\right) \tag{2.10}$$

Thus, it remains to estimate the term $(|lv|^2, \phi_{x_0})$. To this end we will use a trick based on (1.7) with the weight function ϕ_{x_0}, Sobolev's embedding theorem $W^{1,2} \subset L^{q_0}$ for $\frac{1}{q_0} = \frac{1}{2} - \frac{1}{n}$, and Holder inequality with the exponents $p/2$ and $q = (p/2)^* < q_0/2$ (since $p > \max\{2, n\}$). Indeed,

$$\begin{aligned} (2.11)\quad \left(|lv|^2, \phi_{x_0}\right) &\leq C\int_\Omega \phi_{x_0}(x)\|lv, V_{x_0}\|^2_{0,2}\,dx \leq \\ &\leq C_1\int_\Omega \phi_{x_0}(x)\|l, V_x\|^2_{0,p}\|v, V_x\|^2_{0,2q}\,dx \leq C_2\|l(t)\|^2_{b,0,p}\int_\Omega \phi_{x_0}(x)\|v, V_x\|^2_{1,2}\,dx \\ &\leq C_3M^2\left[\left(|v|^2, \phi_{x_0}\right) + \left(|\nabla_x v|^2, \phi_{x_0}\right)\right] \leq C_4M^2F_{x_0}(\theta, v) \end{aligned}$$

Here the domain V_x is the same as in (1.5) and (1.6).

Inserting this estimate into (2.9) we will finally obtain that

$$\partial_t F_{x_0}(\theta, v) + (\alpha - C_5M^2)F_{x_0}(\theta, v) \leq C\left(|h(t)|^2, \phi_{x_0}\right) \tag{2.12}$$

Applying the Gronewal's inequality to (2.12) we derive the estimate

$$F_{x_0}(\theta(t), v(t)) \leq e^{-\delta t}F_{x_0}(\theta(0), v(0)) + C\int_0^t e^{-\delta(t-s)}\left(|h(s)|, \phi_{x_0}\right)\,ds \tag{2.13}$$

where $\delta := \alpha - C_5M^2$.

To complete the proof of the theorem it remains to use the following evident estimates

$$\begin{aligned} (2.14)\quad \|\partial_t v, \Omega\cap B^1_{x_0}\|^2_{0,2} + \|v, \Omega\cap B^1_{x_0}\|^2_{1,2} &\leq C\left(|\partial_t v|^2 + |v|^2 + |\nabla_x v|^2, \phi_{x_0}\right) \leq \\ &\leq C_1F(\partial_t v + \alpha v, v) \leq C_2\left(|\partial_t v|^2 + |v|^2 + |\nabla_x v|^2, \phi_{x_0}\right) \end{aligned}$$

Theorem 2.1 is proved.

Remark 2.1. Note that the estimate (2.6) remains valid if $h \in L^2_b(\mathbb{R}_+, L^2_{\{\varepsilon_0\}}(\Omega))$ and $\xi_v(0) \in E_{\{\varepsilon_0\}}(\Omega)$ if $\varepsilon_0 < \varepsilon$. Note also that if $n > 2$ then Theorem 2.1 remains valid for $p = n$ as well. Indeed, we have used the assumption $p > n$ only in the estimate (2.11) but since $q_0 < \infty$ for $n \geq 3$ then we may take $q = q_0/2$ which corresponds to $p = n$).

Corollary 2.1. *Let $\varepsilon > 0$ be the same as in Theorem 2.1 and let ϕ be a weight function with the rate of growth $\mu < \varepsilon$. Assume also that the initial conditions for $\xi_v(0) \in E_\phi(\Omega)$ and the right-hand side $h \in L^2_b(\mathbb{R}_+, L^2_\phi(\Omega))$. The the problem* (2.1) *has a unique solution*

$$\xi_v \in C(\mathbb{R}_+, E_\phi(\Omega)) \tag{2.15}$$

and the following estimate holds

$$\|\xi_v(t)\|^2_{E_\phi} \le Ce^{-\delta t}\|\xi_v(0)\|^2_{E_\phi} + C\int_0^t e^{-\delta(t-s)}\|h(s)\|^2_{L^2_\phi}\, ds \tag{2.16}$$

where δ is the same as in Theorem 2.1 and the constant C depends only on the corresponding constant in the estimate (2.6) *and on the constants C_ϕ and μ from* (1.1) *and independent of the particular choice of the weight ϕ (satisfying* (1.1)*).*

Moreover, assume that $h \in L^2_b(\mathbb{R}_+, L^2_{b,\phi}(\Omega)$ and $\xi_v(0) \in E_{b,\phi}(\Omega)$. Then, analogously,

$$\|\xi_v(t)\|^2_{E_{b,\phi}} \le Ce^{-\delta t}\|\xi_v(0)\|^2_{E_{b,\phi}} + C(1+e^{-\delta t})\|h\|^2_{L^2_{b,\phi}([0,t]\times\Omega)} \tag{2.17}$$

Indeed, multiplying the estimate (2.6) by $\phi(x_0)$, integrating over $x_0 \in \Omega$ and using the estimates (1.3) and (1.7) we obtain the estimate (2.16). The continuity in (2.15) can be derived by standard arguments (see e.g. [10] and [20]).

The estimate (2.17) can be derived analogously only instead of integrating over $x_0 \in \Omega$ one should take a supremum and use (1.3) with $q = \infty$.

The following Theorem, which establishes the additional regularity of solutions of (2.1) has a fundamental significance for our estimation of ε-entropy.

Theorem 2.2. *Let the right-hand side $h \in L^2_b(\mathbb{R}_+, W^{\kappa,2}_{\{\varepsilon_0\}}(\Omega))$ and the initial conditions $\xi_v(0) \in E^\kappa_{\{\varepsilon_0\}}(\Omega)$ for a certain $0 < \kappa < 1/2$ and $\varepsilon_0 > 0$. Assume also that*

$$l \in L^\infty(\mathbb{R}_+, L^p_b(\Omega)) \quad \text{and} \quad \nabla_x l \in L^\infty(\mathbb{R}_+, L^q_b(\Omega)) \tag{2.18}$$

and the exponents p, q and κ satisfy the following relations

$$1.\ \ \frac{1}{q} \ge \frac{1}{p} + \frac{1}{n},\ \ 2.\ \ \frac{1}{q} \le \frac{1}{2} + \frac{1+\kappa}{n},\ \ 3.\ \ \frac{1+\kappa}{n} \ge \frac{1-\kappa}{p} + \frac{\kappa}{q} \tag{2.19}$$

for $n \ge 3$ (for $n < 3$ the second inequality in (2.19) *may be omitted and the first one should be replaced by $\frac{1}{q} \ge \frac{1}{p} + \frac{1}{2} - \frac{\kappa}{n}$).*

Then the solution ξ_v of (2.1) *belongs to the space $C(\mathbb{R}_+, E^\kappa_{\{\varepsilon_0\}}(\Omega))$ and the following estimate is valid*

$$\begin{aligned} \|v(t), \Omega\cap B^1_{x_0}\|^2_{1+\kappa,2} + \|\partial_t v(t), \Omega\cap B^1_{x_0}\|^2_{\kappa,2} \le \\ \le Ce^{Kt}\int_\Omega e^{-\varepsilon|x-x_0|}\left(\|v(0), \Omega\cap B^1_x\|^2_{1+\kappa,2} + \|\partial_t v(0), \Omega\cap B^1_x\|^2_{\kappa,2}\right) dx + \\ + C\int_0^t e^{K(t-s)}\int_\Omega e^{-\varepsilon|x-x_0|}\|h(s), \Omega\cap B^1_x\|^2_{\kappa,2} dx\, ds \end{aligned} \tag{2.20}$$

for a some positive constant K and $\varepsilon > \varepsilon_0$.

Proof. Let us define a family of cut-off functions $\psi_{x_0}(x) \in C_0^\infty(\mathbb{R}^n)$ in such a way that $\psi_{x_0} = 1$ for $x \in B^2_{x_0} \cap \Omega$ and $\psi_{x_0} = 0$ for $x \in \Omega\backslash V_{x_0}$. Moreover, due to our assumptions on the domains V_{x_0} we may assume that

$$\|\psi_{x_0}\|_{C^2(\mathbb{R}^n)} \le C$$

uniformly with respect to x_0. Let us introduce also the functions $v_{x_0}(t) := v(t)\psi_{x_0}$. Then

$$(2.21)\quad \begin{aligned} &\partial_t^2 v_{x_0} + \gamma\partial_t v_{x_0} - \Delta_x v_{x_0} + \lambda_0 v_{x_0} + l(t,x)v_{x_0} = \\ &\quad = h(t)\psi_{x_0} - \Delta_x\psi_{x_0} v - 2\nabla_x\psi_{x_0}\nabla_x v := h_{x_0}(t), \quad v_{x_0}\big|_{\partial V_{x_0}} = 0 \end{aligned}$$

The proof of the Theorem is based on the following two lemmata.

Lemma 2.2. *Let the previous assumptions hold. Then, for every $w \in W^{1+\kappa}(V_{x_0})$ the following estimate is valid*

$$(2.22)\qquad \|lw, V_{x_0}\|_{\kappa,2} \le C\|w, V_{x_0}\|_{1+\kappa,2}$$

Moreover, the constant C is independent of x_0.

Proof. Let us consider firstly the case $n \ge 3$.

Since $l \in L^p(V_{x_0})$ and $w \in W^{1+\kappa}(V_{x_0}) \subset L^r(V_{x_0})$ with $1/r = 1/2 - (1+\kappa)/n$ then according to Holder inequality

$$(2.23)\qquad \|lw, V_{x_0}\|_{0,k} \le C\|l, V_{x_0}\|_{0,p}\|w, V_{x_0}\|_{1+\kappa} \le C_1\|w, V_{x_0}\|_{1+\kappa,2}$$

where $\frac{1}{k} = \frac{1}{p} + \frac{1}{r}$.

Consider now the function $\nabla_x(lw) = \nabla_x lw + l\nabla_x w$. Since $\nabla_x l \in L^q$ and $w \in L^r$ then $\nabla_x lw \in L^{k'}$ where $\frac{1}{k'} = \frac{1}{q} + \frac{1}{r}$. Analogously, since $l \in L^p$ and $\nabla_x w \in W^{\kappa,2} \subset L^{r'}$ with $1/r' = 1/2 - \kappa/n$ then $l\nabla_x w \in L^{k''}$ with $1/k'' = 1/p + 1/r'$.

Note now that the first and the second conditions of (2.19) guarantee that k' and k'' are well defined (i.e. $k', k'' \ge 1$) and in addition $k' \le k''$. Consequently, we have proved that

$$(2.24)\qquad \|lw, V_{x_0}\|_{1,k'} \le C_1\|w, V_{x_0}\|_{1+\kappa,2}$$

According to the interpolation inequality (see, e.g., [19])

$$(2.25)\qquad \|lw, V_{x_0}\|_{\delta,\tilde{k}} \le C\|lw, V_{x_0}\|_{0,k}^{1-\kappa}\|lw, V_{x_0}\|_{1,k'}^{\kappa} \le C_2\|w, V_{x_0}\|_{1+\kappa,2}$$

with $\frac{1}{\tilde{k}} = \frac{1-\kappa}{k} + \frac{\kappa}{k'}$. It remains to note now that the third assumption of (2.19) implies $\tilde{k} \ge 2$.

Thus, we have proved the lemma if $n \ge 3$. For the case $n \le 2$ the estimate (2.22) can be derived analogously (the only difference that we should put $r = \infty$). (Note also that the assumptions (1.5) on V_{x_0} and assumptions (2.18) on l imply that all constants introduced in the proof of the lemma are in fact independent of x_0). Lemma 2.2 is proved

Lemma 2.3. *Let the previous assumptions hold. Then*

$$\begin{aligned}(2.26)\quad &\|v_{x_0}(t),V_{x_0}\|^2_{1+\kappa,2}+\|\partial_t v_{x_0}(t),V_{x_0}\|^2_{\kappa,2}\le\\ &Ce^{Kt}\left(\|v_{x_0}(0),V_{x_0}\|^2_{1+\kappa,2}+\|\partial_t v_{x_0}(0),V_{x_0}\|^2_{\kappa,2}\right)+\\ &+C\int_0^t e^{K(t-s)}\|h(s),V_{x_0}\|^2_{\kappa,2}+C\int_0^t e^{K(t-s)}\|v(s),V_{x_0}\|^2_{1+\kappa,2}ds\end{aligned}$$

where the constants C and K are independent of x_0.

Proof. Let us introduce the scale of Hilbert spaces $H^l(V_{x_0}):=(-\Delta_x)^{l/2}L^2(V_{x_0})$ with the norms $\|\cdot\|_{H^l}:=\|(-\Delta_x)^{l/2}\cdot\|_{0,2}$ where the Laplacian is endowed by zero Dirihlet boundary conditions on ∂V_{x_0}. Since the domains V_{x_0} are smooth then as known (see e.g., [19], [20]), $H^l(V_{x_0})=W^{l,2}(V_{x_0})$ for $1/2>l>-5/2$, $l\ne -1/2$ and

$$(2.27)\qquad C_l\|w,V_{x_0}\|^2_{l,2}\le\|w,V_{x_0}\|^2_{H^l}=\left(w,(-\Delta_x)^l w\right)\le C'_l\|w,V_{x_0}\|^2_{l,2}$$

Moreover, according our assumptions on domains V_{x_0} the constants C_l and C'_l in (2.27) are independent of x_0.

Let us apply the operator $(-\Delta_x)^{\kappa/2}$ to both sides of the equation (2.21) and denote $\theta_{x_0}(t):=(-\Delta_x)^{\kappa/2}v_{x_0}(t)$. We obtain that

$$(2.28)\qquad\begin{cases}\partial_t^2\theta_{x_0}+\gamma\partial_t\theta_{x_0}-\Delta_x\theta_{x_0}+\lambda_0\theta_{x_0}+\\ \qquad\qquad+(-\Delta_x)^{\kappa/2}(l(t,x)v_{x_0})=(-\Delta_x)^{\kappa/2}h_{x_0}(t):=h_{x_0,\kappa}(t)\\ \theta_{x_0}\big|_{\partial V_{x_0}}=0\\ \theta_{x_0}\big|_{t=0}=(-\Delta_x)^{\kappa/2}v_{x_0},\quad \partial_t\theta_{x_0}\big|_{t=0}=(-\Delta_x)^{\kappa/2}\partial_t v_{x_0}\end{cases}$$

According to the estimates (2.27) and the definition of the function $h_{x_0,\kappa}$ we derive that

$$(2.29)\qquad \|h_{x_0,\kappa},V_{x_0}\|^2_{0,2}\le C\left(\|h,V_{x_0}\|^2_{\kappa,2}+\|v,V_{x_0}\|^2_{1+\kappa,2}\right)$$

Multiplying the equation (2.28) by $\partial_t\theta_{x_0}$ and integrating over Ω we obtain (as usual) that

$$\begin{aligned}(2.30)\quad 1/2\partial_t&\left(\|\partial_t\theta_{x_0},V_{x_0}\|^2_{0,2}+\|\theta_{x_0},V_{x_0}\|^2_{1,2}+\lambda_0\|\theta_{x_0},V_{x_0}\|^2_{0,2}\right)=\\ &=-\Big((-\Delta_x)^{\kappa/2}(lv_{x_0}),\partial_t\theta_{x_0}\Big)+(h_{x_0,\kappa}(t),\partial_t\theta_{x_0})\end{aligned}$$

Applying the inequalities (2.22) and (2.27) to the first term in the right-hand side of (2.30) we derive

$$\begin{aligned}(2.31)\quad &|\Big((-\Delta_x)^{\kappa/2}(lv_{x_0}),\partial_t\theta_{x_0}\Big)|\le C\left(\|lv_{x_0},V_{x_0}\|^2_{\kappa,2}+\|\partial_t\theta_{x_0},V_{x_0}\|^2_{0,2}\right)\le\\ &\le C_1\left(\|v_{x_0},V_{x_0}\|^2_{1+\kappa,2}+\|\partial_t\theta_{x_0},V_{x_0}\|^2_{0,2}\right)\le C_2\left(\|\theta_{x_0},V_{x_0}\|^2_{1,2}+\|\partial_t\theta_{x_0},V_{x_0}\|^2_{0,2}\right)\end{aligned}$$

Inserting (2.31) and (2.29) into the right-hand side of (2.30) we obtain the inequality

$$\begin{aligned}(2.32)\quad \partial_t&\left(\|\partial_t\theta_{x_0},V_{x_0}\|^2_{0,2}+\|\theta_{x_0},V_{x_0}\|^2_{1,2}+\lambda_0\|\theta_{x_0},V_{x_0}\|^2_{0,2}\right)\le\\ &\le K\left(\|\partial_t\theta_{x_0},V_{x_0}\|^2_{0,2}+\|\theta_{x_0},V_{x_0}\|^2_{1,2}+\lambda_0\|\theta_{x_0},V_{x_0}\|^2_{0,2}\right)+\\ &\qquad+C\left(\|h,V_{x_0}\|^2_{\kappa,2}+\|v,V_{x_0}\|^2_{1+\kappa,2}\right)\end{aligned}$$

Applying the Gronewal inequality to (2.32) we derive (2.26) Lemma 2.3 is proved.

Now we are in position to complete the proof of Theorem 2.2. To this end we note that according to our choice of the cut-off functions ψ_{x_0}

$$(2.33)\quad \|v,\Omega\cap B^1_{x_0}\|^2_{1+\kappa,2}+\|\partial_t v,\Omega\cap B^1_{x_0}\|^2_{\kappa,2}\le \\ \le C\left(\|v_{x_0}(t),V_{x_0}\|^2_{1+\kappa,2}+\|\partial_t v_{x_0}(t),V_{x_0}\|^2_{\kappa,2}\right)$$

Multiplying the estimate (2.26) by $e^{-\varepsilon|z-x_0|}$, $\varepsilon>\varepsilon_0$ and integrating over $x_0\in\Omega$ we derive now that

$$(2.34)\quad \int_{x_0\in\Omega} e^{-\varepsilon|z-x_0|}(\|v(t),\Omega\cap B^1_{x_0}\|^2_{1+\kappa,2}+\|\partial_t v(t),\Omega\cap B^1_{x_0}\|^2_{\kappa,2})dx_0\le \\ \le Ce^{Kt}\int_{x_0\in\Omega} e^{-\varepsilon|z-x_0|}\left(\|v_{x_0}(0),V_{x_0}\|^2_{1+\kappa,2}+\|\partial_t v_{x_0}(0),V_{x_0}\|^2_{\kappa,2}\right)dx_0+ \\ +C\int_0^t e^{K(t-s)}\int_{x_0\in\Omega} e^{-\varepsilon|z-x_0|}\|h(s),V_{x_0}\|^2_{\kappa,2}dx_0+ \\ +C\int_0^t e^{K(t-s)}\int_{x_0\in\Omega} e^{-\varepsilon|z-x_0|}\|v(s),V_{x_0}\|^2_{1+\kappa,2}dx_0ds$$

Let us transform the last term into the right-hand side of (2.34). Indeed, according to (1.13) and (1.15)

$$(2.35)\quad \int_{x_0\in\Omega} e^{-\varepsilon|z-x_0|}\|v(s),V_{x_0}\|^2_{1+\kappa,2}dx_0\le \\ \le C\int_{x_0\in\Omega} e^{-\varepsilon|z-x_0|}\int_{x\in V_{x_0}}\|v(s),\Omega\cap B^1_x\|^2_{1+\kappa,2}\,dx\,dx_0\le \\ \le C_1\int_{x_0\in\Omega} e^{-\varepsilon|z-x_0|}\int_{x\in\Omega} e^{-2\varepsilon|x-x_0|}\|v(s),\Omega\cap B^1_x\|^2_{1+\kappa,2}\,dx\,dx_0\le \\ \le C_2\int_{x_0\in\Omega} e^{-\varepsilon|z-x_0|}\|v(s),\Omega\cap B^1_{x_0}\|^2_{1+\kappa,2}dx_0$$

Inserting this estimate to (2.34) and denoting

$$(2.36)\quad R_z(v(s),\partial_t v(s)):= \\ =\int_{x_0\in\Omega} e^{-\varepsilon|z-x_0|}\left(\|v(s),\Omega\cap B^1_{x_0}\|^2_{1+\kappa,2}+\|\partial_t v(s),\Omega\cap B^1_{x_0}\|^2_{1+\kappa,2}\right)dx_0$$

we derive the following inequality

$$(2.37)\quad R_z(v(t),\partial_t v(t))\le Ce^{Kt}R_z(v(0),\partial_t v(0))+ \\ C\int_0^t e^{K(t-s)}R_z(v(s),\partial_t v(s))\,ds+C\int_0^t e^{K(t-s)}\int_{x_0\in\Omega} e^{-\varepsilon|z-x_0|}\|h(s),V_{x_0}\|^2_{\kappa,2}dx_0$$

Applying the Gronewal inequality to (2.37) we obtain that

$$(2.38)\quad R_z(v(t),\partial_t v(t))\le C_1e^{K_1t}R_z(v(0),\partial_t v(0))+ \\ +C_1\int_0^t e^{K_1(t-s)}\int_{x_0\in\Omega} e^{-\varepsilon|z-x_0|}\|h(s),V_{x_0}\|^2_{\kappa,2}dx_0$$

Applying the estimate (1.13) again we derive

$$\|v,\Omega\cap B^1_z\|^2_{1+\kappa,2}+\|\partial_t v,\Omega\cap B^1_z\|^2_{\kappa,2}\le \|v,V_z\|^2_{1+\kappa,2}+\|\partial_t v,V_z\|^2_{\kappa,2}\le \tag{2.39}$$
$$\le C\int_{x_0\in V_{x_0}}\|v,\Omega\cap B^1_{x_0}\|^2_{1+\kappa,2}+\|\partial_t v,\Omega\cap B^1_{x_0}\|^2_{\kappa,2}\,dx_0\le$$
$$\le C_1\int_{x_0\in\Omega}e^{-\varepsilon|z-x_0|}\left(\|v(s),\Omega\cap B^1_{x_0}\|^2_{1+\kappa,2}+\|\partial_t v(s),\Omega\cap B^1_{x_0}\|^2_{1+\kappa,2}\right)dx_0$$

The estimates (2.38) and (2.39) imply (after the replacing $x_0:\to x$ and $z:\to x_0$) the inequality (2.20). Theorem 2.2 is proved.

The following Corollary is a complete analogue of Corollary 2.1.

Corollary 2.2. *Let ε_0, p, q, and κ be the same as in Theorem 2.2 and let ϕ be a weight function with the rate of growth $\mu<\varepsilon_0$. Assume also that the initial conditions for $\xi_v(0)\in E^\kappa_\phi(\Omega)$ and the right-hand side $h\in L^2_b(\mathbb{R}_+,W^{\kappa,2}_\phi(\Omega))$. Then the solution ξ_v of the problem* (2.1) *belong to*

$$\xi_v\in C(\mathbb{R}_+,E^\kappa_\phi(\Omega))$$

and the following estimate holds

$$\|\xi_v(t)\|^2_{E^\kappa_\phi}\le Ce^{Kt}\|\xi_v(0)\|^2_{E^\kappa_\phi}+C\int_0^t e^{K(t-s)}\|h(s)\|^2_{W^{\kappa,2}_\phi}\,ds \tag{2.40}$$

where K is the same as in Theorem 2.2 and the constant C depends only on the corresponding constant in the estimate (2.20) *and on the constants C_ϕ and μ from* (1.1) *and independent of the particular choice of the weight ϕ (satisfying* (1.1)*).*

§3 The nonlinear equation.

In this Section we consider the following nonlinear damped hyperbolic equation in the unbounded domain Ω:

$$\begin{cases}\partial_t^2u+\gamma\partial_t u-\Delta_x u+\lambda_0u+f(u)=g(t), & u\big|_{\partial\Omega}=0\\ u\big|_{t=0}=u_0;\quad \partial_t u\big|_{t=0}=u_0'\end{cases} \tag{3.1}$$

As in previous Section it is assumed that $\gamma,\lambda_0>0$ and the domain Ω satisfies the assumptions (1.5) and (1.6).

We assume also that the nonlinear term $f\in C^2(\mathbb{R},\mathbb{R})$ and satisfies the following conditions

$$\begin{cases}1.\ f'\ge -K\ ,\quad |f'(v)|\le C(1+|v|^{q_1})\text{ with }q_1<2/(n-2)\text{ if }n>2\\ 2.\ |f''(v)|\le C(1+|v|^{q_2}),\text{ where }q_2=1\text{ if }n=3\text{ and }q_2=0\text{ for }n\ge 3\\ 3.\ f=f_1+f_2,\text{ with }f_1(v)v\ge 0,\text{ and }|f_2(v)|+|f_2'(v)|+|f_2''(v)|\le C\end{cases} \tag{3.2}$$

Note also, that without loss of generality we may suppose that $f_2(0)=0$.

We restrict ourselves to consider only *bounded* with respect to $|x|\to\infty$ solutions $\xi_u:=(u,\partial_t u)$ of the equation (3.1), i.e.

$$\xi_u\in L^\infty_{loc}(\mathbb{R}_+,E_b(\Omega)) \tag{3.3}$$

And consequently we assume that $\xi_u(0) := (u_0, u_0') \in E_b(\Omega)$ and $g \in L_b^2(\mathbb{R}_+ \times \Omega)$.

Remark 3.1. Note that (3.3) together with the growth restrictions on $f(u)$ and Sobolev embedding theorem imply that

$$f(u) \in L^\infty_{loc}(\mathbb{R}_+, L^2_b(\Omega))$$

Therefore, the equation (3.1) can be understood in the sense of distributions. Moreover it follows from Theorem 2.1 that the solution ξ_u satisfies (2.5) and consequently the initial conditions are also well posed.

The main result of this Section is the following theorem.

Theorem 3.1. *Let the above assumptions hold. Then for every $\xi_u(0) \in E_b(\Omega)$ the problem* (3.1) *has a unique solution (in the class* (3.3)*) and the following estimate is valid:*

$$\begin{aligned}(3.4)\quad &\|\partial_t u(t), \Omega \cap B^1_{x_0}\|^2_{0,2} + \|u(t), \Omega \cap B^1_{x_0}\|^2_{1,2} + \|F_1(u(t), \Omega \cap B^1_{x_0}\|_{0,1} \le \\ &\le Ce^{-\delta t}\left(|\partial_t v(0)|^2 + |\nabla_x v(0)|^2 + |v(0)|^2 + F_1(v(0)), e^{-\varepsilon|x-x_0|}\right) + \\ &+ C(f_2) + \int_0^t e^{-\delta(t-s)}\left(|g(s)|^2, e^{-\varepsilon|x-x_0|}\right)\,ds\end{aligned}$$

where $F_1(z) := \int_0^z f_1(s)ds \ge 0$, δ, ε are sufficiently small positive constants, and the constant $C(f_2)$ is such that $C(f_2) = 0$ if $f_2 \equiv 0$.

Proof. Let us check for the first the uniqueness of solutions. Indeed, let u_1 and u_2 be two solutions of the problem (3.1). Then their difference $v = u_1 - u_2$ satisfies the equation

$$(3.5)\qquad \partial_t^2 v + \gamma\partial_t v - \Delta_x v + \lambda_0 v + l(t)v = 0,\ \xi_v(0) = 0$$

with $l(t) = l(t,x) := \int_0^1 f'(su_1(t) + (1-s)u_2(t))\,ds$. Note, that according to our growth restrictions on f' and Sobolev embedding Theorem

$$(3.6)\quad \|l, \Omega\|_{L^p_b} \le C(1 + \|u_1, \Omega\|^{q_1}_{L^{pq_1}_b} + \|u_2, \Omega\|^{q_1}_{L^{pq_1}_b}) \le C_1(1 + \|\xi_{u_1}\|^{q_1}_{E_b(\Omega)} + \|\xi_{u_2}\|^{q_1}_{E_b(\Omega)})$$

for $p = \frac{2n}{(n-2)q_1} > \max\{2, n\}$. Therefore the equation (3.5) satisfies the assumptions of Theorem 2.1 and according to this Theorem has a unique zero solution. Thus, $u_1 \equiv u_2$ and consequently the uniqueness is proved.

Let us derive now the estimate (3.4). To this end we introduce a function $\theta = \partial_t u + \alpha u$, multiply the equation (3.1) by $\theta e^{-\varepsilon|x-x_0|} := \theta\phi_{x_0}$ (where α and ε are small positive parameters) and integrate over $x \in \Omega$. Then, arguing as in the proof of Theorem 2.1, we derive that there exists small positive constants α and ε such that

$$\begin{aligned}(3.7)\quad &\partial_t\left[\left(|\theta|^2, \phi_{x_0}\right) + \lambda_0\left(|u|^2, \phi_{x_0}\right) + \left(|\nabla_x u|^2, \phi_{x_0}\right)\right] + \\ &+ \alpha\left[\left(|\theta|^2, \phi_{x_0}\right) + \lambda_0\left(|u|^2, \phi_{x_0}\right) + \left(|\nabla_x u|^2, \phi_{x_0}\right)\right] + \\ &+ 2\left(f(u), \theta\phi_{x_0}\right) \le C\left(|g(t)|^2, \phi_{x_0}\right)\end{aligned}$$

Let us estimate the last term into the left-hand side of (3.7) using the decomposition $f = f_1 + f_2$ and the boundedness of f_2

$$(3.8)\quad (f(u),\theta\phi_{x_0}) = (f_1(u),\partial_t u\phi_{x_0}) + \alpha\,(f_1(u)u,\phi_{x_0}) + (f_2(u),\theta\phi_{x_0}) \geq \\ \geq \partial_t\,(F_1(u),\phi_{x_0}) + \alpha\,(f_1(u)u,\phi_{x_0}) - \alpha/4\left(|\theta|^2,\phi_{x_0}\right) - C(f_2)$$

where $C(f_2)=0$ if $f_2 \equiv 0$.

Note that (rescaling if necessary the positive parameter λ_0) we may assume without loss of generality that $f_1(z)z \geq \beta z^2$ for a small positive β. Using this estimate and the fact that $f_1' \geq -K$ we derive that

$$(3.9)\quad F_1(z) = \int_0^z (f_1(z)+Kz)\,dz - Kz^2/2 \leq f_1(z)z + Kz^2 - Kz^2/2 = \\ = f_1(z) + Kz/2 \leq (1+K/2\beta)f_1(z)z$$

Inserting this estimate to (3.8) we obtain

$$(3.10)\quad (f(u),\theta\phi_{x_0}) \geq \partial_t\,(F_1(u),\phi_{x_0}) + \beta_1\,(F_1(z),\phi_{x_0}) - \alpha/4\left(|\theta|^2,\phi_{x_0}\right) - C(f_2)$$

with $\beta_1 = 2\alpha\beta/(K+2\beta)$.

Estimating the last term into the right-hand side of (3.7) using (3.10) we derive that

$$(3.11)\quad \partial_t\big[\left(|\theta|^2,\phi_{x_0}\right) + \lambda_0\left(|u|^2,\phi_{x_0}\right) + \left(|\nabla_x u|^2,\phi_{x_0}\right) + 2\,(F_1(u),\phi_{x_0})\big] + \\ + \alpha_1\big[\left(|\theta|^2,\phi_{x_0}\right) + \lambda_0\left(|u|^2,\phi_{x_0}\right) + \left(|\nabla_x u|^2,\phi_{x_0}\right) + 2\,(F_1(u),\phi_{x_0})\big] \leq \\ \leq C\left(|g(t)|^2,\phi_{x_0}\right) + C_1(f_2)$$

for a sufficiently small $\alpha_1 > 0$.

Applying now the Gronewal inequality to (3.11) and arguing then as in the end of the proof of Theorem 2.1 we obtain the estimate (3.4).

Let us discuss now the existence of a solution. To this end we need the following Lemma which reflects the well known fact of finite velocity of wave's propagation for hyperbolic equations.

Lemma 3.1. *Let the above assumptions hold. Then, for every $x_0 \in \Omega$, $R \in \mathbb{R}_+$ and $T < R$ the following estimates is valid:*

$$(3.12)\quad \|u(T),\Omega\cap B_{x_0}^{R-T}\|_{1,2}^2 + \|\partial_t u(T),\Omega\cap B_{x_0}^{R-T}\|_{0,2}^2 \leq \\ \leq Ce^{LT}\Big(\|u(0),\Omega\cap B_{x_0}^{R}\|_{1,2}^2 + \|\partial_t u(0),\Omega\cap B_{x_0}^{R}\|_{0,2}^2 + \\ + \|F_1(u(0)),\Omega\cap B_{x_0}^{R}\|_{0,1} + \int_0^T \|g(t),\Omega\cap B_{x_0}^{R-t}\|_{0,2}^2\,dt\Big)$$

for a some constants $C, L > 0$.

Proof. Indeed, multiplying the equation (3.1) by $\partial_t u$, integrating over the 'cone'

$$K(T,R,x_0,\Omega) := \{t \in [0,T],\ x \in \Omega\cap B_{x_0}^{r-t}\}$$

and using the Green's formula and the fact that $F_1 \ge 0$ we obtain (analogously to [14] and [23]) that

$$\int_{\Omega\cap B^{R-T}_{x_0}} |\partial_t u(T)|^2 + |\nabla_x u(T)|^2 + \lambda_0|u(T)|^2 + 2F_1(u(T))\,dx \le \tag{3.13}$$
$$\le \int_{\Omega\cap B^{R}_{x_0}} |\partial_t u(0)|^2 + |\nabla_x u(0)|^2 + \lambda_0|u(0)|^2 + 2F_1(u(0))\,dx +$$
$$+ 2\int_0^T \int_{\Omega\cap B^{R-t}_{x_0}} f_2(u(t))\partial_t u(t) - g(t)\partial_t u(t)\,dx\,dt$$

Since $f_2(0) = 0$ and $|f_2'(u)| \le C$ then $f_2(u) = Q(u)u$ with $|Q(u)| \le C$ and the last term in the right-hand side of (3.13) can be estimated in the following way

$$|f_2(u(t))\partial_t u(t) - g(t)\partial_t u(t)| \le C_1(|u(t)|^2 + |\partial_t u(t)|^2) + |g(t)|^2$$

Inserting this estimate in (3.13) and applying the Gronewal inequality we obtain (3.12) Lemma 3.1 is proved.

Now we are in position to construct a solution for the equation (3.1). To this end we fix an arbitrary large time interval $[0,T]$ and consider for the first the equation (3.1) with compactly supported initial data and the right-hand side. More precisely, we assume that

$$\xi_u(0) = 0 \text{ and } g(t) = 0 \text{ for } |x| > R \tag{3.14}$$

Then, as follows from Lemma 3.1 and from the fact that $F_1(0) = 0$ the solution $\xi_u(t,x) \equiv 0$ outside of the cylinder $[0,T]\times(\Omega\cap B_0^{R+T})$, i.e. without loss of generality we may solve the equation (3.1) in the *bounded* domain $\Omega\cap B_0^{R+T}$. But for bounded domains the existence of a solution is a well known fact (see e.g. [15]). Thus, we have proved the existence of solutions for (3.1) with compactly supported initial data and right-hand side.

Approximating now the arbitrary $\xi_u(0) \in E_b(\Omega)$ by in the space $E_{loc}(\Omega)$ by a sequence $\xi_{u_n}(0)$ of compactly supported initial dates, doing the same with the right-hand side $g \in L^2_b(\mathbb{R}_+ \times \Omega)$ and passing to the limit $n \to \infty$ we obtain a solution of the problem (3.1). Theorem 3.1 is proved.

Corollary 3.1. *Let all assumptions of Theorem 3.1 hold. Then*

$$\|\partial_t u(t), \Omega\cap B^1_{x_0}\|^2_{0,2} + \|u(t), \Omega\cap B^1_{x_0}\|^2_{1,2} \le \tag{3.15}$$
$$\le e^{-\delta t} Q(\|\xi_u(0)\|_{E_b(\Omega)}) \left(|\partial_t v(0)|^2 + |\nabla_x v(0)|^2 + |v(0)|^2, e^{-\varepsilon|x-x_0|}\right) +$$
$$+ C(f_2) + \int_0^t e^{-\delta(t-s)}\left(|g(s)|^2, e^{-\varepsilon|x-x_0|}\right)\,ds$$

for a certain monotonic function $Q = Q_{f_1}$.

Indeed, since $f_1(0) = 0$ (according to the third assumption of (3.2) then $F_1(0) = F_1'(0) = 0$, consequently $F_1(u) = R(u)u^2$ with $|R(u)| \le C(1 + |u|^{q_2})$ and

$$\left(F_1(u), e^{-\varepsilon|x-x_0|}\right) \le C\int_{x\in\Omega} e^{-|x-x_0|}\|R(u)u^2, \Omega\cap B^1_{x_0}\|_{0,1}\,dx \le \tag{3.16}$$
$$\le C\|R(u)\|_{b,0,n}\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|u, \Omega\cap B^1_{x_0}\|^2_{1,2}\,dx \le$$
$$\le Q(\|u\|_{b,1,2})\left(|u|^2 + |\nabla_x u|^2, e^{-\varepsilon|x-x_0|}\right)$$

Here we have used Sobolev embedding theorem, Holder inequality and the fact that

$$\|R(u),\Omega\|_{b,0,n} \le C(1+\|u,\Omega\|^{q_2}_{b,0,q_2n}) \le Q(\|u,\Omega\|_{b,1,2})$$

Note also that $Q \equiv const$ for $n \ge 4$ since $q_2 = 0$.

Inserting this estimate to (3.4) we derive (3.15).

Corollary 3.2. *Let the assumptions of Theorem 3.1 hold. Then the following estimate is valid:*

$$\|\xi_u(t)\|^2_{E_b(\Omega)} \le Q_1(\|\xi_u(0)\|_{E_b(\Omega)})e^{-\delta t} + C(f_2) + C\|g\|^2_{L^2_b(\mathbb{R}_+\times\omega)} \tag{3.17}$$

where Q_1 depends of f_1 and independent of the initial conditions $\xi_u(0)$.

Indeed, applying the supremum with respect to $x_0 \in \Omega$ to both sides of (3.16) and using the estimates (1.17) we obtain (3.17).

Remark 3.2. Note that all results of this Section remains valid for the limit case $q_1 = 2/(n-2)$ but we will essentially use the fact that $q_1 < 2/(n-2)$ in order to construct the attractor for the equation (3.1).

Remark 3.3. Note also that the assertion of Theorem 3.1 and it's proof remains valid for the nonlinear functions $f = f(u,x)$ which depends explicitly on $x \in \Omega$ if this function satisfies the conditions (3.2) for every fixed x and the constant C in (3.2) is independent of x. We will use this fact in Section 12.

Part 2. The attractors.

In this Part we construct the attractors for the equation (0.1).

The autonomous case of the equation (0.1) is considered in Section 4.

The general nonautonomous case is considered in Section 5. We construct there the locally compact attractor $\mathcal{A}$ (i.e. the attractor which is compact only in $E_{loc}(\Omega)$ and give examples which show that this attractor may be not globally (in $E_b(\Omega)$) compact.

The particular case of equations (0.1) for which the locally compact attractor constructed in previous Section will be in a fact the globally compact attractor is studied in Section 6.

§4 The autonomous attractor.

In this Section we consider the attractor for the equation (3.1) under the additional assumption

$$g(t) \equiv g \in L^2_b(\Omega) \tag{4.1}$$

The general case will be considered in the next Section.

The assumption (4.1) together with Corollary 3.2 imply that under the conditions of Section 3 the equation (3.1) generates a semigroup $S_t : E_b(\Omega) \to E_b(\Omega)$ defined by formula

$$S_t u(0) = u(t) \quad \text{where } u(t) \text{ is a solution of (3.1)} \tag{4.2}$$

Moreover, it follows from Corollary 3.2 that this semigroup possesses a bounded absorbing set $\mathcal{B}$ in the space $E_b(\Omega)$, i.e. for any other bounded subset $B \subset \Phi_b(\Omega)$ there exists $T = T(B)$ such that

$$S_t B \subset \mathcal{B} \text{ if } t \geq T$$

It seems natural to consider the attractor of (4.2) in the 'uniform' topology of the space $E_b(\Omega)$ but, in contrast to the case of bounded domains the problem of existence of the compact attractor in a 'uniform' topology (i.e. in $E_b(\Omega)$) is more delicate. The main difficulty is the non compactness of the embedding $E_b^\kappa(\Omega) \subset E_b(\Omega)$ for $\kappa > 0$. That is why we cannot derive the existence of the attractor using only the asymptotically smoothing arguments and should control the behavior of solutions when $|x| \to \infty$ in a more precise way.

In order to do that we assume in addition that the function f_2 which has been introduced in (3.2) equals zero identically and right-hand side

$$g \in \dot{L}_b^2(\Omega) \tag{4.3}$$

Recall, that roughly speaking, (4.3) means that g should decay when $|x| \to \infty$ but the rate of decaying may be arbitrary slow.

Remark 4.1. Note that as it shown in Lemma 4.1 and Remark 4.2, without these assumptions the compact attractor in $E_b(\Omega)$ may not exist (and do not exist for a number of interesting from the physical point of view examples of equations (3.1)). In order to consider such equations we will construct below the locally compact attractor (i.e. the attractor in $E_{loc}(\Omega)$) which exists in general situation.

Recall firstly the definition of the attractor (see e.g. [2] for details).

Definition 4.1. *The set $\mathcal{A} \subset E_b(\Omega)$ is defined to be the attractor of the semigroup S_t if the following assumptions hold:*

1. The set $\mathcal{A}$ is compact in $E_b(\Omega)$.

2 The set $\mathcal{A}$ is strictly invariant with respect to S_t, i.e.

$$S_t\mathcal{A} = \mathcal{A} \textit{ for } t \geq 0$$

3. The set $\mathcal{A}$ is the attracting set for S_t in $E_b(\Omega)$, i.e. for every neighborhood $\mathcal{O}(\mathcal{A})$ of $\mathcal{A}$ in the topology of the space $E_b(\Omega)$ and for every bounded in uniform topology subset $B \subset E_b(\Omega)$ there exists $T = T(\mathcal{O}, B)$ such that

$$S_t B \subset \mathcal{O}(\mathcal{A}) \textit{ if } t \geq T$$

Theorem 4.1. *Let the assumptions of Theorem 3.1 hold with $f_2 \equiv 0$ and let the right-hand side g satisfy* (4.3). *Then the semigroup S_t, defined by* (4.2), *possesses the (globally compact) attractor $\mathcal{A}$ in the sense of Definition 4.1 which has the following structure:*

$$\mathcal{A} = \mathcal{K}\big|_{t=0} \tag{4.4}$$

where we denote by $\mathcal{K}$ the set of all solutions of (3.1), *defined and bounded for all $t \in \mathbb{R}$ ($\sup_{t\in\mathbb{R}} \|u(t)\|_{E_b(\Omega)} < \infty$).*

The proof of this Theorem will be given in Section 6 for a more general nonautonomous case.

Let us discuss now the problem of the globally compact attractor's existence in the case where the $f_2 \neq 0$ or the right-hand side g does not decay when $|x| \to \infty$. For simplicity we restrict ourselves to consider only the case $\Omega = \mathbb{R}^n$.

Lemma 4.1. *Let $\Omega = \mathbb{R}^n$ and the right-hand side g be independent of x. Let us suppose also that the attractor $\mathcal{A}$ of the problem* (3.1) *is compact in the space $E_b(\Omega)$. Then*

$$\mathcal{A} \subset AP_{E_b}(\mathbb{R}^n) \tag{4.5}$$

Here we denote by $AP_{E_b}(\mathbb{R}^n)$ the space of almost-periodic functions (in Stepanov sense) which belong to $E_b(\mathbb{R}^n)$

Proof. Let $\xi \in \mathcal{A}$. Then by definition, we should verify that the hull

$$H(\xi) = \{T_x^h u_0, h \in \mathbb{R}^n\}_{E_b(\mathbb{R}^n)}, \text{ where } (T_x^h \xi)(x) = \xi(x+h) \tag{4.6}$$

is compact in $E_b(\mathbb{R}^n)$. (Here and below we denote by $\{\cdot\}_X$ the closure in the topology of the space X.)

Note that our equation is invariant with respect to T_x^h since $\xi \in \mathcal{A}$ implies $H(\xi) \subset \mathcal{A}$. But according to our assumptions $\mathcal{A}$ is compact in $E_b(\Omega)$ and consequently the hull $H(\xi)$ is compact in $E_b(\mathbb{R}^n)$. Lemma 4.1 is proved.

Remark 4.2. It is worth to emphasize now that the obtained embedding $\mathcal{A} \subset AP(\mathbb{R}^n)$ is not natural. Indeed, consider the equation

$$\partial_t^2 u + \gamma \partial_t u - \Delta_x u + u^3 - \alpha^2 u = 0 \tag{4.7}$$

in $\mathbb{R}^n$, Then, the equilibria point $u_0(x) = \alpha \tanh\left(\frac{\alpha}{2^{1/2}} x_1\right)$ evidently belongs to the attractor $\mathcal{A}$ (if it exists) but not almost periodic. Thus, according to Lemma 4.1, the equation (4.7) does not possess a globally compact attractor (in $E_b(\Omega)$).

That is why we will construct now the *locally* compact attractor (i.e. the attractor which attracts bounded in $E_b(\Omega)$ sets in the topology of the space $E_{loc}(\Omega)$) for the equation (3.1) which in fact more natural for the unbounded domains (see e.g. [14], [21], [28]).

Definition 4.2. *The set $\mathcal{A} = \mathcal{A}^{loc} \subset E_b(\Omega)$ is defined to be the locally compact attractor of the semigroup S_t (the $(E_b(\Omega), E_{loc}(\Omega))$-attractor in the notations of* [2]*) if the following assumptions hold:*

1. The set $\mathcal{A}$ is compact in $E_{loc}(\Omega)$.

2 The set $\mathcal{A}$ is strictly invariant with respect to S_t, i.e.

$$S_t \mathcal{A} = \mathcal{A} \text{ for } t \geq 0$$

3. The set $\mathcal{A}$ is the attracting set for S_t in local topology, i.e. for every neighborhood $\mathcal{O}(\mathcal{A})$ of $\mathcal{A}$ in the topology of the space $E_{loc}(\Omega)$ and for every bounded in uniform topology subset $B \subset E_b(\Omega)$ there exists $T = T(\mathcal{O}, B)$ such that

$$S_t B \subset \mathcal{O}(\mathcal{A}) \text{ if } t \geq T$$

Recall that the first condition means that the restriction $\mathcal{A}\big|_{\Omega_1}$ is compact in $E(\Omega_1)$ for every *bounded* $\Omega_1 \subset \Omega$.

Analogously, the third condition means that for every *bounded* $\Omega_1 \subset \Omega$, every bounded B in $E_b(\Omega)$ and every $E(\Omega_1)$-neighborhood $\mathcal{O}(\mathcal{A}\big|_{\Omega_1})$ of the restriction $\mathcal{A}\big|_{\Omega_1}$ there exists $T = T(\Omega_1, \mathcal{O}, B)$ such that

$$(S_t B)\big|_{\Omega_1} \subset \mathcal{O}(\mathcal{A}\big|_{\Omega_1}) \text{ if } t \geq T$$

Theorem 4.2. *Let the assumptions of Theorem 3.1 be valid. Then the semigroup S_t, defined by* (4.2)*, possesses the locally compact attractor $\mathcal{A}$ in the sense of Definition 4.2 which has the structure* (4.4)*.*

The proof of this theorem will be given in the next Section for a more general nonautonomous case.

§5 The nonautonomous locally compact attractor.

In this Section we consider the general case of the equation (3.1) where the right-hand side $g = g(t)$ depends on t. In order to construct the locally compact attractor for the nonautonomous equation (the nonautonomous analogue of Theorem 4.2) we will use the approach, developed in [4], [6] and [28].

Together with our initial equation we will consider simultaneously a family of equation of the type (3.1), obtained from the initial one by positive shifting along the t axis and by taking a closure in the corresponding topology.

To be more precise we consider the family of problems of type (3.1)

$$\partial_t^2 u + \gamma\partial_t u - \Delta_x u + \lambda_0 u + f(u) = \widehat{g}(t), \quad \widehat{g} \in \mathcal{H}^+(g) \tag{5.1}$$

where in contrast to the case of almost-periodic functions (see (4.6)) we define the hull by taking the closure of the set $\{T_h g, h \in \mathbb{R}_+\}$ ($(T_h)g(t) = (T_h^t g) = g(t+h)$) in a *local* topology of $L^2_{loc}(\mathbb{R}_+ \times \Omega) := L^2_{loc}(\mathbb{R}_+ L^2_{loc}(\Omega))$:

$$\mathcal{H}^+(g) := \{T_h g, h \in \mathbb{R}_+\}_{L^2_{loc}(\mathbb{R}_+\times\Omega)} \tag{5.2}$$

The main requirement to the right-hand side g of the initial equation (3.1) is: the hull $\mathcal{H}^+(g)$ is compact in the space $L^2_{loc}(\mathbb{R}_+ \times \Omega)$. The functions, which satisfy this assumption, is called translation-compact in $L^2_{loc}(\mathbb{R}_+, L^2_{loc})$ (following to [4]).

More general, a set of functions $\Sigma \in L^2_{loc}(\mathbb{R}_+, L^2_{loc}(\Omega))$ is called translation compact if their hull

$$\mathcal{H}^+(\Sigma) := \{T_h \Sigma, h \in \mathbb{R}_+\}_{L^2_{loc}(\mathbb{R}_+\times\Omega)} \tag{5.3}$$

is compact in $L^2_{loc}(\mathbb{R}_+ \times \Omega)$.

Let us formulate now a number of necessary and sufficient conditions for sets to be translation compact. The following evident proposition reduce this problem to the case where Ω is bounded.

Proposition 5.1. *The set $\Sigma \in L^2_{loc}(\mathbb{R}_+ \times \Omega)$ is translation-compact in L^2_{loc} if and only if the restriction $\Sigma\big|_{\Omega_1}$ is translation-compact in $L^2_{loc}(\mathbb{R}_+, L^2(\Omega_1))$ for every bounded subdomain $\Omega_1 \subset \Omega$.*

Indeed, by definition $\mathcal{H}^+(\Sigma)$ is compact in $L^2_{loc}(\mathbb{R}_+ \times\Omega)$ if and only if $\mathcal{H}^+(\Sigma\big|_{\Omega_1}) = \mathcal{H}^+(\Sigma)\big|_{\Omega_1}$ is compact in $L^2_{loc}(\mathbb{R}_+, L^2(\Omega_1))$ for every bounded $\Omega_1 \subset \Omega$.

Thus it remains to formulate necessary and sufficient conditions for the translation compactness for the case where Ω_1 is bounded.

Note for the first that by definition a set $\Sigma \in L^2_{loc}(\mathbb{R}_+ \times \Omega_1)$ is translation-compact in $L^2_{loc}(\mathbb{R}_+, L^2(\Omega_1))$ if and only if the set

$$\{(T_h\Sigma)\big|_{t\in[0,1]}, h \in \mathbb{R}_+\} \subset\subset L^2([0,1] \times \Omega_1) \tag{5.4}$$

(is precompact in the space $L^2([0,1] \times \Omega_1)$). Therefore, every translation-compact set $\Sigma \in L^2_{loc}(\mathbb{R}_+ \times \Omega_1)$ belongs to $L^2_b(\mathbb{R}_+, L^2(\Omega_1))$ and bounded in this space.

The following Proposition gives these conditions in the spirit of Arcela-Ascoli theorem.

Proposition 5.2. *Let $\Omega_1 \subset \Omega$ be a bounded domain. Then a set $\Sigma \in L^2_b(\mathbb{R}_+ \times \Omega_1)$ is translation-compact in $L^2_{loc}(\mathbb{R}_+, L^2(\Omega_1))$ if and only if the following conditions hold:*

(a) for any fixed $t > 0$ the set $\{\int_s^{t+s} g(z)dz,\ s \in \mathbb{R}_+, g \in \Sigma\}$ is precompact in the space $L^2(\Omega_1)$;

(b) there exists a function $\beta(s)$, $s \geq 0$, $\beta(s) \to 0$ as $s \to 0$, such that

$$\int_t^{t+1} \|g(z) - g(z+l)\|^2_{L^2(\Omega_1)} dz \leq \beta(|l|) \quad \forall t \in \mathbb{R}_+,\ t+l \in \mathbb{R}_+,\ g \in \Sigma$$

The proof of Proposition 5.2 is given in [5].

Corollary 5.1. *Let Ω_1 be a bounded domain in $\mathbb{R}^n$ and let*

$$\Sigma \in W_b^{\alpha,2}(\mathbb{R}_+ \times \Omega_1)),\ \alpha > 0 \tag{5.5}$$

and bounded in this space. Then Σ is translation-compact in $L^2_{loc}(\mathbb{R}_+, L^2(\Omega_1))$.

Remark 5.1. Note that the assumption (5.5) is not necessary for the translation-compactness. Indeed, any periodic, quasiperiodic, or almost periodic function is evidently translation-compact. Moreover, if g is translation-compact in $L^2_{loc}(\mathbb{R}_+ \times \Omega)$ and $g_1 \in L^2_{loc}(\mathbb{R}_+ \times \Omega)$ satisfy the condition $\operatorname{supp} g \subset [0,T] \times \Omega$ then $g + g_1$ is also translation-compact. Thus, in contrast to the concept of almost periodicity, the translation-compactness is some kind of regularity condition when $t \to \infty$.

The following Proposition shows the relations between the translation- compactness and smoothness.

Proposition 5.3. *Let Ω_1 be bounded domain. Let $TC_2(\mathbb{R}_+ \times \Omega_1)$ be the closure of the set $C^1_b(\mathbb{R}_+ \times \Omega_1)$ in the space $L^2_b(\mathbb{R}_+ \times \Omega_1)$*

$$TC_2(\mathbb{R}_+ \times \Omega_1) = \left\{C^1_b(\mathbb{R}_+ \times \Omega_1)\right\}_{L^2_b(\mathbb{R}_+ \times \Omega_1)} \tag{5.6}$$

Then g is translation-compact in $L^2_{loc}(\mathbb{R}_+ \times \Omega_1)$ if and only if $g \in TC_2(\mathbb{R}_+ \times \Omega_1)$.

The proof of this proposition is given in [28].

Let us return now to the family of equations (5.1). Define a semigroup $\{\mathbb{S}_t, t \geq 0\}$, acting on the extended phase space $E_b \times \mathcal{H}^+(g)$, by formula

$$\mathbb{S}_t(\xi_u(0), \widehat{g}) := (\xi_{u_{\widehat{g}}}(t), T_t\widehat{g}) \tag{5.7}$$

where $u_{\widehat{g}}(t)$ is the solution of the problem (3.1) with the right-hand $\widehat{g} \in \mathcal{H}^+(g)$ and $\xi_u(0) \in E_b(\Omega)$.

Theorem 5.1. *Let the assumptions of Theorem 3.1 hold and let the right-hand side g be translation-compact in $L^2_{loc}(\mathbb{R}_+ \times \Omega)$. Then the semigroup $\mathbb{S}_t$ possesses the $(E_b(\Omega) \times \mathcal{H}^+(g), E_{loc}(\Omega) \times \mathcal{H}^+(g))$-attractor $\mathbb{A}$ (see Definition 4.2).*

Proof. According to the abstract theorem which gives the sufficient conditions for the attractor's existence (see e.g. [2]) we should verify the following assumptions.

1. The semigroup $\mathbb{S}_t$ is continuous for every fixed $t \geq 0$ in the topology of $E_{loc} \times \mathcal{H}^+(g)$ on every bounded subset of $E_b(\Omega)$.

2. This semigroup possesses the attracting set which is compact in $E_{loc}(\Omega) \times \mathcal{H}^+(g)$.

The first assertion is evident (see Theorem 3.1 and Theorem 2.1). Thus, it remains to verify the second one. To this end we need the following Lemma.

Lemma 5.1. *Consider the family of linear problems*

$$\begin{cases} \partial_t^2 v + \gamma\partial_t v - \Delta_x v + \lambda_0 v = h(t), \quad h \in \Sigma \\ v\big|_{\partial\Omega} = 0; \quad \xi_v\big|_{t=0} = 0 \end{cases} \tag{5.8}$$

where $\gamma, \lambda_0 > 0$. Assume that the functional set Σ is bounded in $L^2_b(\mathbb{R}_+ \times \Omega)$ and translation compact in $L^2_{loc}(\mathbb{R}_+ \times \Omega)$. Then the set

$$L(\Sigma) := \{\xi_{v,h}(T) : T \in \mathbb{R}_+,\ h \in \Sigma\} \tag{5.9}$$

(where $\xi_{h,v}$ means the solution of (5.8) *with the right-hand h) is precompact in the space $E_{loc}(\Omega)$.*

Proof. Indeed, consider a sequence $\xi_{v,h_n}(t_n)$, $h_n \in \Sigma$, $t_n \in \mathbb{R}_+$. Our aim is to extract the converging in $E_{loc}(\Omega)$ subsequence from this sequence.

It is sufficient to consider only the following two cases:

1. $t_n \to T_0 \in \mathbb{R}_+$.
2. $t_n \to \infty$.

Let us consider the first case. Since Σ is precompact in $L^2_{loc}(\mathbb{R}_+, L^2_{loc})$ then without loss of generality we may assume that $h_n \to h$ in $L^2([0, T_0+1], L^2_{loc})$. We claim that

$$\lim_{n\to\infty} \xi_{v,h_n}(t_n) = \xi_{v,h}(T_0) \tag{5.10}$$

Indeed, according to Theorem 2.1, for every $R > 0$

$$\|\xi_{v,h_n}(t_n) - \xi_{v,h}(t_n)\|^2_{E(\Omega\cap B^R_0)} \le C_R \int_0^{t_n} \left(|h_n(s) - h(s)|^2, e^{-\varepsilon|x-x_0|}\right) ds \to 0$$

(since $h_n \to h$ in $L^2([0, T_0+1], L^2_{loc})$ and the sequence h_n is bounded in $L^2_b(\mathbb{R}_+ \times \Omega)$). Moreover, since $\xi_{v,h}(t)$ is continuous in L^2_{loc} (due to Theorem 2.1) then

$$\|\xi_{v,h}(t_n) - \xi_{v,h}(T_0)\|_{E(\Omega\cap B^R_0)} \to 0$$

Thus, (5.10) is proved if $t_n \to T_0 \in \mathbb{R}_+$.

Let us consider now case $t_n \to \infty$. Since the set Σ is translation compact in L^2_{loc} then without loss of generality we may assume that

$$\widehat{h}_n := T_{t_n} h_n \to h \tag{5.11}$$

in the space $L^2_{loc}(\mathbb{R} \times \Omega)$ (here we have used the fact that $t_n \to +\infty$). Note also that $h \in L^2_b(\mathbb{R} \times \Omega)$. Denote by $\theta(t) = \theta_h(t)$, $t \in \mathbb{R}$ a unique (due to the estimate (2.6)) solution of the equation

$$\partial_t^2 \theta + \gamma\partial_t\theta - \Delta_x\theta + \lambda_0\theta = h$$

which defined for every $t \in \mathbb{R}$ and bounded. We claim that

$$\lim_{n\to\infty} \xi_{v,h_n}(t_n) = \xi_\theta(0) \tag{5.12}$$

Indeed, according to the estimate (2.6), and the evident fact that $(T_{t_n}\xi_{v,h_n})(0) = \xi_{v,h_n}(t_n)$

$$(5.13)\quad \|\xi_{v,h_n}(t_n) - \xi_\theta(0)\|^2_{E(\Omega\cap B_0^R)} \le Ce^{-\delta t_n}\|\xi_\theta(-t_n)\|^2_{E_b(\Omega)} + \\ + C_R\int_{-t_n}^0 e^{\delta s}\left(|\widehat{h}_n(s) - h(s)|^2, e^{-\varepsilon|x-x_0|}\right) ds$$

Note that the second integral in the right-hand side of (5.13) tends to 0 according to (5.11) and due to the fact that the sequence $T_{t_n}h_n$ is bounded in $L^2_b(\mathbb{R}\times\Omega)$. The first one tends to 0 since $\|\xi_\theta(-t_n)\|_{E_b}$ is uniformly bounded and $t_n \to \infty$ when $n \to \infty$. Lemma 5.1 is proved.

Now we are in a position to complete the proof of the theorem.

Note for the first that the estimate (3.17) together with the evident fact

$$(5.14)\qquad \|\widehat{g}\|_{L^2_b(\mathbb{R}_+\times\Omega)} \le \|g\|_{L^2_b(\mathbb{R}_+\times\Omega)} \text{ for } \widehat{g}\in\mathcal{H}^+(g)$$

imply that the set

$$(5.15)\qquad \mathbb{B} := \{\|\xi_u\|_{E_b} \le R\}\times\mathcal{H}^+(g)$$

will be the attracting (and even the absorbing) set for the semigroup (5.7) if R large enough. (but this set is not compact in $E_{loc}\times\mathcal{H}^+(g)$).

Thus, it remains to construct the compact attracting set for the semigroup $\mathbb{S}_t$ *only* for the initial data belonging to the absorbing set $\mathbb{B}$. To this end we represent every solution u of the problem (5.1) as a sum of two functions

$$(5.16)\qquad u(t) = v(t) + w(t)$$

where the function v satisfies the linear equation

$$(5.17)\qquad \partial_t^2 v + \gamma\partial_t v + \lambda_0 v - \Delta_x v = 0,\quad \xi_v(0) = \xi_u(0)$$

and the function w satisfies the equation

$$(5.18)\qquad \partial_t^2 w + \gamma\partial_t w + \lambda_0 w - \Delta_x w = -f(u) + \widehat{g}(t),\quad \xi_w(0) = 0$$

According to Theorem 2.1, the function v decays exponentially when $t\to\infty$:

$$(5.19)\qquad \|\xi_v(t)\|^2_{E_b} \le Ce^{-\delta t}\|\xi_u(0)\|^2_{E_b}$$

Let us study the equation (5.18). We claim that the set

$$(5.20)\qquad \Sigma := \{-f(u) + \widehat{g}:\quad u \text{ is a solution of (5.1)},\ (\xi_u(0),\widehat{g})\in\mathbb{B}\}$$

is bounded in $L^2_b(\mathbb{R}_+\times\Omega)$ and translation compact in $L^2_{loc}(\mathbb{R}_+\times\Omega)$. Indeed, since the exponent q_1 in the conditions (3.2) is *strictly* less then the limit one $2/(n-2)$ then it is not difficult to derive using Holder inequality and Sobolev embedding theorem that there exists $\kappa = \kappa(q_1, n) > 0$ such that

$$(5.21)\quad \|f(u),\Omega\|_{b,0,2(1+\kappa)} + \|f(u),\Omega\|_{b,1,1+\kappa} + \\ + \|f'(u)\partial_t u,\Omega\|_{b,0,1+\kappa} \le C(1 + \|\xi_u\|^{q_1}_{E_b(\Omega)})$$

Consequently, the set

$$(5.22) \qquad \Sigma_1 := \{-f(u): \; u \text{ is a solution of (5.1) }, \; (\xi_u(0), \widehat{g}) \in \mathbb{B}\}$$

is bounded in $L_b^{2(1+\kappa)}(\mathbb{R}_+ \times \Omega) \cap W_b^{1+\kappa,2}(\mathbb{R}_+ \times \Omega)$ and therefore, according to the interpolation theorem, Σ_1 is bounded in the space $W_b^{\kappa,2}(\mathbb{R}_+ \times \Omega)$. Corollary 5.1 implies now that Σ_1 is translation compact in $L^2_{loc}(\mathbb{R}_+ \times \Omega)$.

Note that

$$\mathcal{H}^+(\Sigma) \subset \mathcal{H}^+(\Sigma_1) + \mathcal{H}^+(g)$$

consequently (since g is translation compact) the set Σ is also translation compact in $L^2_{loc}(\mathbb{R}_+ \times \Omega)$.

Applying now the result of Lemma 5.1 to the equation (5.18) we derive that the set $L(\Sigma)$, defined by (5.9) is precompact in $E_{loc}(\Omega)$. The estimate (5.19) implies now that the set

$$\mathbb{K} = \{L(\Sigma)\}_{E_{loc}(\Omega)} \times \mathcal{H}^+(g)$$

be the compact in $E_{loc} \times \mathcal{H}^+(g)$ attracting set for the semigroup $\mathbb{S}_t$, defined by (5.7). Theorem 5.1 is proved.

Definition 5.1. *The projection $\mathcal{A} = \Pi_1 \mathbb{A}$ to the first component of the attractor $\mathbb{A}$ is called the (uniform) attractor of the family* (5.1) *or the (nonautonomous) attractor of the equation* (3.1).

Corollary 5.1. *Let the assumptions of Theorem 6.1 hold. Then the equation* (3.1) *possesses the attractor $\mathcal{A}$.*

Remark 5.2. There exists the internal definition of the attractor $\mathcal{A}$ without using the corresponding semigroup in the extended phase space. Namely, the set $\mathcal{A}$ is called the uniform attractor of the family (5.1) if the following conditions hold:

1. $\mathcal{A} \subset E_b$ is compact in E_{loc}.

2. For every bounded $B \subset E_b$ and every neighborhood $\mathcal{O}(\mathcal{A})$ of $\mathcal{A}$ in the topology of E_{loc} there exists $T = T(\mathcal{O}, B)$, such that

$$\xi_u(t) \in \mathcal{O}(\mathcal{A})$$

for every solution $\xi_u(t)$ of the equation (5.1) with $\xi_u(0) \in B$, the right-hand side $h \in \mathcal{H}^+(g)$ and $t \geq T$.

3. The set $\mathcal{A}$ is minimal set which satisfy the condition 1 and 2.

It is proved in [4] that the attractor thus defined coincides with the attractor, defined above.

We study now the structure of the obtained attractor $\mathcal{A}$. To this end we need the next definitions

Definition 5.2. *Let $\omega(g)$ be the attractor (ω-limit set) of the semigroup $\{T_h, h \in \mathbb{R}_+\}$, acting in the compact metric space $\mathcal{H}^+(g)$, i.e. (see* [2], [4], [16]*)*

$$(5.23) \qquad \omega(g) = \cap_{h \geq 0} \left\{ \cup_{s \geq h} T_s \mathcal{H}^+(g) \right\}_{L^2_{loc}(\mathbb{R}_+ \times \Omega)}$$

Definition 5.3. *Let us denote by $Z(g)$ the set of functions $\widehat{\xi} \in L^2_b(\mathbb{R} \times \Omega)$ which satisfy the condition:*

$$\Pi_+(T_h\widehat{\xi}) \subset \omega(g) \text{ for every } h \in \mathbb{R} \tag{5.24}$$

where Π_+ is the restriction operator to the semiaxis $t \in \mathbb{R}_+$.

It is known (see e.g. [5]) that the sets $\omega(g)$ and $Z(g)$ are not empty and compact in the spaces $L^2_{loc}(\mathbb{R}_+ \times \omega)$ and $L^2_{loc}(\mathbb{R} \times \Omega)$ correspondingly. Moreover, for every $\xi \in \omega(g)$ there exists $\widehat{\xi} \in Z(g)$ such that $\Pi_+\widehat{\xi} = \xi$, i.e

$$\Pi_+ Z(g) = \omega(g) \tag{5.25}$$

Theorem 5.2. *Let the assumptions of Theorem 5.1 hold. Then the attractor $\mathcal{A}$ of the equation (3.1) has the following structure*

$$\mathcal{A} = \Pi_0 \cup_{\widehat{\xi} \in Z(g)} \mathcal{K}_{\widehat{\xi}} \tag{5.26}$$

where $\mathcal{K}_{\widehat{\xi}}$ is the union of all solutions $\widehat{u}$ of the equation (3.1) with the right-hand side $\widehat{\xi} \in Z(g)$ which are defined for every $t \in \mathbb{R}$ and bounded with respect to $t \in \mathbb{R}$ (as usual $\Pi_0 u \equiv u(0)$).

Theorem 5.2 is a corollary of general theorem which describes the structure of nonautonomous attractors (see [4], [5]).

§6 The globally compact attractor.

In this Section we consider the particular case of the problem (3.1) where the nonlinear term $f \equiv f_1$ (i.e. $f_2 \equiv 0$ in the conditions (3.2) and the right-hand side $g(t)$ decays when $|x| \to \infty$. To be more rigorous we assume that

$$g \in L^2_b(\mathbb{R}_+, \dot{L}^2_b(\Omega)) \tag{6.1}$$

and translation compact in the space $L^2_{loc}(\mathbb{R}_+, \dot{L}^2_b(\Omega))$, i.e. it's hull $\mathcal{H}^+(g)$, defined by (5.2) is compact in the space $L^2_{loc}(\mathbb{R}_+, \dot{L}^2_b(\Omega))$. Note, that according to the compactness criteria in $\dot{L}^2_b(\Omega)$ (see Proposition 1.3 and Remark 1.2), a function $g \in L^2_b(\mathbb{R}_+, \dot{L}^2_b(\Omega))$ is translation compact in $L^2_{loc}(\mathbb{R}_+, \dot{L}^2_b)$ if and only if it is translation compact in $L^2_{loc}(\mathbb{R}_+ \times \Omega)$ and the following condition is valid:

$$\int_T^{T+1} \|g(t), \Omega \cap B^1_{x_0}\|^2_{0,2}\, dt \le \Psi(|x_0|), \quad \text{for all } T \in \mathbb{R}_+ \text{ and } x_0 \in \Omega \tag{6.2}$$

where the monotonic function $\Psi(R)$, $R \in \mathbb{R}_+$ tends to 0 when $R \to \infty$. Moreover if the assumption (6.2) holds then the topologies on $\mathcal{H}^+(g)$ induced by the embedding to $L^2_{loc}(\mathbb{R}_+ \times \Omega)$ and by the embedding to $L^2_{loc}(\mathbb{R}_+, \dot{L}^2_b(\Omega))$ will coincide.

The main result of this Section is the following theorem.

Theorem 6.1. *Let the assumptions of Theorem 3.1 hold and let in addition $f_2 \equiv 0$ and the right-hand side g is translation compact in $L^2_{loc}(\mathbb{R}_+, \dot{L}^2_b(\Omega))$. Then the locally compact attractor $\mathbb{A}$ of the the semigroup* (5.7) *which has been constructed in Theorem 5.1 will be compact in $E_b(\Omega) \times \mathcal{H}^+(g)$ and will coincide with the globally compact attractor of the semigroup* (5.7) *acting on the extended phase space $E_b(\Omega) \times \mathcal{H}^+(g)$ (see Definition 4.1).*

Proof. Let us verify for the first that the set $\mathbb{A}$ is compact in $E_b(\Omega) \times \mathcal{H}^+(g)$ or which is the same the set $\mathcal{A} := \Pi_1 \mathbb{A}$ (see Definition 5.1) is compact in $E_b(\Omega)$. Since due to Theorem 5.1 the set $\mathcal{A}$ is compact in $E_{loc}(\Omega)$ then it remains to verify that

$$\|\xi_u\|^2_{E(\Omega \cap B^1_{x_0})} \le \Psi_1(|x_0|) \text{ for every } \xi_u \in \mathcal{A} \tag{6.3}$$

with $\Psi_1(R) \to 0$ when $R \to \infty$. Indeed, let $\xi_u \in \mathcal{A}$. Then according to Theorem 5.2 there exists $\widehat{g} \in Z(g)$ and the complete bounded solution $\xi_u(t)$ (i.e. $\xi_u \in L^\infty(\mathbb{R}, E_b(\Omega))$ such that $\xi_u = \xi_u(0)$. The estimate (3.15) implies now that

$$\|\xi_u(0)\|^2_{E_b(\Omega \cap B^1_{x_0})} \le e^{-\delta t} Q(\|\xi_u(-t)\|_{E_b}) + C \int_{-t}^0 e^{\delta s} \left(|\widehat{h}(s)|^2, e^{-\varepsilon|x-x_0|}\right) \tag{6.4}$$

Passing to the limit $t \to -\infty$ in (6.4) and using the fact that the solution $\xi_u(t)$ is bounded we derive the estimate

$$\|\xi_u(0)\|^2_{E_b(\Omega \cap B^1_{x_0})} \le C \int_{-\infty}^0 e^{\delta s} \left(|\widehat{h}(s)|^2, e^{-\varepsilon|x-x_0|}\right) \tag{6.5}$$

Since g is translation compact in $L^2_{loc}(\mathbb{R}_+, \dot{L}^2_b(\Omega))$ then (6.2) is valid and consequently (by the definition of $Z(g)$) the function $\widehat{h} \in Z(g)$ satisfies (6.2) with the same function Ψ (but for every $T \in \mathbb{R}$ instead of $T \in \mathbb{R}_+$). Thus, according to Proposition 1.4 and Remark 1.2,

$$C \int_{-\infty}^0 e^{\delta s} \left(|\widehat{h}(s)|^2, e^{-\varepsilon|x-x_0|}\right) \le \Psi_1(|x_0|) \tag{6.6}$$

where Ψ_1 depends only on Ψ, δ and ε (and independent of the concrete choice of ξ_u). The estimates (6.5) and (6.6) imply (6.3). Therefore the attractor $\mathcal{A}$ is compact in $\dot{E}_b(\Omega)$.

Let us verify now that the set $\mathbb{A}$ attracts bounded subsets of $E_b(\Omega)$ in a uniform topology (i.e. in $E_b(\Omega)$). Indeed, assume that it is not true. Then there exists a sequence of the right-hand sides $h_n \in \mathcal{H}^+(g)$, a sequence of solutions $\xi_{u_n}(t)$ of the equation (5.1) with the right-hand side h_n and $(\xi_{u_n}(0), h_n) \in \mathbb{B}$ and sequence of time moments $t_n \to \infty$ such that

$$\text{dist}_{E_b}(\xi_{u_n}(t_n), \mathcal{A}) \ge \varepsilon_0 > 0 \tag{6.7}$$

Note, that since $\mathcal{A}$ is the locally compact attractor then there exists $\xi \in \mathcal{A}$ such that $\xi_{u_n}(t_n) \to \xi$ when $n \to \infty$. The latter means that for every $R > 0$

$$\|\xi_{u_n}(t_n) - \xi\|_{E_b(\Omega \cap B^R_0)} \to 0 \text{ for } n \to \infty \tag{6.8}$$

Note also that according to the estimates (3.15) and (6.6)

(6.9)
$$\limsup_{n\to\infty}\|\xi_{u_n}(t_n)\|^2_{E_b(\Omega\setminus B_0^R)}\le C\sup_{|x_0|\ge R}\sup_{n\in\mathbb{N}}\int_0^\infty e^{-\delta s}\left(|h_n(s)|^2,e^{-\varepsilon|x-x_0|}\right)\le\Psi_1(R)$$

Recall that $\Psi_1(R)\to 0$ when $R\to\infty$.

The estimates (6.8) and (6.9) (together with the fact that $\xi\in\dot{E}_b(\Omega)$ implies that

$$\|\xi_{u_n}(t_n)-\xi\|_{E_b(\Omega)}\to 0 \text{ when } n\to\infty$$

which contradicts the assumption (6.7). Theorem 6.1 is proved.

Part 3. Kolmogorov's ε-entropy and attractors.

This part is devoted to study Kolmogorov's ε-entropy of the attractors constructed above.

For the reader convenience we recall firstly (in Section 7) the definition of the entropy and give some examples of asymptotic behavior of this entropy for the typical sets in the functional spaces.

The estimates for differences of solutions of (0.1) which have a fundamental significance for our entropy estimations are obtained in Section 8.

The upper bounds of the entropy in general case of locally compact attractors are given in Section 9.

These estimates are essentially improved in Section 10 for the particular case where (0.1) possesses the globally compact attractor.

And finally in Section 11 using the infinite dimensional unstable manifolds technique developed in [10], [28] we obtain the lower bounds of the entropy for the case $\Omega=\mathbb{R}^n$ and prove the sharpness of upper estimates obtained before.

§7 Definitions and typical examples.

In this Section we recall briefly the definition of ε-entropy and give the upper and lower estimates of it when $\varepsilon\to 0$ for the typical sets in functional spaces. For the detailed study of this concept see [18], [20].

Definition 7.1. *Let $\mathbb{M}$ be a metric space and let K be precompact subset of it. For a given $\varepsilon>0$ let $N_\varepsilon(K)=N_\varepsilon(K,\mathbb{M})$ be the minimal number of ε-balls in $\mathbb{M}$ which cover the set K (this number is evidently finite by Hausdorff criteria). By definition, Kolmogorov's ε-entropy of K in $\mathbb{M}$ is the following number*

(7.1)
$$\mathbb{H}_\varepsilon(K)=\mathbb{H}_\varepsilon(K,\mathbb{M})\equiv\ln N_\varepsilon(K)$$

Example 7.1. Let K be compact n-dimensional Lipschitz manifold in $\mathbb{M}$. Then the evident estimates imply that

(7.2)
$$C_1\left(\frac{1}{\varepsilon}\right)^n\le N_\varepsilon(K)\le C_2\left(\frac{1}{\varepsilon}\right)^n$$

and consequently

(7.3)
$$\mathbb{H}_\varepsilon(K)=(n+\overline{\overline{o}}(1))\ln\frac{1}{\varepsilon}$$

when $\varepsilon\to 0$.

This example justifies the following definition

Definition 7.2. *The fractal (box-counting) dimension of the set $K \subset\subset \mathbb{M}$ is defined to be the following number:*

$$\dim_F(K) = \dim_F(K, \mathbb{M}) = \limsup_{\varepsilon \to 0} \frac{\mathbb{H}_\varepsilon(K)}{\ln \frac{1}{\varepsilon}} \tag{7.4}$$

Note that the fractal dimension $\dim_F(K) \in [0, \infty]$ is defined for any compact set in $\mathbb{M}$ but may be not integer if K is not a manifold.

Example 7.2. Let $\mathbb{M} = [0, 1]$ and let K be the ternary Cantor set in $\mathbb{M}$. Then it is not difficult to obtain that

$$C_1 \left(\frac{1}{\varepsilon}\right)^d \le N_\varepsilon(K) \le C_2 \left(\frac{1}{\varepsilon}\right)^d \quad , \quad d = \frac{\ln 2}{\ln 3} \tag{7.5}$$

and consequently $\dim_F(K) = d = \frac{\ln 2}{\ln 3}$.

Consider now the examples of infinite dimensional sets (i.e. $\dim_F(K) = \infty$).

The following two examples give the typical asymptotic for the entropy in the spaces of analytical functions.

Example 7.3. Let K be the set of all analytic functions f in a ball $B(R)$ of radius $R > 1$ in $\mathbb{C}^n$ such that $\|f\|_{L^\infty(B(R))} \le 1$ and let $\mathbb{M}$ be the space $C(B^{Re})$, where $B^{Re} = \{z \in \mathbb{C}^n \ : \operatorname{Im} z_i = 0\,, |z| \le 1\}$. Thus, K consists of all functions from $C(B^{Re})$ which can be extended holomorphically to the ball $B(R) \subset \mathbb{C}^n$ and the C-norm of this extension is not greater then one. Then

$$C_1 \left(\ln \frac{1}{\varepsilon}\right)^{n+1} \le \mathbb{H}_\varepsilon(K, \mathbb{M}) \le C_2 \left(\ln \frac{1}{\varepsilon}\right)^{n+1} \tag{7.6}$$

For the proof of this estimate see [18].

Example 7.4. Let $\mathbb{M}$ be the same as in previous example and let K be the set of all functions f in $\mathbb{M}$ which can be extended to the entire function $\widehat{f}$ in $\mathbb{C}^n$ which satisfy the estimate

$$|\widehat{f}(z)| \le K_1 e^{K_2|z|}, \quad z \in \mathbb{C}^n \tag{7.7}$$

Then, as proved in [18],

$$C_1 \frac{\left(\ln \frac{1}{\varepsilon}\right)^{n+1}}{\left(\ln \ln \frac{1}{\varepsilon}\right)^n} \le \mathbb{H}_\varepsilon(K) \le C_2 \frac{\left(\ln \frac{1}{\varepsilon}\right)^{n+1}}{\left(\ln \ln \frac{1}{\varepsilon}\right)^n} \tag{7.8}$$

The next example gives the typical asymptotic for the entropy in the class of Sobolev spaces in bounded domains.

Example 7.5. Let Ω be smooth bounded domain in $\mathbb{R}^n$ and

$$W^{l_1,p_1}(\Omega) \subset\subset W^{l_2,p_2}(\Omega)\ ,\ 0 \le l_i < \infty,\ 1 < p_i < \infty,\ l_1 > l_2$$

i.e., according to the embedding theorem $\frac{l_1}{n} - \frac{1}{p_1} > \frac{l_2}{n} - \frac{1}{p_2}$.

Let now $\mathbb{M} = W^{l_2,p_2}(\Omega)$ and K be the unitary ball in $W^{l_1,p_1}(\Omega)$. Then

$$C_1 \left(\frac{1}{\varepsilon}\right)^{\frac{n}{l_1 - l_2}} \le \mathbb{H}_\varepsilon(K) \le C_2 \left(\frac{1}{\varepsilon}\right)^{\frac{n}{l_1 - l_2}} \tag{7.9}$$

The proof of this estimate can be found in [20].

The following proposition, which will be essentially used in the next Section gives the estimate of the constants C_i in (7.9) in dependence on the 'size' of Ω in the particular case $E_b^\kappa(\Omega) \subset E_b(\Omega)$

Proposition 7.1. *Let Ω be a bounded domain, which satisfies the conditions* (1.5) *and* (1.6), $\mathbb{M} = E_b(\Omega)$ *and let* $\mathbb{K}$ *be the unitary ball in* $E_b^\kappa(\Omega)$, $\kappa > 0$. *Then*

$$\mathbb{H}_\varepsilon(\mathbb{K}) \le C \operatorname{vol}(\Omega) \left(\frac{1}{\varepsilon}\right)^{n/\kappa} \quad \textit{for } \varepsilon < \varepsilon_0 \tag{7.10}$$

where $\operatorname{vol}(\Omega)$ *is n-dimensional volume of Ω. Moreover constants C and ε_0 in* (7.10) *depends only on K and R_0 from the assumptions* (1.5) *and* (1.6).

The proof of this Theorem is completely analogous to the proof of [28, Proposition 7.1]. That is why we omit it here.

Let us consider now the class of functions which will be used in the next Sections in order to obtain the lower bounds of ε-entropy of attractors.

Definition 7.3. *Let us denote by $\mathbb{B}_\sigma(\mathbb{R}^n)$ the subspace of $L^\infty(\mathbb{R}^n)$ which consists of all functions ϕ with the Fourier transform $\widehat{\phi}$ satisfying the condition*

$$\operatorname{supp} \widehat{\phi} \subset [-\sigma, \sigma]^n \tag{7.11}$$

It is well-known that every function $\phi \in \mathbb{B}_\sigma$ can be extended to entire function $\tilde{\phi}(z) \in A(\mathbb{C}^n)$ which satisfy the estimate

$$\sup_{x \in \mathbb{R}^n} |\tilde{\phi}(x + iy)| \le C \|\phi, \mathbb{R}^n\|_{0,\infty} e^{\sigma \sum_{i=1}^n |y_i|} \tag{7.12}$$

Moreover, every function $\phi \in L^\infty$, which possesses the entire extension $\tilde{\phi}$ which satisfies (7.12) belongs in fact to the space $\mathbb{B}_\sigma$.

Example 7.6. Let $K = B(0, 1, \mathbb{B}_\sigma)$, $\mathbb{M} = L_b^2(B_0^R)$. Then

$$\mathbb{H}_\varepsilon(B(0, 1, \mathbb{B}_\sigma), L_b^2(B_0^R)) \le C(R + K \ln \frac{1}{\varepsilon})^n \ln \frac{1}{\varepsilon} \tag{7.13}$$

Moreover C and K are independent of R.

For the proof of this estimate see for instance [28]. We formulate in conclusion the lower bounds for the entropy form Example 7.6.

Proposition 7.2. *The following estimate is valid for $R \ge R_0$ and $\varepsilon < \varepsilon_0$*

$$\mathbb{H}_\varepsilon \left(B(0, 1, \mathbb{B}_\sigma), L_b^2(B_0^R)\right) \ge C R^n \ln \frac{1}{\varepsilon} \tag{7.14}$$

where the constant C is independent of R and ε.

For the proof of (7.14) see for instance [18], or [28]. Thus, the estimate (7.13) is sharp for $R \sim \ln \frac{1}{\varepsilon}$ and $R >> \ln \frac{1}{\varepsilon}$. For the case $R << \ln \frac{1}{\varepsilon}$ we formulate only the following result.

Proposition 7.3. *For every $\delta > 0$ there exists $C_\delta > 0$ such that*

$$\mathbb{H}_\varepsilon \left(B(0, 1, \mathbb{B}_\sigma), L^2(B_0^1)\right) \ge C_\delta \left(\ln \frac{1}{\varepsilon}\right)^{n+1-\delta} \tag{7.15}$$

And consequently, the estimate (7.13) *is sharp for the case* $R << \ln \frac{1}{\varepsilon}$ *also.*

The estimate (7.15) has been obtained in [28, Theorem 9.2] for the space $C(B^1_{x_0})$. The estimate (7.15) for the space $L^2(B^1_{x_0})$ can be easily derived from it's analogue in the space $C(B^1_{x_0})$. Indeed, according to the interpolation inequality (see e.g. [20])

$$\|u, B^1_{x_0}\|_{0,\infty} \le C\|u, B^1_{x_0}\|^{1/2}_{0,2}\|u, B^1_{x_0}\|^{1/2}_{N,2} \tag{7.16}$$

for $N > n$. Moreover, it follows from the definition of the class $\mathbb{B}_\sigma$ that

$$\|u, \mathbb{R}^n\|_{b,N,2} \le C_{\sigma,N}\|u, \mathbb{R}^n\|_{b,0,2} \tag{7.17}$$

(Particularly, the norms of the spaces $C_b(\mathbb{R}^n)$ and $L^2_b(\mathbb{R}_n)$ are equivalent on $\mathbb{B}_\sigma$.)

The estimates (7.16) and (7.17) imply that

$$\|u_1 - u_2, B^1_{x_0}\|_{0,\infty} \le C\|u_1 - u_2, B^1_{x_0}\|^{1/2}_{0,2} \tag{7.18}$$

for all $u_1, u_2 \in K := B(0, 1, \mathbb{B}_\sigma)$ and consequently

$$\mathbb{H}_\varepsilon\left(K, L^2(B^1_{x_0})\right) \ge \mathbb{H}_{C\varepsilon^{1/2}}\left(K, C(B^1_{x_0})\right)$$

Proposition 7.3 is proved.

§8 The estimates for differences of solutions.

In this Section we derive a number of estimates for differences between the solutions belonging to the attractor or to the appropriate neighborhood of it. We will use these estimates in the next Sections in order to estimate the ε-entropy of the attractor.

Theorem 8.1. *Let u_1, u_2 be two solutions of the equation* (3.1) *with the right-hand sides g_1 and g_2 respectively and let the assumptions of Theorem 3.1 hold. Then the difference $v(t) = u_1(t) - u_2(t)$ can be represented in the following form*

$$u_1(t) - u_2(t) = \mathcal{P}_{u_1,u_2}(t) + \mathcal{R}_{u_1,u_2}(t) \tag{8.1}$$

where

$$\begin{aligned} &\|\mathcal{P}_{u_1,u_2}(t), \Omega \cap B^1_{x_0}\|^2_{1+\kappa,2} + \|\partial_t\mathcal{P}_{u_1,u_2}(t), \Omega \cap B^1_{x_0}\|^2_{\kappa,2} \le \\ &\le Ce^{Kt}\int_0^t \left(|g_1(s) - g_2(s)|^2, e^{-\varepsilon|x-x_0|}\right) ds + \\ &\quad Ce^{Kt}\left(|v(0)|^2 + |\nabla_x v(0)|^2 + |\partial_t v(0)|^2, e^{-\varepsilon|x-x_0|}\right) \end{aligned} \tag{8.2}$$

with the appropriate $0 < \kappa < 1/2$ *and*

$$\begin{aligned} &\|\mathcal{R}_{u_1,u_2}(t), \Omega \cap B^1_{x_0}\|^2_{1,2} + \|\partial_t\mathcal{R}_{u_1,u_2}(t), \Omega \cap B^1_{x_0}\|^2_{0,2} \le \\ &\le C\int_0^t \left(|g_1(s) - g_2(s)|^2, e^{-\varepsilon|x-x_0|}\right) ds + \\ &\quad Ce^{-\delta t}\left(|v(0)|^2 + |\nabla_x v(0)|^2 + |\partial_t v(0)|^2, e^{-\varepsilon|x-x_0|}\right) \end{aligned} \tag{8.3}$$

Moreover, the constants $\delta, \varepsilon > 0$ depend only on the equation (3.1) *and constants K and C depends on the norms $\|\xi_{u_i}(0)\|_{E_b}$ (but independent of the concrete choice of u_1 and u_2).*

Proof. Note for the first that the function v satisfies the equation

$$\partial_t^2 v + \gamma\partial_t v + \lambda_0 v - \Delta_x v + l(t)v = h(t) \tag{8.4}$$

where $h = g_1 - g_2$ and $l(t) = \int_0^1 f'(su_1(t) + (1-s)u_2(t))\,ds$ which has the form of (2.1). Note also, that according to the assumptions (3.2), one can easily derive using Holder inequality and Sobolev embedding Theorem that $l \in L_b^p(\Omega)$ and $\nabla_x l \in L_b^q(\Omega)$ with

$$p = \begin{cases} 2n/(n-2)q_1 > n \text{ for } n \geq 3 \\ < \infty \text{ for } n = 2 \\ \infty \text{ for } n = 1 \end{cases} \qquad q = \begin{cases} 2 \text{ for } n = 1 \text{ or } n > 3 \\ 3/2 \text{ for } n = 3 \\ < 2 \text{ for } n = 2 \end{cases} \tag{8.5}$$

Moreover, for these exponents

$$\|l(t), \Omega\|_{0,p,b} + \|l(t), \Omega\|_{b,1,q} \leq Q(\|\xi_{u_i}(0)\|_{E_b}) \tag{8.6}$$

for the appropriate function Q which depends only on f and n.

Let us decompose the function v in a sum $v = w + \theta$ where $w(t)$ satisfies the equation

$$\partial_t^2 w + \gamma\partial_t w + \lambda_0 w - \Delta_x w = h, \quad \xi_w(0) = \xi_v(0) \tag{8.7}$$

and the function θ satisfies

$$\partial_t^2 \theta + \gamma\partial_t \theta + \lambda_0 \theta - \Delta_x \theta = -l(t)v, \quad \xi_\theta(0) = 0 \tag{8.8}$$

and denote $\mathcal{P}_{u_1,u_2}(t) := \theta(t)$, $\mathcal{R}_{u_1,u_2}(t) := w(t)$.

Applying Theorem 2.1 to the equation (8.7) we obtain the estimate (8.3). Thus, it remains to prove (8.4). To this end we need the following analogue of Lemma 2.2.

Lemma 8.1. *Let $0 < \kappa < 1/2$, the function $l \in L^p(V_{x_0}) \cap W^{1,q}(V_{x_0})$ and the exponents p, q, κ satisfy the inequalities*

$$1.\ \frac{1}{p} + \frac{1}{n} < \frac{1}{q} < \frac{1}{2} + \frac{1}{n}; \qquad 2.\ \frac{1}{n} > \frac{1-\kappa}{p} + \frac{\kappa}{q} \tag{8.9}$$

for $n \geq 2$ (in the case $n = 1$ we should replace the first inequality of (8.9) *by $1/q \geq 1/p + 1/2$). Then the following estimate is valid*

$$\|lv, V_{x_0}\|_{\kappa,2} \leq C\|v, V_{x_0}\|_{1,2} \tag{8.10}$$

where the constant C depends on $\|l\|_{L_b^p \cap W_b^{1,q}}$ and independent of x_0.

The proof of this lemma is completely analogous to the proof of Lemma 2.3 so we omit it here.

Inserting the exponents p and q computed in (8.5) into the conditions (8.9) we derive that there exists sufficiently small positive $\kappa = \kappa(n)$ such that (8.9) and consequently (2.19) are valid. Applying Theorem 2.2 to the equation (8.8) and using Lemma 8.1 we obtain that

$$
\begin{aligned}
(8.11) \quad & \|\theta(t), \Omega \cap B^1_{x_0}\|^2_{1+\kappa,2} + \|\partial_t \theta(t), \Omega \cap B^1_{x_0}\|^2_{\kappa,2} \leq \\
& \leq C \int_0^t e^{K(t-s)} \int_\Omega e^{-\varepsilon|x-x_0|} \|l(s)v(s), V_{x_0}\|^2_{\kappa,2}\, dx\, ds \leq \\
& \leq C_1 \int_0^t e^{K(t-s)} \int_\Omega e^{-\varepsilon|x-x_0|} \|v(s), V_{x_0}\|^2_{1,2}\, dx\, ds
\end{aligned}
$$

Applying Theorem 2.1 to the equation (8.4) we obtain the estimate

$$
\begin{aligned}
(8.12) \quad & \|v(t), \Omega \cap B^1_{x_0}\|^2_{1,2} \leq C e^{K_1 t} \left(|v(0)|^2 + |\nabla_x v(0)|^2 + |\partial_t v(0)|^2, e^{-\varepsilon_1|x-x_0|} \right) + \\
& + C \int_0^t e^{K_1(t-s)} \left(|h(s)|^2, e^{-\varepsilon_1|x-x_0|} \right)
\end{aligned}
$$

Inserting the estimate (8.12) in the inequality (8.11) and using the estimates (1.15) we derive the estimate (8.2). Theorem 8.1 is proved.

Corollary 8.1. *Let the assumptions of Theorem 8.1 hold and let ϕ be the weight function with the rate of growth μ (see Definition 1.1). Then the functions $\mathcal{P} := \mathcal{P}_{u_1,u_2}$ and $\mathcal{R} := \mathcal{R}_{u_1,u_2}$ possess the following estimates*

$$
(8.13) \quad \begin{cases} \|\xi_\mathcal{P}(t)\|^2_{E^\kappa_{b,\phi}(\Omega)} \leq C e^{Kt} \|\xi_v(0)\|^2_{E_{b,\phi}(\Omega)} + C e^{Kt} \|g_1 - g_2\|^2_{L^2_{b,\phi}([0,t]\times\Omega))} \\ \|\xi_\mathcal{R}(t)\|^2_{E_{b,\phi}(\Omega)} \leq C e^{-\delta t} \|\xi_v(0)\|^2_{E_{b,\phi}(\Omega)} + C \|g_1 - g_2\|^2_{L^2_{b,\phi}([0,t]\times\Omega))} \end{cases}
$$

where the constant C depends only on C_ϕ and μ introduced in (1.1) *and is independent of the concrete choice of the weight ϕ.*

The proof of this corollary is analogous to the proof of Corollaries 2.1 and 2.2.

Let us consider now the particular case of equations (3.1) which has been introduced in §6 ($f_2 \equiv 0$ and g is translation compact in $L^2_{loc}(\mathbb{R}_+, \dot{L}^2_b(\Omega))$) where the equation possesses the globally compact attractor in $E_b(\Omega)$. The following theorem gives an improved version of the decomposition (8.1).

Theorem 8.2. *Let the assumptions of Theorem 6.1 hold. Then there exist the constants $\varepsilon, \mu > 0$ such that for every $(\xi_{u_1}(0), g_1), (\xi_{u_2}(0), g_2) \in \mathcal{V}_\mu(\mathbb{A})$ (where the neighborhood of $\mathcal{V}_\mu(\mathcal{A})$ of the attractor $\mathcal{A}$ is understood in the uniform topology of $E_b(\Omega) \times \mathcal{H}^+(g)$) the following decomposition is valid*

$$
(8.14) \qquad u_1 - u_2 = \mathcal{P}'_{u_1,u_2} + \mathcal{R}'_{u_1,u_2}
$$

and the functions $\mathcal{P}' := \mathcal{P}'_{u_1,u_2}$ and $\mathcal{R}' := \mathcal{R}'_{u_1,u_2}$ satisfy the estimates

$$
(8.15) \quad \begin{cases} \|\xi_{\mathcal{P}'}(t)\|^2_{E^\kappa_{b,e^{+\varepsilon|x|}}(\Omega)} \leq C e^{Kt} \|\xi_v(0)\|^2_{E_b(\Omega)} + C e^{Kt} \|g_1 - g_2\|^2_{L^2_b([0,t]\times\Omega)} \\ \|\xi_{\mathcal{R}'}(t)\|^2_{E_b(\Omega)} \leq C e^{-\delta t} \|\xi_v(0)\|^2_{E_b(\Omega)} + C \|g_1 - g_2\|^2_{L^2_b([0,t]\times\Omega)} \end{cases}
$$

where the constant C depends only on the equation and κ the same as in previous theorem.

Proof. Note for the first that without loss of generality we may assume that $f(0) = f'(0) = 0$. Indeed, $f(v).v \geq 0$ implies that $f'(0) \geq 0$. Rescaling if necessary the parameter $\lambda_0 > 0$ we may satisfy the condition $f'(0) = 0$. Then, according to the assumptions (3.1) and Sobolev embedding theorem

$$\|l(t), V_{x_0}\|_{0,p} \leq Q\left(\max_{i=1,2}\|\xi_{u_i}(t)\|_{E_b}\right)\max_{i=1,2}\|\xi_{u_i}(t)\|_{E(V_{x_0})} \tag{8.16}$$

with the appropriate function Q depending on f and the same exponent p as in (8.5).

Let us introduce the cut off function $\psi_K(x) \in C_0^\infty(\mathbb{R}^n)$ such that $\psi_K(x) = 1$ for $|x| \leq K$ and $\psi_K(x) = 0$ for $|x| > K + 1$ where the large parameter $K > 0$ will be fixed below.

We decompose the solution v of (8.4) in the following way $v = v_1 + v_2$ where

$$\partial_t v_1 + \gamma\partial_t v_1 - \Delta_x v_1 + \lambda_0 v_1 + (1 - \psi_K)l(t)v_1 = h(t);\ \ \xi_{v_1}(0) = \xi_v(0) \tag{8.17}$$

and the function v_2 satisfies the equation

$$\partial_t v_2 + \gamma\partial_t v_2 - \Delta_x v_2 + \lambda_0 v_2 + (1 - \psi_K)l(t)v_2 = -\psi_K l(t)v;\ \ \xi_{v_2}(0) = 0 \tag{8.18}$$

Denote $\mathcal{P}'_{u_1,u_2}(t) := v_2(t)$, $\mathcal{R}'_{u_1,u_2}(t) = v_1(t)$.

We are going to apply Theorem 2.1 to the equation (8.17). Moreover, we are going to fix K large enough to guarantee the constant $\delta = \alpha - CM^2$ introduced in Theorem 2.1 (with $M := \|(1 - \psi_K)l\|_{L^\infty(L^p_b)}$) be positive. Indeed, since (according to Theorem 6.1) the attractor $\mathcal{A} \in \dot{E}_b(\Omega)$ and possesses the uniform 'tale' estimate (6.3), then it is not difficult do derive (using also the estimates (3.15) and (6.6)) that for every $\mu > 0$ there exists a E_b-neighborhood $\mathcal{V}_\mu(\mathbb{A})$ of the extended attractor $\mathbb{A}$ such that

$$\begin{cases} 1.\ \ \mathbb{S}_t\mathcal{V}_\mu(\mathbb{A}) \subset \mathcal{V}_\mu(\mathbb{A}) \\ 2.\ \ \limsup_{K\to\infty}\|\xi\|_{E_b(\Omega\setminus B_0^K)} \leq \mu \text{ for every } \xi \in \Pi_1\mathcal{V}_\mu(\mathbb{A}) \end{cases} \tag{8.19}$$

and consequently there exists $K = K(\mu)$ such that

$$\|\mathcal{O}_\mu(\mathcal{A})\|_{E_b(\Omega\setminus B_0^K)} \leq 2\mu \tag{8.20}$$

where we denote $\mathcal{O}_\mu(\mathcal{A}) := \Pi_1\mathcal{V}_\mu(\mathbb{A})$.

The estimate (8.16) implies now that

$$\begin{aligned} M = M(\mu) &:= \|(1 - \psi_{K(\mu)})l\|_{L^\infty(\mathbb{R}_+, L^p_b(\Omega))} \leq \\ &\leq \|l\|_{L^\infty(\mathbb{R}_+, L^p_b(\Omega\setminus B_0^{K(\mu)}))} \leq C\|\mathcal{O}_\mu(\mathcal{A})\|_{E_b(\Omega\setminus B_0^{K(\mu)})} \leq 2C\mu \end{aligned} \tag{8.21}$$

Note that (8.21) implies that the constant $\delta = \alpha - CM^2$ will be positive if μ small enough.

We will assume below that the constants μ and $K = K(\mu)$ is fixed small enough to guarantee that $\delta > 0$ and will consider the initial data $(\xi_{u_i}(0), g_i)$ only belonging to the neighborhood $\mathcal{V}_\mu(\mathbb{A})$.

Applying the estimate (2.17) to the equation (8.17) we obtain the second estimate of (8.15). Thus, it remains to obtain the first one.

Applying the estimate of Theorem 2.2 to the equation (8.18) and using Lemma 8.1 together with the fact that $\psi_K(x) = 0$ if $|x| > K+1$ we derive that

$$\text{(8.22)}\quad \|v_2(t), \Omega\cap B^1_{x_0}\|^2_{1+\kappa,2} + \|\partial_t v_2(t), \Omega\cap B^1_{x_0}\|^2_{\kappa,2} \le$$
$$\le Ce^{Kt}\int_0^t\int_\Omega e^{-\varepsilon|x-x_0|}\|\psi_K l(s)v(s), V_{x_0}\|^2_{\kappa,2}\,dx\,ds \le$$
$$\le C_1e^{Kt}\int_0^t\int_{\Omega\cap B_0^{K+1}} e^{-\varepsilon|x-x_0|}\|v(s), V_{x_0}\|^2_{1,2}\,dx\,ds \le$$
$$\le C_2 e^{Kt-\varepsilon|x_0|}\int_0^t \|v(s), \Omega\cap B_0^{K+1}\|^2_{1,2}\,dx\,ds$$

Multiplying the estimate (8.22) by $e^{\varepsilon|x_0|}$ and estimating the integral in the right-hand side by (8.12) we derive that

$$\text{(8.23)}\quad e^{\varepsilon|x_0|}\left(\|v_2(t), \Omega\cap B^1_{x_0}\|^2_{1+\kappa,2} + \|\partial_t v_2(t), \Omega\cap B^1_{x_0}\|^2_{\kappa,2}\right) \le$$
$$\le Ce^{K_1t}\left(|v(0)|^2 + |\nabla_x v(0)|^2 + |\partial_t v(0)|^2, e^{-\varepsilon|x-x_0|}\right) +$$
$$+ Ce^{K_1t}\int_0^t\left(|h(s)|^2, e^{-\varepsilon|x-x_0|}\right)\,ds$$

Applying the supremum over $x_0 \in \Omega$ to the both sides of (8.23) we obtain the first estimate of (8.15). Theorem 8.2 is proved.

Remark 8.1. The main advantage of the decomposition (8.15) over the decomposition (8.13) is the compactness of the embedding $E^\kappa_{b,e^{\varepsilon|x|}}(\Omega) \subset E_b(\Omega)$ (the embedding $E^\kappa_{b,\phi}(\Omega) \subset E_{b,\phi}(\Omega)$ is evidently not compact).

§9 The entropy for the locally compact attractor: the upper bounds.

In this Section we obtain the upper estimates of ε-entropy for the locally compact attractor $\mathcal{A}$ of the equation (3.1) constructed in Section 5. Recall that this attractor compact only in F-space $E_{loc}(\Omega)$ but not in the uniform topology of $E_b(\Omega)$. That's why we will estimate the entropy of the restrictions $\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}$. (The entropy of globally compact attractor constructed in Section 6 will be considered in the next Section.)

The main result of this Section is the following theorem.

Theorem 9.1. *Let the assumptions of Theorem 5.1 be valid and let*

$$\text{(9.1)}\qquad \mathrm{vol}_{\Omega,x_0}(R) := \mathrm{vol}(\Omega\cap B^R_{x_0})$$

Then for every $R\in\mathbb{R}_+$, $x_0\in\Omega$ and $0<\varepsilon<1$

$$\text{(9.2)}\quad \mathbb{H}_\varepsilon\left(\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}, E_b(\Omega\cap B^R_{x_0})\right) \le C\,\mathrm{vol}_{\Omega,x_0}(R + K\ln\frac{1}{\varepsilon})\ln\frac{1}{\varepsilon} +$$
$$+ \mathbb{H}_{\varepsilon/L}\left(\omega(g)\big|_{[0,K\ln\frac{1}{\varepsilon}]\times\Omega\cap B^{R+K\ln\frac{1}{\varepsilon}}_{x_0}}, L^2_b([0,K\ln\frac{1}{\varepsilon}]\times\Omega\cap B^{R+K\ln\frac{1}{\varepsilon}}_{x_0})\right)$$

where the constants C, K and L are independent of R and $x_0 \in \Omega$.

Proof. Define a family of weight functions with the rate of growth 1 by the following formula

$$\phi_{R,x_0}(x) = \begin{cases} e^{R-|x-x_0|} & \text{if } |x-x_0| \geq R \\ 1 & \text{if } |x-x_0| \leq R \end{cases} \tag{9.3}$$

Note that we have defined these weight functions in such a way that

$$\mathbb{H}_\varepsilon \left(\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}, E_b(\Omega \cap B^R_{x_0}) \right) \leq \mathbb{H}_\varepsilon \left(\mathcal{A}, E_{b,\phi_{R,x_0}}(\Omega) \right) \tag{9.4}$$

Hence, instead of estimating the entropy of the restriction $\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}$ we will estimate below the entropy of the attractor in weighted Sobolev spaces $E_{b,\phi_{R,x_0}}(\Omega)$.

Note that the attractor $\mathcal{A}$ is bounded in $E_b(\Omega)$ (see Section 5) since there exists a ball $B(\varepsilon_0, 0, E_b(\Omega))$ which contains $\mathcal{A}$ (here and in the following we denote by $B(R, u_0, V)$ the ball of radius R in the space V centered in u_0 and in the case when $u_0 \notin V$ the ball $B(R, u_0, V)$ means $u_0 + B(R, 0, V)$). Let $u_1(t)$ and $u_2(t)$ be two solutions of the family (5.1) with the right-hand sides g_1 and g_2 respectively such that $\xi_{u_i}(0) \in B(2\varepsilon_0, 0, E_b(\Omega))$. Then, according Theorem 8.1 and Corollary 8.1 the difference $v(t) = u_1(t) - u_2(t)$ can be represented in the form $v(t) = \mathcal{P}(t) + \mathcal{R}(t)$. Moreover, there exists a sufficiently large time moment $T_0 > 0$ (for simplicity we assume below that $T_0 = 1$) such that

$$\begin{cases} \|\xi_\mathcal{P}(1)\|_{E^\kappa_{b,\phi_{R,x_0}}(\Omega)} \leq C\|\xi_v(0)\|_{E_{b,\phi_{R,x_0}}(\Omega)} + C\|g_1 - g_2\|_{L^2_{b,\phi_{R,x_0}}([0,1]\times\Omega)} \\ \|\xi_\mathcal{R}(1)\|_{E_{b,\phi_{R,x_0}}(\Omega)} \leq 1/8\|\xi_v(0)\|_{E_{b,\phi_{R,x_0}}(\Omega)} + C\|g_1 - g_2\|_{L^2_{b,\phi_{R,x_0}}([0,1]\times\Omega)} \end{cases} \tag{9.5}$$

Here the constant C in (8.5) is independent of $\xi_{u_1}, \xi_{u_2} \in B(2\varepsilon_0, 0, E_b)$. Moreover, since

$$\phi_{R,x_0}(x+y) \leq e^{|x|}\phi_{R,x_0}(y)$$

then this constant is independent of R and x_0 also.

This decomposition admits to obtain the following recurrent formula for the ε-entropy of the attractor

Lemma 9.1. *The following recurrent inequality is valid*

$$\mathbb{H}_{\varepsilon/2^k} \left(\mathcal{A}, E_{b,\phi_{R,x_0}} \right) \leq \mathbb{H}_\varepsilon \left(\mathcal{A}, E_{b,\phi_{R,x_0}} \right) + \\ + k \ln M_k(\varepsilon) + \mathbb{H}_{\varepsilon/(L2^k)} \left(\omega(g), L^2_{b,\phi_{R,x_0}}([0,k]\times\Omega) \right) \tag{9.6}$$

where $\varepsilon \leq \varepsilon_0$ and

$$\ln M_k(\varepsilon) \leq C \operatorname{vol}_{\Omega,x_0}(R + K \ln \frac{2^k}{\varepsilon}) \tag{9.7}$$

Moreover, the constants C, K and L is independent of k, R, ε and x_0.

Proof. Denote by $U_g(t,s) : E_b(\Omega) \to E_b(\Omega)$, $t \geq s$ the solving operator for the problem (3.1) with the right-hand side g) (i.e $\xi_u(t) = U_g(t,s)\xi_u(s)$ where $\xi_u(t)$ is a solution of (3.1)). Denote also by $\mathcal{O}_\mu(S,V)$ the μ-neighborhood of the set S in the space V.

Let $\{B(\xi_0^i,\varepsilon,E_{b,\phi})\}$, $i=1,\cdots,N_0(\varepsilon)$ be the initial ε-covering of $\mathcal{A}$. We will call this system of balls by the ε-system of 0th order. Let us fix also the $\varepsilon/C2^{k+2}$-covering of the set $w(g)\big|_{[0,k]\times\Omega}$. Let h_j, $j=1,\cdots,N_g(\varepsilon)$ be the centers of this covering. Having the ε-system of 0th order we construct now the $\varepsilon/2$-system of the 1st order. To this end we construct firstly the system

$$(9.8)\quad O_{\varepsilon/4}(B(v_1^{i,j},C\varepsilon+\varepsilon/2^{k+2}),E^\kappa_{b,\phi}),E_{b,\phi});\quad i=1,\cdots,N_0(\varepsilon),\ j=1,\cdots,N_g(\varepsilon)$$

of special sets in $E_{b,\phi}$ and with the constant C the same as in the estimate (9.5) by taking

$$(9.9)\qquad v_1^{i,j}=U_{h_j}(1,0)\xi_0^i$$

Recall now that the set $B(v_1^{i,j},C\varepsilon+\varepsilon/2^{k+2}),E^\kappa_{b,\phi})\cap\mathcal{A}$ is compact in $E_{b,\phi}$ and consequently it can be covered by the finite number of $\varepsilon/4$-balls. Let

$$(9.10)\qquad M_1(\varepsilon)=\max\nolimits_{i,j}N_{\varepsilon/4}\left(\mathcal{A}\cap B(v_1^{i,j},C\varepsilon+\varepsilon/2^{k+2}),E^\kappa_{b,\phi}),E_{b,\phi}\right)$$

Let us cover now every ball $B(v_1^{i,j},C\varepsilon+\varepsilon/2^{k+2}),E^\kappa_{b,\phi})$ from (9.8) by $\le M_1(\varepsilon)$ $\varepsilon/4$-balls in $E_{b,\phi}$. Thus, increasing twicely the radiuses in the obtained covering we will construct the $\varepsilon/2$-system $B(\xi_1^{i,j},\varepsilon/2,E_{b,\phi})$, $i=1,\cdots,N_1'(\varepsilon/2)$, $j=1,\cdots,N_g(\varepsilon)$ of the 1st order which covers all of the sets (9.8) and has the number of balls

$$(9.11)\qquad N_1(\varepsilon/2)\le M_1(\varepsilon)N_0(\varepsilon)N_g(\varepsilon)$$

Note that by definition $B(\xi_1^{i,j},\varepsilon/2,E_{b,\phi})$ belongs to the covering of $B(v_1^{i_1,j},C\varepsilon+\varepsilon/2^{k+2}),E^\kappa_{b,\phi})$ (with the same j!).

Having the $\varepsilon/2$-system of the 1st order, we construct the $\varepsilon/4$-system of 2nd order. To this end we consider the system of sets

$$(9.12)\qquad \mathcal{O}_{\varepsilon/8}(B(v_2^{i,j},C\varepsilon/2+\varepsilon/2^{k+2}),E^\kappa_{b,\phi}),E_{b,\phi})$$

centered in $v_2^{i,j}=U_{h_j}(2,1)\xi_1^{i,j}$. Note that in contrast to the first step we will not change the function h_j any more, i.e. if $B(u_1^{i,j},\varepsilon/4,E_{b,\phi})$ belongs to the covering of $B(v^{i_1,j},C\varepsilon+\varepsilon/2^{k+2},E^\kappa_{b,\phi})$ with $v_1^{i_1,j}=U_{h_j}(1,0)\xi_0^{i_1}$ then we apply the operator $U_h(2,1)$ to $\xi_1^{i,j}$ *only*(!) with $h=h_j$. Covering now every $(C\varepsilon/2+\varepsilon/2^{k+2})$-ball in $E^\kappa_{b,\phi}$ buy the finite number of $\varepsilon/8$-balls, and increasing (as before) twicely the radiuses, we obtain the $\varepsilon/4$-system $B(\xi_2^{i,j},\varepsilon/4,E_{b,\phi})$ of the 2nd order. Analogously to (9.10) we define

$$M_2(\varepsilon)=\max\nolimits_{i,j}N_{\varepsilon/8}\left(\mathcal{A}\cap B(v_2^{i,j},C\varepsilon/2+\varepsilon/2^{k+2},E^\kappa_{b,\phi}),E_{b,\phi}\right)$$

Then the number of $\varepsilon/4$-balls in the covering of 2nd order not exceed

$$N_2(\varepsilon/4)\le M_2(\varepsilon)M_1(\varepsilon)N_0(\varepsilon)N_g(\varepsilon)$$

Iterating the above procedure we obtain finally the $\varepsilon/2^k$-system $B(\xi_k^{i,j},\varepsilon/2^k,E_{b,\phi})$ of kth order and the number of balls in this system not exceed

$$N_k(\varepsilon/2^k)\le M_1(\varepsilon)\cdots M_k(\varepsilon)N_0(\varepsilon)N_g(\varepsilon) \tag{9.13}$$

where

$$M_l(\varepsilon)=\max_{i,j}N_{\varepsilon/2^{l+1}}\left(\mathcal{A}\cap B(v_l^{i,j},C\varepsilon/2^{l-1}+\varepsilon/2^{k+2},E^{\kappa}_{b,\phi}),E_{b,\phi}\right) \tag{9.14}$$

We claim that $\varepsilon/2^k$-system of kth order covers $\mathcal{A}$. Indeed, let $\xi\in\mathcal{A}$. Then due to (5.26) there exists $\xi_0\in\mathcal{A}$ and $h\in\omega(g)$ such that $\xi=\xi_k=U_h(k,0)\xi_0$. Let us find the indexes i and j such that

$$\|\xi_0-\xi_0^i\|_{E_{b,\phi_{R,x_0}}}\le\varepsilon\quad,\quad\|h-h_j\|_{L^2_{b,\phi_{R,x_0}}([0,k]\times\Omega)}\le\varepsilon/C2^{k+2} \tag{9.15}$$

It is possible to do due to our assumptions. Let $\xi_l=U_h(l,0)\xi_0$, $l=1,\cdots,k$. Then, according to the estimates (9.5) and (9.15),

$$\xi_v(1):=\xi_1-v_1^{i,j}=\xi_{\mathcal{P}}(1)+\xi_{\mathcal{R}}(1) \tag{9.16}$$

and

$$\|\xi_{\mathcal{R}}(1)\|_{E_{b,\phi}}\le\varepsilon/8+\varepsilon/2^{k+2}\le\varepsilon/4;\quad\|\xi_{\mathcal{P}}(1)\|_{E^{\kappa}_{b,\phi}}\le C\varepsilon+\varepsilon/2^{k+2} \tag{9.17}$$

and consequently $\xi_1\in\mathcal{O}_{\varepsilon/4}(B(v_1^{i,j},C\varepsilon+\varepsilon/2^{k+2},E^{\kappa}_{b,\phi}),E_{b,\phi})$. Therefore, there exists i_1, such that $\xi_1\in B(\xi_1^{i_1,j},\varepsilon/2,E_{b,\phi})$. Applying the estimate (9.5) again we obtain that $\xi_2\in\mathcal{O}_{\varepsilon/8}(B(v_2^{i_1,j},C\varepsilon/2+\varepsilon/2^{k+2},E^{\kappa}_{b,\phi}),E_{b,\phi})$ and consequently there exists i_2 such that $\xi_2\in B(\xi_2^{i_2,j},\varepsilon/4,E_{b,\phi})$. Arguing analogously, we obtain finally that $\xi=\xi_k\in B(\xi_k^{i_k,j},\varepsilon/2^k,E_{b,\phi})$. Since $\xi\in\mathcal{A}$ is arbitrary then the $\varepsilon/2^k$-system of kth order covers $\mathcal{A}$.

Note also that for every $l\in[0..k]$ we have used in the system of the lth order *only* the balls which intersect the attractor. Consequently, on every step of iteration we may eliminate the balls which do not intersect the attractor. Since $\varepsilon\le\varepsilon_0$ then we may assume that $\xi_l^{i,j}\in\mathcal{O}_{\varepsilon_0}(\mathcal{A},E_b)\subset B(2\varepsilon_0,0,E_b)$ for every i,j,l and consequently we really can apply the estimates (9.5) to the difference (9.16).

Thus, the estimate (9.13) implies now that

$$\mathbb{H}_{\varepsilon/2^k}\left(\mathcal{A},E_{b,\phi_{R,x_0}}\right)\le\sum_{i=1}^{k}\ln M_i(\varepsilon)+ \\ +\mathbb{H}_{\varepsilon}\left(\mathcal{A},E_{b,\phi_{R,x_0}}\right)+\mathbb{H}_{\varepsilon/C2^{k+2}}\left(w(g),L^2_{b,\phi_{R,x_0}}([0,k]\times\Omega)\right) \tag{9.18}$$

To complete the proof of the lemma it remains to estimate the numbers $M_i(\varepsilon)$.

It follows from Corollary 3.2 that

$$\|\mathcal{A}\|_{E_b(\Omega)}\le K_1 \tag{9.19}$$

Consequently, according to the estimate $\phi_{R,x_0}(x)^{1/2} \le \frac{\varepsilon}{2^{l+3}K_1}$ if $|x-x_0| > R + K\ln\frac{2^{l+2}}{\varepsilon} \le R + K_1 \ln\frac{2^k}{\varepsilon} \equiv R_k(\varepsilon)$, we obtain that

(9.20)

$$M_l(\varepsilon) \le \max_{i,j} N_{\varepsilon/2^{l+1}}\left(\mathcal{A}\cap B(v_l^{i,j}, C\varepsilon/2^{l-1}+\varepsilon/2^{k+2}, E^{\kappa}_{b,\phi_{R,x_0}}), E_{b,\phi_{R,x_0}}\right) \le$$
$$\max_{i,j} N_{\varepsilon/2^{l+2}}\left(\mathcal{A}\cap B(v_l^{i,j}, C\varepsilon/2^{l-2}, E^{\kappa}_{b,\phi_{R,x_0}}(\Omega\cap B^{R_k(\varepsilon)}_{x_0})), E_{b,\phi_{R,x_0}}(\Omega\cap B^{R_k(\varepsilon)}_{x_0})\right)$$
$$\le N_{\varepsilon/2^{l+2}}\left(B(0, C\varepsilon/2^{l-2}, E^{\kappa}_{b,\phi_{R,x_0}}(\Omega\cap B^{R_k(\varepsilon)}_{x_0})), E_{b,\phi_{R,x_0}}(\Omega\cap B^{R_k(\varepsilon)}_{x_0})\right) \le$$
$$\le N_{1/(16C)}\left(B(0,1, E^{\kappa}_{b,\phi_{R,x_0}}(\Omega\cap B^{R_k(\varepsilon)}_{x_0})), E_{b,\phi_{R,x_0}}(\Omega\cap B^{R_k(\varepsilon)}_{x_0})\right)$$

Thus, it remains to estimate the entropy of the unitary $E^{\kappa}_{b,\phi_{R,x_0}}(\Omega\cap B^{R_k(\varepsilon)}_{x_0})$-ball in the space $E_{b,\phi_{R,x_0}}(\Omega\cap B^{R_k(\varepsilon)}_{x_0})$. To this end we need the smooth version of weight function ϕ_{R,x_0}.

Let the function $\psi_{R,x_0}(x)\in C_0^\infty(\mathbb{R}^n)$ satisfy the assumptions

$$\left\{\begin{array}{l} 1.\ \ \psi_{R,x_0}(x) = \phi_{R,x_0}(x) \text{ if } |x-R|>1 \\ 2.\ \ \nabla_x\psi_{R,x_0} \le \psi_{R,x_0};\ \ \nabla^2_x\psi_{R,x_0} \le \psi_{R,x_0} \\ 3.\ \ C_1'\phi_{R,x_0} \le \psi_{R,x_0} \le C_2'\phi_{R,x_0}\end{array}\right. \tag{9.21}$$

It is not difficult to verify that such functions exist. Moreover the constants C_1' and C_2' are independent of R.

Proposition 9.1. *Let $F:\xi\to\psi^{1/2}_{R,x_0}\xi$. Then F realizes the linear isomorphizm between E_b and $E_{b,\phi_{R,x_0}}$ and also between E^κ_b and $E^\kappa_{b,\phi_{R,x_0}}$. Moreover*

$$C_1\|\xi\|_{E_{b,\phi_{R,x_0}}(\Omega\cap B^{R_k(\varepsilon)})} \le \|F\xi\|_{E_b(\Omega\cap B^{R_k(\varepsilon)})} \le C_2\|\xi\|_{E_{b,\phi_{R,x_0}}(\Omega\cap B^{R_k(\varepsilon)})} \tag{9.22}$$

where constants C_1 and C_2 are independent of R_k, R, and x_0 and the analogous fact hold for the spaces E^κ_b.

The proof of this fact can be obtained directly using the assumptions (9.22) and the explicit expression for the norms $W^{1+\kappa,2}$ given by (1.12) (see also [28]).

According to Proposition 9.1 instead of estimating the entropy in weighted Sobolev spaces $E_{b,\phi_{R,x_0}}$ it is sufficient to estimate it in the spaces E_b, i.e.

$$M_l(\varepsilon) \le N_{1/(16C)}\left(B(0,1,E^\kappa_{b,\phi_{R,x_0}}(\Omega\cap B^{R_k(\varepsilon)}_{x_0})), E_{b,\phi_{R,x_0}}(\Omega\cap B^{R_k(\varepsilon)}_{x_0})\right) \le$$
$$\le N_{1/(C_3)}\left(B(0,1,E^\kappa_b(\Omega\cap B^{R_k(\varepsilon)}_{x_0})), E_b(\Omega\cap B^{R_k(\varepsilon)}_{x_0})\right) \tag{9.23}$$

Applying the estimate (7.10) to the right-hand side of the estimate (9.23) we obtain finally

$$\ln M_l(\varepsilon) \le C_4 \operatorname{vol}_{\Omega,x_0}\left(R + K\ln\frac{2^k}{\varepsilon}\right) \tag{9.24}$$

Lemma 9.1 is proved.

Now we are in position to complete the proof of Theorem 9.1.

According to the definition of ε_0

$$\mathbb{H}_{\varepsilon_0}\left(\mathcal{A}, L^2_{b,\phi_{R,x_0}}(\Omega)\right) = 0$$

for any R and x_0. Let us apply now the recurrent estimate (9.6) with $\varepsilon = \varepsilon_0$. Then we will have

$$\mathbb{H}_{\varepsilon_0/2^k}\left(\mathcal{A}, E_{b,\phi_{R,x_0}}\right) \le Ck\,\mathrm{vol}_{\Omega,x_0}\left(R + K\ln\frac{2^k}{\varepsilon_0}\right) + \\ + \mathbb{H}_{\varepsilon_0/2^kL}\left(\omega(g), L^2_{b,\phi_{R,x_0}}([0,k]\times\Omega)\right) \tag{9.25}$$

Let us fix an arbitrary $\beta < \varepsilon_0$ and take $k = k(\beta)$ such that

$$\frac{\varepsilon_0}{2^{k-1}} \ge \beta \ge \frac{\varepsilon_0}{2^k} \text{ and consequently } 2^k \le \frac{2\varepsilon_0}{\beta} \tag{9.26}$$

Then (9.25) and (9.26) imply that

$$\mathbb{H}_\beta\left(\mathcal{A}, E_{b,\phi_{R,x_0}}\right) \le \mathbb{H}_{\varepsilon_0/2^k}\left(\mathcal{A}, E_{b,\phi_{R,x_0}}\right) \le \\ \le Ck\,\mathrm{vol}_{\Omega,x_0}\left(R + K\ln\frac{2^k}{\varepsilon_0}\right) + \mathbb{H}_{L_1\varepsilon_0/2^k}\left(\omega(g), L^2_{b,\phi_{R,x_0}}([0,k]\times\Omega)\right) \le \\ \le C_1\,\mathrm{vol}_{\Omega,x_0}\left(R + K_1\ln\frac{2}{\beta}\right)\ln\frac{1}{\beta} + \mathbb{H}_{\beta/L_2}\left(\omega(g), L^2_{b,\phi_{R,x_0}}([0,K_1\ln\frac{1}{\beta}]\times\Omega)\right) \tag{9.27}$$

Using the now the fact that $\|\omega(g)\|_{L^2_b(\mathbb{R}_+\times\Omega)} \le C$ and arguing as in the proof of (9.20) we derive that

$$\mathbb{H}_\varepsilon\left(\mathcal{A}, E_b(\Omega\cap B^R_{x_0})\right) \le C\,\mathrm{vol}_{\Omega,x_0}\left(R + K\ln\frac{1}{\varepsilon}\right)\ln\frac{1}{\varepsilon} + \\ + \mathbb{H}_{\varepsilon/L_2}\left(\omega(g), L^2_b([0,K\ln\frac{1}{\varepsilon}]\times\Omega\cap B^{R+K\ln\frac{1}{\varepsilon}}_{x_0})\right) \tag{9.28}$$

Theorem 9.1 is proved.

We consider now a number of corollaries of the main Theorem 9.1.

Corollary 9.1. Let the equation (3.1) be autonomous ($g = g(x)$). Then

$$\mathbb{H}_\varepsilon\left(\mathcal{A}, E_b(\Omega\cap B^R_{x_0})\right) \le C\,\mathrm{vol}_{\Omega,x_0}\left(R + K\ln\frac{1}{\varepsilon}\right)\ln\frac{1}{\varepsilon} \tag{9.29}$$

Particularly, if $\Omega = R^n$ then $\mathrm{vol}_{\Omega,x_0}(r) = cr^n$ and consequently

$$\mathbb{H}_\varepsilon\left(\mathcal{A}, E_b(B^R_{x_0})\right) \le C\left(R + K\ln\frac{1}{\varepsilon}\right)^n\ln\frac{1}{\varepsilon} \tag{9.30}$$

Taking $R = \ln\frac{1}{\varepsilon}$ we obtain that

$$\mathbb{H}_\varepsilon\left(\mathcal{A}, E_b(B^{\ln\frac{1}{\varepsilon}}_{x_0})\right) \le C_1\left(\ln\frac{1}{\varepsilon}\right)^{n+1} \tag{9.31}$$

Note that the estimate (9.30) gives the same type of upper bounds for $R = 1$ and $R = \ln \frac{1}{\varepsilon}$.

Corollary 9.2. Let Ω be a bounded domain. Then Theorem 9.1 implies the estimate

$$\mathbb{H}_\varepsilon\left(\mathcal{A}, E_b(\Omega)\right) \le C\,\mathrm{vol}(\Omega)\ln\frac{1}{\varepsilon} + \mathbb{H}_{\varepsilon/L}\left(\omega(g), L^2_b([0, K\ln\frac{1}{\varepsilon}] \times \Omega)\right) \tag{9.32}$$

which extends the estimate, obtained in [6] to the case of hyperbolic equations. Particularly if the equation (3.1) is autonomous ($g = g(x)$), then the estimate (9.32) reflects the well-known fact that in this case the attractor $\mathcal{A}$ has the finite fractal dimension.

Corollary 9.3. Let $\Omega = \mathbb{R}^k \times \omega^{n-k}$ be a cylindrical domain where ω is bounded. Then the estimate (9.30) gives the following bound of the ε-entropy of the autonomous attractor

$$\mathbb{H}_\varepsilon\left(\mathcal{A}, E_b(\Omega \cap B^R_{x_0})\right) \le C\left(R + K\ln\frac{1}{\varepsilon}\right)^k \ln\frac{1}{\varepsilon} \tag{9.33}$$

Following to [7], [18] and [28] one may define the entropy per unit volume for the attractor $\mathcal{A}$.

Definition 9.1. *Let $\mathcal{A} \subset E_b(\Omega)$ be a compact set in the space $E_{loc}(\Omega)$. Then the ε-entropy per unit volume is defined to be the following number*

$$\overline{\mathbb{H}}_\varepsilon(\mathcal{A}) = \limsup_{R\to\infty} \frac{\mathbb{H}_\varepsilon\left(\mathcal{A}, E_b(\Omega \cap B^R_0)\right)}{\mathrm{vol}_{\Omega,0}(R)} \tag{9.34}$$

Corollary 9.4. Let the equation (3.1) be autonomous. Then

$$\overline{\mathbb{H}}_\varepsilon(\mathcal{A}) \le C\ln\frac{1}{\varepsilon} \tag{9.35}$$

Indeed, the estimate (9.35) is an immediate corollary of the estimate (9.30) and trivial assertion

$$\lim_{R\to\infty} \frac{\mathrm{vol}_{\Omega,x_0}(R + C_1)}{\mathrm{vol}_{\Omega,x_0}(R)} = 1 \tag{9.36}$$

To formulate the result for the entropy per unit volume for the nonautonomous case we need the following definition (see [28])

Definition 9.2. *Let the entropy per unit volume of the right-hand side be the following number*

$$\overline{\mathbb{H}}_\varepsilon(g) = \limsup_{R\to\infty} \frac{\mathbb{H}_{\varepsilon/L}\left(\omega(g), L^2_b([0, K\ln\frac{1}{\varepsilon}] \times \Omega \cap B_0^{R+K\ln\frac{1}{\varepsilon}})\right)}{\mathrm{vol}_{\Omega,0}(R)} \tag{9.37}$$

Corollary 9.5. Let $\overline{\mathbb{H}}_\varepsilon(g) < \infty$. Then

$$\overline{\mathbb{H}}_\varepsilon(\mathcal{A}) \le C\ln\frac{1}{\varepsilon} + \overline{\mathbb{H}}_\varepsilon(g) \tag{8.38}$$

Corollary 9.6. Note that if $g(t,x) = \phi(t)g_0(x)$ where $g_0 \in L^p_b(\Omega)$ and ϕ is translation-compact in $L^\infty_{loc}(\mathbb{R}_+)$ then $\overline{\mathbb{H}}_\varepsilon(g) \equiv 0$ and consequently

$$\overline{\mathbb{H}}_\varepsilon(\mathcal{A}) \le C \ln \frac{1}{\varepsilon} \tag{9.39}$$

Remark 9.1. Let the right-hand side g be (t,x)-almost-periodic in $C(\mathbb{R}\times\mathbb{R}^n)$. Then, it not difficult to verify that $\overline{\mathbb{H}}_\varepsilon(g) = 0$ and consequently the assertion of Corollary 9.6 remains valid for such right-hand sides.

Definition 9.3. *Let $h_{sp}(\mathcal{A})$ be the following number*

$$h_{sp}(\mathcal{A}) = \limsup_{\varepsilon\to 0} \frac{\overline{\mathbb{H}}_\varepsilon(\mathcal{A})}{\ln\frac{1}{\varepsilon}} \tag{9.40}$$

Corollary 9.7. Let the assumptions of Corollary 9.4 or 9.6 or Remark 9.1 hold. Then

$$h_{sp}(\mathcal{A}) < \infty \tag{8.41}$$

Remark 9.2. The number $h_{sp}(\mathcal{A})$ can be interpreted as some quantitative characteristic of the phenomena of *spatial* chaotisity of the dynamical system, generated by the equation (3.1) (see [28]).

§10 The entropy for the globally compact attractor: the upper bounds.

In this Section we consider the particular case of the equation (3.1) where $f_2 \equiv 0$ and the right-hand side g is translation compact in $L^2_{loc}(\mathbb{R}_+, \dot{L}^2_b(\Omega))$. Using the decomposition (8.14) and estimates (8.15) we essentially improve the entropy estimate (9.2) obtained in the previous Section. Particularly we will prove that if the right-hand side $g(t)$ is in the appropriate sense finite dimensional (i.e. g is autonomous or quasiperiodic with respect to t) then the attractor $\mathcal{A}$ has a finite fractal dimension in $E_b(\Omega)$.

The main result of this Section is the following theorem

Theorem 10.1. *Let the assumptions of Theorem 6.1 hold. Then the entropy of the attractor $\mathcal{A}$ possesses the following estimate*

$$\mathbb{H}_\varepsilon(\mathcal{A}, E_b(\Omega)) \le C_1 \ln\frac{1}{\varepsilon} + C_2 + \mathbb{H}_{\varepsilon/L}\left(\omega(g), L^2_b([0, K\ln\frac{1}{\varepsilon}]\times\Omega)\right) \tag{10.1}$$

where the constants C_i, K and L are independent of ε.

Proof. Let $\mu > 0$ and the μ-neighborhood $\mathcal{O}_\mu(\mathcal{A}) = \mathcal{O}_\mu(\mathcal{A}, E_b(\Omega)$ be the same as in Theorem 8.2. Then, according to Theorem 8.2, we may find a sufficiently large time moment T_0 (for simplicity we assume that $T_0 = 1$ such that for every $\xi_{u_i} \in \mathcal{O}_\mu(\mathcal{A})$ $i = 1,2$ the difference $v(t) = u_1(t) - u_2(t)$ of the corresponding solutions of the equation (3.1) with the right-hand sides g_1 and g_2 respectively can be represented in the following form $\xi_v(t) = \xi_{u_1}(t) - \xi_{u_2}(t) = \xi_{\mathcal{P}'}(t) + \xi_{\mathcal{R}'}(t)$ and

$$\begin{cases} \|\xi_{\mathcal{P}'}(1)\|_{E^\kappa_{b,e^{\delta|x|}}(\Omega)} \le C\left(\|\xi_v(0)\|_{E_b(\Omega)} + \|g_1 - g_2\|_{L^2_b([0,1]\times\Omega)}\right) \\ \|\xi_{\mathcal{R}'}(1)\|_{E_b(\Omega)} \le 1/8\|\xi_v(0)\|_{E_b(\Omega)} + C\|g_1 - g_2\|_{L^2_b([0,1]\times\Omega)} \end{cases} \tag{10.2}$$

where the constants C and $\delta > 0$ depends only on the equation (3.1).

As in previous Section this decomposition admits to obtain the recurrent inequality for the entropy of the attractor.

Lemma 10.1. *Let the above assumptions hold. Then for every $\varepsilon \le \mu$ and every $k \in \mathbb{N}$ the following estimate is valid:*

$$\mathbb{H}_{\varepsilon/2^k}\left(\mathcal{A}, E_b(\Omega)\right) \le \mathbb{H}_{\varepsilon}\left(\mathcal{A}, E_b(\Omega)\right) + Ck + \mathbb{H}_{\varepsilon/(L2^k)}\left(\omega(g), L^2_b([0,k]\times\Omega)\right) \tag{10.3}$$

where the constants C and L depends only on the equation.

The proof of this Lemma is analogous to the proof of Lemma 9.1 but the fact of compactness of the embedding $E_{b,e^{\delta|x|}}(\Omega) \subset E_b(\Omega)$ admits us to simplify the proof and verify that the numbers $M_k(\varepsilon)$ are independent of ε and k.

Indeed, let $B(\xi_0^i, \varepsilon, E_b(\Omega)$, $i = 1, \cdots, N_0(\varepsilon)$ be the initial ε-covering of the attractor $\mathcal{A}$. Arguing as in the proof of Lemma 9.1 with formal replacing $E_{b,\phi}$ by E_b and $E^{\kappa}_{b,\phi}$ by $E_{b,e^{\delta|x|}}$ we derive the $\varepsilon/2^k$-covering of the attractor with the number of balls

$$N_k(\varepsilon/2^k) \le M_1(\varepsilon)\cdots M_k(\varepsilon) N_0(\varepsilon) N_g(\varepsilon) \tag{10.4}$$

where $N_g(\varepsilon)$ is the number of balls in the $\varepsilon/C2^{k+2}$ covering of $\omega(g)$ in $L^2_b([0,k]\times\Omega)$ and the multipliers $M_l(\varepsilon)$ are defined by (9.14) (where the symbol $E^{\kappa}_{b,\phi}$ is replaced by $E^{\kappa}_{b,e^{\delta|x|}}$ and the symbol $E_{b,\phi}$ – by E_b.

Note now that in contrast to the case of Lemma 9.1 the embedding $E^{\kappa}_{b,e^{\delta|x|}}(\Omega) \subset E_b(\Omega)$ is compact and consequently the multipliers $M_l(\varepsilon)$ can be estimated in the following way

$$\begin{aligned} M_l(\varepsilon) := \max\nolimits_{i,j} N_{\varepsilon/2^{l+1}}\left(\mathcal{A}\cap B(v_l^{i,j}, C\varepsilon/2^{l-1} + \varepsilon/2^{k+2}, E^{\kappa}_{b,e^{\delta|x|}}), E_b\right) \le \\ \le \max\nolimits_{i,j} N_{\varepsilon/2^{l+1}}\left(B(v_l^{i,j}, C\varepsilon/2^{l-1} + \varepsilon/2^{k+2}, E^{\kappa}_{b,e^{\delta|x|}}), E_b\right) \le \\ \le N_{\varepsilon/2^{l+2}}\left(B(0, C\varepsilon/2^{l-2}, E^{\kappa}_{b,e^{\delta|x|}}(\Omega)), E_b(\Omega)\right) \\ \le N_{1/(16C)}\left(B(0,1,E^{\kappa}_{b,e^{\delta|x|}}(\Omega)), E_b(\Omega)\right) := C_1 \end{aligned} \tag{10.5}$$

Thus, (10.5) implies that the multipliers $M_l(\varepsilon)$ are independent of ε and l. Inserting this estimate in (10.4) we obtain the estimate (10.3). Lemma 10.1 is proved.

The estimate (10.1) can be derived from the result of Lemma 10.1 in the same way as we derive the estimate (9.2) from the result of Lemma 9.1 (the only difference that we should start with $\varepsilon = \mu$ which gives the additional term $C_2 := \mathbb{H}_{\mu}(\mathcal{A}, E_b)$ in the estimate (10.1). Theorem 10.1 is proved.

We give now a number of corollaries of the estimate (10.1).

Corollary 10.1. *Let the equation* (3.1) *be autonomous (i.e. $g(t) \equiv g$ is independent of t) and let the assumptions of Theorem 4.1 hold. Then the attractor $\mathcal{A}$ has a finite fractal dimension*

$$\dim_F(\mathcal{A}) \le C_1 \tag{10.6}$$

where C_1 is the same as in (10.1).

To formulate the corollaries for the nonautonomous case we need the following definition.

Definition 10.1. *Let the function $g(t)$ be translation compact in $L^2_{loc}(\mathbb{R}_+, \dot{L}^2_b(\Omega))$. The fractal dimension of the right-hand side g is defined to be the following number*

$$\dim_F(g) := \limsup_{\varepsilon\to 0} \frac{\mathbb{H}_{\varepsilon/L}(w(g), L^2_b([0, K\ln 1/\varepsilon]\times\Omega))}{\ln\frac{1}{\varepsilon}} \tag{10.7}$$

where the constants K and L are the same as in (10.1).

Corollary 10.2. *Let the assumptions of Theorem 10.1 hold and let also the right-hand side g have a finite fractal dimension. Then the attractor $\mathcal{A}$ also has the finite fractal dimension and*

$$\dim_F(\mathcal{A}) \le C_1 + \dim_F(g) \tag{10.8}$$

where C_1 is the same as in (10.6).

Let us consider now the examples of the right-hand sides which have the finite fractal dimension. For the first we consider the so called quasiperiodic case.

Let $\mathbb{T}^k := \mathbb{R}^k/\mathbb{Z}^k$ be k-dimensional torus, $\alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{T}^k$ be the vector of rationally independent frequences and S^α_t ($S^\alpha_t(p) := (\alpha t + p)\, mod\, \mathbb{Z}^k$, $p \in \mathbb{T}^k$) be a standard linear flow on $\mathbb{T}^k$. A function $g \in C(\mathbb{R}, \dot{L}^2_b(\Omega))$ is called quasiperiodic with k independent frequences if there exists a function $G \in C(\mathbb{T}^k, \dot{L}^2_b(\Omega))$ and a vector of frequences α such that

$$g(t) = G(S^\alpha_t(0)) \tag{10.9}$$

Proposition 10.1. *Let the function g be quasiperiodic in $\dot{L}^2_b(\Omega)$ and let the function G in the representation* (10.9) *belong to the space $C^1(\mathbb{T}^k, \dot{L}^2_b(\Omega))$. Then*

$$\dim_F(g) \le k \tag{10.10}$$

Indeed, since $\|S^\alpha_t(p_1) - S^\alpha_t(p_2)\|_{\mathbb{T}^k} = \|p_1 - p_2\|_{\mathbb{T}^k}$ then the hull $\mathcal{H}^+(g)$ is compact in the space $C_b(\mathbb{R}_+, \dot{L}^2_b(\Omega))$. Moreover, since $G \in C^1$ then then

$$\mathbb{H}_{\varepsilon/L}(w(g), L^2_b([0, K\ln\frac{1}{\varepsilon}]\times\Omega)) \le \mathbb{H}_{\varepsilon/L}(\mathcal{H}^+(g), L^2_b(\mathbb{R}\times\Omega)) \le \\ \le \mathbb{H}_{\varepsilon/CL}(\mathcal{H}^+(S^\alpha_t(0)), C_b(\mathbb{R}, \mathbb{T}^k)) \le \mathbb{H}_{\varepsilon/CL}(\mathbb{T}^k, \mathbb{R}^k) \le (k + \bar{\bar{o}}(1))\ln\frac{1}{\varepsilon} \tag{10.11}$$

Corollary 10.3. *Let the assumptions of Theorem 10.1 hold, let also the right-hand side be quasiperiodic with k independent frequences and the function G in the representation* (10.9) *belong to the space. $C^1(\mathbb{T}^k, \dot{L}^2_b(\Omega))$. Then the attractor $\mathcal{A}$ has the finite dimension and*

$$\dim_F(\mathcal{A}) \le C_1 + k \tag{10.12}$$

where the constant C_1 is the same as in (10.6).

Remark 10.1. Note that the assumption $G \in C^1$ is essential for the estimate (10.12). Indeed, it is not difficult to construct a *continuous* (but not Holder continuous) function $G \in C(\mathbb{T}^k, \dot{L}^2_b(\Omega))$ (even for the periodic case $k = 1$) such that $\dim_F(g) = \infty$. If the function G is Holder continuous with the exponent $\alpha < 1$ then we may claim only that $\dim_F(g) \le \frac{k}{\alpha}$.

The result of Proposition 10.1 can be generalized in the following way.

Let M be a compact metric space with the finite fractal dimension $\dim_F(M)$ and let $S_t : M \to M$ be a semigroup acting on M. Assume also that the semigroup S_t has a finite Liapunov exponent $\mu \ge 0$, i.e.

$$\operatorname{dist}_M(S_t(m_1), S_t(m_2)) \le e^{\mu t}\operatorname{dist}_M(m_1, m_2) \quad \text{for } m_1, m_2 \in M \tag{10.13}$$

Proposition 10.2. *Let* (10.13) *be valid and let the function* $G \in C(M, \dot{L}^2_b(\Omega))$ *be globally Lipschitz continuous on* M. *Then for every* $m \in M$ *the function*

$$g(t) := G(S_t(m)) \tag{10.14}$$

has the finite fractal dimension. Moreover,

$$\dim_F(g) \le \dim_F(M)(1 + K\mu) \tag{10.15}$$

where K *is the same as in* (10.1).

Indeed, since G is globally Lipschitz continuous then

$$\mathbb{H}_{\varepsilon/L}(\omega(g), L^2_b([0,T(\varepsilon)] \times \Omega) \le \mathbb{H}_{\varepsilon/CL}(\mathcal{H}^+(S_t(m)), L^\infty([0,T(\varepsilon)], M)) \tag{10.16}$$

where $T(\varepsilon) := K \ln \frac{1}{\varepsilon}$. The estimate (10.13) implies that

$$\begin{aligned} \mathbb{H}_{\varepsilon/CL}(\mathcal{H}^+(S_t(m)), L^\infty([0,T(\varepsilon)], M)) \le \mathbb{H}_{e^{-\mu T(\varepsilon)}\varepsilon/CL}(M, M) \le \\ \le \left(\dim_F(M) + \overline{\overline{o}}(1)\right) \left(\ln \frac{1}{\varepsilon} + \mu T(\varepsilon) + \ln(CL)\right) \end{aligned} \tag{10.17}$$

The estimates (10.16) and (10.17) imply (10.15).

Corollary 10.4. *Let the assumptions of Theorem 10.1 hold, let also the right-hand side* g *satisfy the assumptions of Proposition 10.2. Then the attractor* $\mathcal{A}$ *has the finite dimension and*

$$\dim_F(\mathcal{A}) \le C_1 + \dim_F M\,(1 + \mu K) \tag{10.18}$$

where the constant C_1 *is the same as in* (10.6) *and* K *is the same as* (10.1).

§11 Unstable manifolds and lower bounds of entropy.

In previous Sections the upper bounds for the ε-entropy of the attractor of the equation (3.1) has been obtained. In this Section, using the technique of infinite dimensional unstable manifolds, developed in [10], [28] we obtain the lower bounds of the entropy for rather wide class of equations of the type (3.1).

We assume throughout of this Section that $\Omega = \mathbb{R}^n$ and the equation (3.1) has the form

$$\partial_t^2 u + \gamma \partial_t u = \Delta_x u + \alpha^2 u - f(u), \quad f(0) = 0, \quad f'(0) = 0, \quad \alpha > 0 \tag{11.1}$$

and the nonlinear term $f \in C^2$ satisfies the conditions

$$\begin{cases} 1.\ f(u).u \ge -C + \beta |u|^2, \quad \beta > \alpha^2 \\ 2.\ f'(u) \ge -K; \quad |f'(u)| \le C(1 + |u|^{q_1}); \\ 3.\ |f''(u)| \le C(1 + |u|^{q_2}) \end{cases} \tag{11.2}$$

where the exponents q_i are the same as in (3.2).

Moreover since $f(0) = f'(0) = 0$ it is natural to assume also that there exists $0 < \eta \le \min\{1, q_1\}$ such that

$$|f'(u)| \le C|u|^\eta \phi_\eta(u), \quad |\phi_\eta(u)| \le C_1(1 + |u|^{q_1 - \eta}) \tag{11.3}$$

To construct the unstable manifold of the equation (11.1) near the equilibria point $u \equiv 0$ we study for the first the backward solutions for the linear ($f \equiv 0$) nonhomogeneous equation (11.1) with the right-hand side $h(t)$.

Definition 11.1. *Let $\beta > 0$. Then we define the space $\mathbb{L}_\beta(E_b)$ by the following expression*

$$\mathbb{L}_\beta(E_b) = \{\xi \in C(\mathbb{R}_-, E_b(\Omega)) : \|\xi\|_{\mathbb{L}_\beta} := \sup_{t\le 0} e^{-\beta t}\|\xi(t)\|_{E_b(\mathbb{R}^n)} < \infty\} \tag{11.4}$$

The space $\mathbb{L}_\beta(L^2_b) := \mathbb{L}_\beta(\mathbb{R}_-, L^2_b(\Omega))$ can be defined analogously.

Proposition 11.1. *Let $\mu \in \mathbb{R}^n$ and*

$$\lambda_+(\mu) = \frac{-\gamma + \sqrt{\gamma^2 + 4(\alpha^2 - |\mu|^2)}}{2}; \quad \lambda_-(\mu) = \frac{-\gamma - \sqrt{\gamma^2 + 4(\alpha^2 - |\mu|^2)}}{2} \tag{11.5}$$

Assume also that $\beta > \lambda_+(0)$ and $h \in \mathbb{L}_\beta(L^2_b)$. Then the equation

$$\partial_t^2 v + \gamma\partial_t v = \Delta_x v + \alpha^2 v + h(t), \quad t \le 0 \tag{11.6}$$

possesses the unique solution $v \in \mathbb{L}_\beta(E_b)$ and consequently defines the linear operator $\mathbb{T}_\beta : \mathbb{L}_\beta(L^2_b) \to \mathbb{L}_\beta(E_b)$, $v(t) = (\mathbb{T}_\beta h)(t)$.

Proof. Indeed, the equation (11.5) can be rewritten in the following form:

$$\partial_t \xi_v(t) = A\xi_v(t) + \tilde{h}(t) \tag{11.7}$$

where $A = \begin{pmatrix} 0 & ; & 1 \\ \Delta_x + \alpha^2 & ; & -\gamma \end{pmatrix}$ and $\tilde{h} = \begin{pmatrix} 0 \\ h \end{pmatrix}$.

It is not difficult to verify that the spectrum $\sigma(A)$ has the following form

$$\sigma(A) = \{\operatorname{Re} z = -\frac{\gamma}{2}\} \cup \{\operatorname{Im} z = 0;\ \lambda_-(0) \le \operatorname{Re} z \le \lambda_+(0)\} \tag{11.8}$$

Thus $\beta > \lambda_+(0)$ implies that $\beta > \operatorname{Re}\sigma(A)$ and consequently the unique solution of (11.7) from $\mathbb{L}_\beta(E_b)$ is given by the following expression

$$\xi_v(t) := \int_{-\infty}^t e^{A(t-s)}\tilde{h}(s)\, ds$$

Proposition 11.1 is proved.

Proposition 11.2. *Let $\mu_0 > 0$ satisfy the inequality $\lambda_+(\sqrt{n}\mu_0) > 0$. Then for every $u_0 \in \mathbb{B}_{\mu_0}$ the backward hyperbolic problem*

$$\begin{cases} \partial_t^2 v + \gamma\partial_t v = \Delta_x v + \alpha^2 v, \quad t \le 0 \\ v\big|_{t=0} = u_0, \quad v\big|_{\partial\Omega} = 0 \end{cases} \tag{11.9}$$

possesses a solution $v \in \mathbb{L}_{\beta_0}$ with $0 < \beta_0 < \lambda_+(\sqrt{n}\mu_0)$. Moreover this solution defines a linear operator $\mathbb{P}_{\beta_0} : \mathbb{B}_{\mu_0} \to \mathbb{L}_{\beta_0}(E_b)$, $v(t) = (\mathbb{P}_{\beta_0} u_0)(t)$ and consequently, we have the operator $M : \mathbb{B}_{\mu_0} \to E_b(\Omega)$ defined by

$$M(u_0) := \mathbb{P}_{\beta_0}(u_0)(0) \tag{11.10}$$

Indeed, this solution can be defined by formula

$$\widehat{v}(t,\mu) = e^{\lambda_+(\mu)t}\widehat{u}_0(\mu) \tag{11.11}$$

where $\widehat{v}(t,\mu)$ is $x \to \mu$ Fourier transform.

Now we are in position to study the neighborhood of zero equilibria point for the nonlinear equation.

Theorem 11.1. *Let the nonlinearity f satisfy* (11.2), (11.3) *and $\mathcal{A}$ be the attractor of the equation* (11.1) *which exists according to Theorem 4.2. Then there exist $\mu_0 = \mu_0(\alpha)$, $\delta_0 = \delta_0(f, \alpha)$ and C^1-map*

$$\mathcal{U}_0 \, : \, B(0, \delta_0, \mathbb{B}_{\mu_0}) \to \mathcal{A} \tag{11.12}$$

Moreover,

$$\|\mathcal{U}_0(u_0) - M(u_0)\|_{E_b(\mathbb{R}^n)} \le C\|u_0, \mathbb{R}^n\|^{1+\eta}_{L^2_b(\mathbb{R}^n)} \tag{11.13}$$

for every $u_0 \in B(0, \delta_0, \mathbb{B}_{\mu_0})$ and η defined in (11.3).

Proof. The proof of this Theorem is based on the implicit function theorem and on the following lemma.

Lemma 11.1. *Let $f(0) = f'(0) = 0$ and satisfy the assumptions of* (11.3). *Then for every $\beta > 0$ the Nemitskij operator $Fu = f(u)$ belongs to $C^1(\mathbb{L}_\beta(E_b), \mathbb{L}_{(1+\eta)\beta}(L^2_b))$.*

Proof. Since $f(0) = f'(0) = 0$ then (11.3) implies $f(u) = u^{1+\eta}\phi(u)$ with $|\phi(u)| \le C_2(1+|u|^{q_1-\eta})$ and consequently the Nemitskij operator $F : \mathbb{L}_\beta \to \mathbb{L}_{(1+\eta)\beta}$. Indeed, according to Holder inequality and Sobolev embedding theorem

$$\begin{aligned} \|f(u)\|_{L^2_b} &\le C\| \, |u|^{1+\eta}(1 + |u|^{q_1-\eta})\|_{L^2_b} \le \\ &\le C_1\|u, \mathbb{R}^n\|^{1+\eta}_{b,0,2(q_1+1)}\left(1 + \|u, \mathbb{R}^n\|^{q_1-\eta}_{b,0,2(q_1+1)}\right) \le \\ &\le C_2\|u, \mathbb{R}^n\|^{1+\eta}_{b,1,2}Q(\|u, \mathbb{R}^n\|_{b,1,2}) \le C_2\|\xi_u\|^{1+\eta}_{E_b}Q(\|\xi_u\|_{E_b}) \end{aligned} \tag{11.14}$$

Multiplying (11.14) by $e^{-(1+\eta)\beta t}$ and taking the supremum over $t \le 0$ we derive that

$$\|f(u)\|_{C_{(1+\eta)\beta}(L^2_b)} \le C_2\|\xi_u\|^{1+\eta}_{C_\beta(E_b)}Q(\|\xi_u\|_{C_0(E_b)}) \tag{11.15}$$

Thus, $F : C_\beta(E_b) \to C_{(1+\eta)\beta}(L^2_b)$. The differentiability of this map can be verified analogously. Lemma 11.1 is proved.

Let us fix $\beta > 0$ in such a way that $\beta < \lambda_+(0)$ but $(1+\eta)\beta > \lambda_+(0)$ and $\mu_0 > 0$ such that $\lambda_+(\sqrt{n}\mu_0) > \beta$ and rewrite the equation (11.1) near the equilibria point $u = 0$ in the following form

$$u + \mathbb{T}_{(1+\eta)\beta}Fu = \mathbb{P}_\beta u_0, \quad u \in \mathbb{L}_\beta(E_b) \tag{11.16}$$

where $u_0 \in \mathbb{B}_{\mu_0}$. Note that every solution of (11.16) is simultaneously a solution of the equation (11.1) hence it is sufficient to solve (11.16) in $\mathbb{L}_\beta(E_b)$. Note also that due to Propositions 11.1, 11.2 and Lemma 11.16 all operators in (11.16) are well defined.

We will solve the equation (11.16) using the implicit function theorem. To this end we introduce a function $\mathcal{F} : \mathbb{L}_\beta(E_b) \times \mathbb{B}_{\mu_0} \to \mathbb{L}_\beta(E_b)$ by formula

$$\mathcal{F}(u, u_0) = u + \mathbb{T}_{(1+\eta)\beta}Fu - \mathbb{P}_\beta u_0 \tag{11.17}$$

It follows from Propositions 11.1, 11.2 and from Lemma 11.1 that $\mathcal{F} \in C^1(\mathbb{L}_\beta(E_b)\times \mathbb{B}_{\mu_0}, \mathbb{L}_\beta(E_b))$ and $D_u\mathcal{F}(0,0) = Id$. Hence due to the implicit function theorem (see [26] for instance) there exists a neighborhood $B(0,\delta_0,\mathbb{B}_{\mu_0})$ and a C^1-function

$$\mathcal{U} : B(0,\delta_0,\mathbb{B}_{\mu_0}) \to \mathbb{L}_\beta(E_b) \tag{11.18}$$

such that $\mathcal{F}(\mathcal{U}(u_0),u_0) \equiv 0$ and consequently $\mathcal{U}(u_0)(t)$ is a backward solution of the problem (11.1). The equation (11.16), the estimate (11.15) and the evident fact that $\mathcal{U}(0) = 0$ imply now that

$$\begin{aligned} \|\mathcal{U}(u_0) - \mathbb{P}_\beta u_0\|_{\mathbb{L}_\beta(E_b)} \le C\|f(\mathcal{U}(u_0))\|_{\mathbb{L}_{(1+\eta)\beta}(L^2_b)} \le \\ \le C_1\|\mathcal{U}(u_0)\|^{1+\eta}_{\mathbb{L}_\beta(E_b)} \le C_2\|u_0\|^{1+\eta}_{\mathbb{B}_{\mu_0}} \end{aligned} \tag{11.19}$$

Let us define now $\mathcal{U}_0(u_0) = \mathcal{U}(u_0)\big|_{t=0}$. Then (11.19) together with the equality $(\mathbb{P}_\beta u_0)(0) = M(u_0)$ imply the estimate (11.13). The assertion $\mathcal{U}_0(B(0,\delta_0,\mathbb{B}_{\mu_0})) \subset \mathcal{A}$ can be easily derived from the attractor's description (4.4) and from the fact that every backward solution $U(u_0)(t)$ can be extended (due to Theorem 3.1) to the complete bounded solution (i.e. which is defined for $t \in \mathbb{R}$ and bounded in E_b) Theorem 11.1 is proved.

Corollary 11.1. *Let $u_0^1, u_0^2 \in B(0,\delta,\mathbb{B}_{\mu_0})$ and $\delta \le \delta_0$. Then for every $R > 0$*

$$\|\mathcal{U}_0(u_0^1) - \mathcal{U}_0(u_0^2)\|_{E_b(B_0^R)} \ge \|u_0^1 - u_0^2\|_{L^2_b(B_0^R)} - C\delta^{1+\eta} \tag{11.20}$$

with C independent of R.

Indeed,

$$\begin{aligned} &\|\mathcal{U}_0(u_0^1) - \mathcal{U}_0(u_0^2)\|_{E_b(B_0^R)} \ge \\ &\ge \|M(u_0^1) - M(u_0^2)\|_{E_b(B_0^R)} - \|\mathcal{U}_0(u_0^1) - M(u_0^1)\|_{E_b(\mathbb{R}^n)} - \|\mathcal{U}_0(u_0^2) - M(u_0^2)\|_{E_b(\mathbb{R}^n)} \ge \\ &\ge \|u_0^1 - u_0^2\|_{W^{1,2}_b(B_0^R)} - C_1(\|u_0^1\|^{1+\eta}_{\mathbb{B}_{\mu_0}} + \|u_0^2\|^{1+\eta}_{\mathbb{B}_{\mu_0}}) \ge \|u_0^1 - u_0^2\|_{L^2_b(B_0^R)} - 2C_1\delta^{1+\eta} \end{aligned}$$

The estimate (11.20) admits to obtain the lower bounds for the ε-entropy for the attractor of the equation (11.1). Indeed, let $\varepsilon > 0$ be small enough, $\delta = \left(\frac{\varepsilon}{2C}\right)^{1/(1+\eta)}$ and $u_0^1, u_2^0 \in B(0,\delta,\mathbb{B}_{\mu_0})$ such that

$$\|u_0^1 - u_0^2\|_{L^2_b(B_0^R)} \ge \varepsilon \tag{11.21}$$

Then it follows from (10.12) that

$$\|\mathcal{U}(u_0^1) - \mathcal{U}(u_0^2)\|_{E_b(B_0^R)} \ge \varepsilon/2 \tag{11.22}$$

The estimates (11.21),(11.22) together with the assertion (11.12) imply that

$$\begin{aligned} \mathbb{H}_{\varepsilon/4}\left(\mathcal{A}, E_b(B_0^R)\right) \ge \mathbb{H}_\varepsilon\left(B(0,\left(\frac{\varepsilon}{2C}\right)^{1/(1+\eta)},\mathbb{B}_{\mu_0}), L^2_b(B_0^R)\right) = \\ = \mathbb{H}_{(2C)^{1/(1+\eta)}\varepsilon^{\eta/(1+\eta)}}\left(B(0,1,\mathbb{B}_{\mu_0}), L^2_b(B_0^R)\right) \end{aligned} \tag{11.23}$$

The last estimate together with (7.14) and (7.15) gives the lower bounds for the ε-entropy (the upper bounds have been obtained in Theorem 9.1).

Corollary 11.2. *Let $\mathcal{A}$ be the attractor of the equation* (11.1) *and let $\varepsilon < \varepsilon_0$ and $R \ge \ln\frac{1}{\varepsilon}$. Then*

$$C_1 R^n \ln\frac{1}{\varepsilon} \le \mathbb{H}_\varepsilon\left(\mathcal{A}, E_b(B_0^R)\right) \le C_2 R^n \ln\frac{1}{\varepsilon} \tag{11.24}$$

Particularly, $0 < C_1 \ln\frac{1}{\varepsilon} \le \overline{\mathbb{H}}_\varepsilon(\mathcal{A}) \le C_2 \ln\frac{1}{\varepsilon}$.

Corollary 11.3. *Let $\mathcal{A}$ be the attractor of the equation* (11.1)*. Then for every $\delta > 0$ there exists C_δ such that*

$$\mathbb{H}_\varepsilon\left(\mathcal{A}, E(B_0^1)\right) \ge C_\delta \left(\ln\frac{1}{\varepsilon}\right)^{n+1-\delta} \tag{11.25}$$

Example 11.1. The simplest example of the equation (11.1) for which the estimates (11.24) and (11.25) are valid is the following equation in $\mathbb{R}^n$

$$\partial_t^2 u + \gamma\partial_t u - \Delta_x u = \alpha^2 u - u|u|^{1+\eta}, \quad \alpha > 0 \tag{11.26}$$

where $\eta < 2/(n-2)$.

Part 4. The regular attractor.

In this part we give the detailed study of the autonomous attractor in the particular case where this attractor is compact in $E_b(\Omega)$. We hope also that this investigation clarifies why in this case the attractor occurred to be finite dimensional and looks very similar to the attractors in bounded domains.

The spatial asymptotic (up to the exponentially decaying terms) for the solutions on the attractor are given in Section 12.

The regular structure of the attractor is obtained in Section 13. Moreover using the good structure of the attractor we prove there that every solution of (0.1) stabilizes to one of the equilibria points when $t \to \infty$.

§12 The spatial asymptotic for the attractor

In this Section we study the behavior of solutions of the autonomous equation (3.1) belonging to the globally compact attractor (under the assumptions of Theorem 4.1) when $|x| \to \infty$. Recall that we have already known (due to the fact that $\mathcal{A} \in \dot{E}_b(\Omega)$) that all solutions belonging to the attractor converge to zero when $|x| \to \infty$ but the rate of this converges could be arbitrary small (if g converges to zero slowly enough). The following Theorem gives the asymptotic for this convergence up to *exponentially* small terms.

Theorem 12.1. *Let the assumptions of Theorem 4.1 hold. Assume also that in the case $n > 3$ the right-hand side g satisfies the additional condition*

$$g \in \dot{L}_b^s(\Omega) \text{ where } s > \frac{n}{2} \tag{12.1}$$

Then, there exist an equilibria point $u_0 \in \dot{C}(\Omega) \cap \dot{W}^{1,2}(\Omega)$ and the constants $\delta > 0$ and $C > 0$ depending only on the equation such that for every $(u, v) \in \mathcal{A}$ the following estimate is valid:

$$\|u - u_0, \Omega \cap B_{x_0}^1\|_{1,2}^2 + \|v, \Omega \cap B_{x_0}^1\|_{0,2}^2 \le Ce^{-\delta|x_0|} \tag{12.2}$$

Proof. For the proof of this Theorem we need the following Lemmata

Lemma 12.1. *Let the function $f \in C$ satisfy the assumption $f(u).u \geq 0$ and $g \in L^s_b(\Omega)$ with $s > \frac{n}{2}$ and $s \geq 2$. Then the equation*

$$\Delta_x u_0 - \lambda_0 u_0 - f(u) = g, \quad u_0\big|_{\partial\Omega} = 0 \tag{12.3}$$

has at least one solution $u_0 \in W^{2,s}_b(\Omega) \subset C_b(\Omega)$. Moreover, every solution of this equation possesses the following estimate:

$$\|u_0, \Omega \cap B^1_{x_0}\|^s_{2,s} \leq C \int_\Omega e^{-\varepsilon|x-x_0|}|g(s)|^s \, dx \tag{12.4}$$

where the constants C and ε depend only on the equation (12.3).

Particularly, if $g \in \dot{L}^s_b(\Omega)$ then $u_0 \in \dot{C}_b(\Omega)$.

The proof of the estimate (12.4) is based on a maximum principle applied to the function $w = u^2$ and can be obtained as in [10] (where this result has been proved for a more complicated case of parabolic equation).

The last assertion of the lemma follows immediately from (12.4) and Proposition 1.4.

Lemma 12.2. *Let $f \in C^1$ satisfy the assumptions*

$$\begin{cases} 1. \ \ f(u).u \geq \alpha|u| \cdot |f(u)| \\ 2. \ \ f(u).u \geq \alpha|u|^2, \quad \text{with } \alpha > 0 \\ 3. \ \ f'(0) > 0 \end{cases} \tag{12.5}$$

Then there exists $\mu > 0$ such that for every $u \in \mathbb{R}$ and every v such that $|v| \leq \mu$ the following estimate is valid:

$$[f(u+v) - f(v)] \cdot u \geq 0 \tag{12.6}$$

Proof. Assume for the first that $|u| \geq R >> 1$ and $|v| \leq 1$. Then according to (12.5)

$$\begin{aligned} &[f(u+v) - f(v)] \cdot u = f(u+v).(u+v) - vf(u+v) - f(v)u \geq \\ &\geq \alpha|u+v| \cdot |f(u+v)| - |f(u+v)| - C|u| \geq (\alpha(R-1) - 1)|f(u+v)| - C|u| \geq \\ &\geq \alpha(|u| - 1)(\alpha(R-1) - 1) - C|u| \geq \\ &\geq (\alpha(\alpha(R-1) - 1) - C)R - \alpha(\alpha(R-1) - 1) \geq 0 \end{aligned} \tag{12.7}$$

for R large enough.

Note also that $f'(0) > 0$ implies that there exists $\mu_0 > 0$ such that $f'(w) \geq 0$ if $|w| \leq 2\mu_0$. Then for every $u, v \in \mathbb{R}$ which satisfy $|u| \leq \mu_0$ and $|v| \leq \mu_0$ we will have

$$[f(u+v) - f(v)] \cdot u = \int_0^1 f'(v + su)\, ds|u|^2 \geq 0 \tag{12.8}$$

Let us assume now that the assertion of the lemma is wrong. Then there exist a sequence $v_n \to 0$ and $u_n \in \mathbb{R}$ such that

$$[f(u_n + v_n) - f(v_n)] \cdot u_n < 0 \tag{12.9}$$

Without less of generality we may assume that $u_n \to \infty$ or $u_n \to 0$ or $u_n \to u_0 \neq 0$. The first assumption contradicts (12.7) and the second one – to (12.8). Thus it remains to consider the third assertion. But, passing to the limit $n \to \infty$ in (12.9) and using the fact that $f(0) = 0$ we derive that

$$f(u_0).u_0 \leq 0$$

which contradicts the second assumption of (12.5) (since $u_0 \neq 0$). Lemma 12.1 is proved.

Now we are in a position to complete the proof of the theorem. To this end we consider an arbitrary equilibria point $u_0(x)$ of (3.1) which exists according to Lemma 12.1 and rewrite this equation with respect to a new variable $w(t) = u(t) - u_0$:

$$\begin{cases} \partial_t^2 w + \gamma \partial_t w + \lambda_0 w - \Delta_x w + F(x, w) = 0 \\ w\big|_{t=0} = w_0; \quad \partial_t w\big|_{t=0} = w_0' \end{cases} \tag{12.10}$$

with $F(x, w) := F(u_0(x), w) := f(w + u_0(x)) - f(u_0(x))$ and instead of studying the attractor $\mathcal{A}$ of (3.1) we will study the attractor $\widehat{\mathcal{A}}$ of (12.10). Evidently,

$$\mathcal{A} = \{(u_0, 0)\} + \widehat{\mathcal{A}} \tag{12.11}$$

Note that according to the assumptions (3.2) (with $f_2 \equiv 0$) we may assume that our nonlinearity satisfies also the assumptions of Lemma 12.2. Indeed, since $f(u).u \geq 0$ then $f(u).u = |u| \cdot |f(u)|$ and $f'(0) \geq 0$. Rescaling if necessary the parameter λ_0 if necessary we may add αu with $0 < \alpha << 1$ to f and assume also that $f'(0) > 0$ and $f(u).u \geq \alpha |u|^2$.

Thus, according to Lemma 12.2 there exists $\mu > 0$ such that $F(u_0, w).w \geq 0$ for every $w \in \mathbb{R}$ if $|u_0| \leq \mu$.

Since $u_0 \in \dot{C}_b(\Omega)$ then there exists $R > 0$ such that $u_0(x) < \mu$ if $|x| > R$ and consequently

$$F(x, w).w \geq 0 \text{ if } |x| \geq R \tag{12.12}$$

Let $\chi(z)$ be the Heaviside function ($\chi(z) = 1$ if $z > 0$ and $\chi(z) = 0$ if $z < 0$ and define functions

$$F_1(x, w) := \chi(|x| - R) F(x, w), \quad F_2(x, w) := \chi(R - |x|) F(x, w) \tag{12.13}$$

Then,

$$F_1(x, w).w \geq 0, \quad F_2(x, w) \equiv 0 \text{ if } |x| > R \tag{12.14}$$

We rewrite the equation (12.10) in the following form

$$\partial_t^2 w + \gamma \partial_t w + \lambda_0 w - \Delta_x w + F_1(x, w) = -F_2(x, w) \equiv h_w(t) \tag{12.15}$$

Note that the function $F_1(x, w)$ evidently satisfies all *growth* restrictions from (3.2) and consequently (12.15) satisfies the assumptions of Corollary 3.1 with $C(f_2) \equiv 0$

(see also Remark 3.3). Assume now that $\xi_w(0) \in \widehat{\mathcal{A}}$. Then according to (4.4) there exists complete bounded trajectory $\xi_w(t)$, $t \in \mathbb{R}$. Moreover, according to Theorem 3.1

$$\|\xi_w(t)\|_{E_b} \le C \tag{12.16}$$

Since the function F_2 satisfies the growth assumption $|F_2(x,w)| \le C(1+|w|^{q_1})$ with $q_1 < 1 + 2/(n-2)$ then (12.16) implies that

$$\|h_w(t), \Omega \cap B^1_{x_0}\|^2_{0,2} \le C_1\chi(R+1-|x|) \tag{12.17}$$

Applying the estimate (3.15) with $C(f_2) = 0$ to the equation (12.15) and using the estimate (12.17) we derive that

(12.18)

$$\|w(0), \Omega \cap B^1_{x_0}\|^2_{1,2} + \|\partial_t w(0), \Omega \cap B^1_{x_0}\|^2_{0,2} \le C\int_{-\infty}^0 e^{\varepsilon s}\left(|h_w(s)|^2, e^{-\varepsilon|x-x_0|}\right)\,ds \le$$
$$\le C_3\int_\Omega e^{-\varepsilon|x-x_0|}\chi(R+1-|x|)\,dx \le C_4 e^{-\varepsilon|x_0|}$$

The estimate (12.18) implies (12.2). Theorem 12.1 is proved.

§13 Stabilization of solutions and attractor's regularity.

In this Section we will study the structure of the globally compact attractor $\mathcal{A}$ of the equation (3.1) under the assumptions of Theorem 12.1. Particularly, we will prove that under these assumptions this attractor consists generically of a finite collection of finite dimensional unstable manifolds and will show that as in the case of bounded domain Ω every solution $u(t)$ of (3.1) with $\xi_u(0) \in E_b(\Omega)$ stabilizes when $t \to +\infty$ to the appropriate equilibria point u_0.

Note that in contrast to the case of bounded domains in our situation the standard Liapunov's energy functional

$$\Phi(u, \partial_t u) := \int_\Omega 1/2\left(|\partial_t u|^2 + |\nabla_x u|^2 + \lambda_0|u|^2\right) + F(u) + g.u\,dx \tag{13.1}$$

where $F(u) = \int_0^u f(v)\,dv$, is not well posed in our phase space $E_b(\Omega)$ (it equals infinity for generic $\xi_u \in E_b$). Moreover if g decays slow enough when $|x| \to \infty$ this functional is not well posed even on the attractor $\mathcal{A}$. Thus, the problem is more complicated then in the case of bounded domains (see e.g. [2]) and some additional arguments are required.

Based on the spatial asymptotic of the attractor derived in previous Section we construct the modified Liapunov functional which will be well posed on the *attractor* $\mathcal{A}$ and using this functional we prove the regularity theorem for this attractor. Having this good structure of the attractor we obtain then that for generic g every solution of (3.1) with $\xi_u(0) \in E_b(\Omega)$ stabilizes to the appropriate equilibria point.

We start with a theorem which describes the generic structure of the equilibria points set $\mathcal{R}_g$ for the equation (3.1).

Theorem 13.1. *Let the nonlinear function f satisfies the assumptions*

$$\text{(13.2)} \qquad 1.\ \ f \in C^1(\mathbb{R},\mathbb{R});\quad 2.\ \ f(u).u \geq 0;\quad 3.\ \ f'(0)=0$$

Then the set of $g \in \dot{L}^s_b(\Omega)$ (where the exponent s is the same as in Theorem 12.1) for which the equation

$$\text{(13.3)} \qquad \Delta_x u_0 - \lambda_0 u_0 - f(u_0) = g$$

has a finite number of hyperbolic solutions $\mathcal{R}_g = \{u_0^1, \cdots, u_0^N\}$ is open and dense in $\dot{L}^2_b(\Omega)$.

Proof. We are going to apply the infinite dimensional version of Sard-Smale theorem (see e.g. [2]). To this end we introduce the nonlinear operator

$$\text{(13.4)} \qquad \mathbb{F} : \dot{W}^{2,s}_b(\Omega) \to \dot{L}^s_b(\Omega); \quad \mathbb{F}(u) := \Delta_x u - \lambda_0 u - f(u)$$

Note that since $f(0)=0$ and $\dot{W}^{2,s}_b(\Omega) \subset \dot{C}_b(\Omega)$ then (13.4) is well defined.

According to the Sard-Smale theorem (see [2]) we should verify that

1. The operator $\mathbb{F}$ is properly supported, i.e. the inverse image of every compact set is also compact.

2. For every $v \in \dot{W}^{2,s}_b(\Omega)$ the Frechet derivative $D_u\mathbb{F}(v)$ is a Fredholm operator with zero index.

3. The Frechet derivative $D_u\mathbb{F}(v)$ depends continuously on v.

Let us verify the first assumption. Indeed, let K_g be a compact set in $\dot{L}^s_b(\Omega)$. Then according to the compactness criteria (see Proposition 1.3) we have a uniform 'tale' estimate for K_g. Then the estimate (12.4) together with Proposition 1.4 imply that the set $K_u := \mathbb{F}^{-1}(K_g)$ is bounded in $W^{2,s}_b(\Omega)$ and also possesses the uniform 'tale' estimate. Thus it remains to prove only that the restrictions $K_u\big|_{\Omega\cap B^1_{x_0}}$ are compact in $W^{2,s}(\Omega \cap B^1_{x_0})$ for every $x_0 \in \Omega$. Let ϕ_{x_0} be a cut-off function which equals one on $\Omega \cap B^1_{x_0}$ and zero is $x \notin V_{x_0}$ (where V_{x_0} is defined in (1.5) and (1.6)). The the equation which defines the set K_u can be rewritten in the form

$$\text{(13.5)} \quad \Delta_x(\phi_{x_0}u) - \lambda_0(\phi_{x_0}u) = \phi_{x_0}f(u) + \phi_{x_0}g + 2\nabla_x\phi_{x_0}\nabla_x u + \Delta_x\phi_{x_0}u; \quad g \in K_g$$

Note that since K_u is bounded in $W^{2,s}_b(\Omega)$ then $f(K_u)\big|_{V_{x_0}}$, $\nabla_x K_u\big|_{V_{x_0}}$ and $K_u\big|_{V_{x_0}}$ is compact in $L^s(V_{x_0})$. Moreover, since K_g is compact in $\dot{L}^s_b$ then the set $K_g\big|_{V_{x_0}}$ is also compact in $L^s(V_{x_0})$. Thus the right-hand side of (13.5) is compact in $L^s(V_{x_0})$ and consequently $K_u\big|_{\Omega\cap B^1_{x_0}}$ is also compact in $W^{2,s}(V_{x_0})$. Therefore, K_u is compact.

Let us verify now the second assertion. By definition the Frechet derivative of $\mathbb{F}$ has the form

$$\text{(13.6)} \qquad D_u\mathbb{F}(v) = \Delta_x - \lambda_0 - f'(v)$$

Note that the operator $\Delta_x - \lambda_0$ is invertible ($\lambda_0 > 0$) and the multiplication operator $f'(v)$ is compact as the operator acting from $\dot{W}^{2,s}_b$ to $\dot{L}^s_b$ for every $v \in \dot{W}^{2,s}_b$ (since $f'(0) = 0$ and consequently $f'(v) \in \dot{C}_b(\Omega)$ for every $v \in \dot{W}^{2,s}_b(\Omega)$). Thus, the Frechet derivative $D_u\mathbb{F}$ can be represented as a sum of invertible and compact operators and consequently $D_u\mathbb{F}$ is Fredholm with zero index.

The third assumption can be verified directly using the explicit formula (13.6).

Applying now the Sard-Smale theorem to the operator (13.4) we obtain the assertion of the theorem. Theorem 13.1 is proved.

Corollary 13.1. *Let* $u_0 \in \mathcal{R}_g$*. Then the essential spectrum of the operator* $D_u\mathbb{F}(u_0)$ *satisfies the inequality*

$$\mathrm{Re}\,\sigma_{ess}(D_u\mathbb{F}) \le -\lambda_0 < 0 \tag{13.7}$$

and consequently there are only the finite number of eigenvalues of this operator with positive real part, i.e.

$$\mathrm{Ind}_{u_0} := \#\{\lambda \in \sigma(D_u(\mathbb{F}(u_0))) : \ \mathrm{Re}\,\lambda > 0\} < \infty \tag{13.8}$$

Indeed, $\Delta_x - \lambda_0$ is negative and $f'(u_0)$ is compact.

The following theorem establishes the regular structure of the attractor $\mathcal{A}$ of the equation (3.1).

Theorem 13.2. *Let the assumptions of Theorem 12.1 hold. Then for generic right-hand sides g the attractor* $\mathcal{A}_g$ *of the equation* (3.1) *can be represented in the following form*

$$\mathcal{A}_g = \cup_{u_0\in\mathcal{R}_g}\mathcal{M}^+(u_0) \tag{13.9}$$

where $\#\mathcal{R}_g < \infty$ *and* $\mathcal{M}^+(u_0)$ *are the finite dimensional unstable* C^1*-submanifolds of the equilibria points* u_0 *in* $E_b(\Omega)$*. Moreover, every* $\mathcal{M}^+(u_0)$ *is diffeomorphed to* $\mathbb{R}^\kappa$ *with*

$$\kappa = \dim\mathcal{M}^+(u_0) = \mathrm{Ind}_{u_0} < \infty \tag{13.10}$$

Proof. For the first we construct the modified Liapunov function which will be finite and continuous on the attractor $\mathcal{A}$. To this end we consider an arbitrary equilibria point u_0 which exists according to Lemma 12.1 and introduce the function $w(t) = u(t) - u_0$. Then this function satisfies the equation

$$\partial_t^2 w + \gamma\partial_t w - \Delta_x w + \lambda_0 w + (f(w+u_0) - f(u_0)) = 0 \tag{13.11}$$

Lemma 13.1. *Let the above assumptions hold. Then the following functional is finite on the attractor of* (13.11) *and continuous in the topology of* $E_b(\Omega)$*:*

$$\widehat{\Phi}(w,\partial_t w) := \int_\Omega 1/2\left(|\partial_t w|^2 + |\nabla_x w|^2 + \lambda_0|w|^2\right) + \widehat{F}(w)\,dx \tag{13.12}$$

where $\widehat{F}(w) := \int_0^w (f(w+u_0) - f(u_0))\,dw$*.*

Moreover, the following equality is valid:

$$\widehat{\Phi}(w(t),\partial_t w(t)) - \widehat{\Phi}(w(0),\partial_t w(0)) = \gamma\int_0^t \|\partial_t w(s),\Omega\|_{0,2}^2\,ds \tag{13.13}$$

Proof. Indeed, according to Theorem 12.1, $\xi_w \in E_{b,e^{\delta|x|}}(\Omega) \subset E(\Omega)$ and consequently the linear terms in (13.12) are well defined. Note also that $\widehat{F}(w) = w^2\phi(x,w)$ where the function ϕ satisfies the growth assumption $|\phi(x,w)| \le C(1 + |w|^{q_1}))$ with $q_1 < 2/(n-2)$ (since f' satisfies it and u_0 is continuous), consequently the function $\widehat{F}(w)$ is well defined and continuous on $W^{1,2}(\Omega)$.

Thus, the functional (3.12) is well defined and continuous on $E(\Omega)$. Note now that it is not difficult to verify using the estimate (12.2) that the topologies of the spaces $E(\Omega)$ and $E_b(\Omega)$ coincide on $\widehat{\mathcal{A}} := \mathcal{A} - \{(u_0, 0)\}$ and therefore the functional (13.12) is continuous on the attractor $\widehat{\mathcal{A}}$ in the topology of $E_b(\Omega)$.

The equality (13.13) can be verified as for the case of bounded domains (see e.g. [2]). Lemma 13.1 is proved.

Corollary 13.2. *Let $\xi_u(t)$, $t \in \mathbb{R}$ be a complete bounded trajectory (which automatically belong to the attractor). Then*

$$\int_{-\infty}^{\infty} \|\partial_t u(s), \Omega\|_{0,2}^2 \, ds < \infty \tag{13.14}$$

Indeed,the estimate (13.14) follows from the facts that $\partial_t u = \partial_t w$, the functional (13.12) is bounded on $\widehat{\mathcal{A}}$ and from the equality (13.13).

The dissipation integral (13.14) admits as usual to prove the stabilization of solutions on the attractor in the case where the number of equilibria points $\mathcal{R}_g$ is finite (see also [2], [16]).

Lemma 13.2. *Let the above assumptions hold. Assume also that the set $\mathcal{R}_g$ of equilibria points for* (3.1) *is finite (generic assumption according to Theorem 13.1). Then every complete bounded trajectory $\xi_u(t)$, $t \in \mathbb{R}$ of* (3.1) *stabilizes if $t \to \pm\infty$, i.e. there are the equilibria points $u_0^+, u_0^- \in \mathcal{R}_g$, $u_0^+ \neq u_0^-$ such that*

$$\lim_{t\to\pm\infty} \|\xi_u(t) - (u_0^\pm, 0)\|_{E_b(\Omega)} = 0; \tag{13.15}$$

Proof. Let us prove the stabilization only when $t \to +\infty$ (The case $t \to -\infty$ can be considered analogously).

Let $\xi_u(t) \in \mathcal{A}$, $t \in \mathbb{R}$ be an arbitrary complete bounded solution of (3.1). Consider the ω-limit set $\omega(\xi_u(0))$ of the point $\xi_u(0)$ in the space $E_b(\Omega)$. Since $\mathcal{A}$ is compact in $E_b(\Omega)$ this set exists and nonempty (see e.g. [16]).

We are going to prove that

$$\omega(\xi_u(0)) \subset \mathcal{R}_g \tag{13.16}$$

Indeed, let $\xi_{u_0}(t) \in \omega(\xi_u(t))$, $t \in \mathbb{R}$. Then by the definition of ω-limit set for every $\tau \in \mathbb{R}$ there exists a sequence $t_n = t_n(\tau) \to +\infty$ when $n \to \infty$ such that

$$\xi_u(t_n) \to \xi_{u_0}(\tau) \text{ in } E_b(\Omega) \text{ when } n \to \infty \tag{13.17}$$

Note that due to Theorems 2.1, 3.1 and estimate (2.17) we have the estimate

$$\|\xi_u(t_n + s) - \xi_{u_0}(\tau + s)\|_{E_b(\Omega)}^2 \leq Ce^{Ks}\|\xi_u(t_n) - \xi_u(\tau)\|_{E_b(\Omega)}^2 \tag{13.18}$$

where the constants C and K depends only on the equation.

The estimate (13.18) implies particularly that

$$\xi_u(t_n + \cdot) \to \xi_{u_0}(\tau + \cdot) \text{ in the space } C([\tau, \tau + 1], E_b(\Omega)) \tag{13.19}$$

Observe now that according to (13.14)

$$\int_0^1 \|\partial_t u(t_n + s), \Omega\|_{0,2}^2 \, ds = \int_{t_n}^{t_n+1} \|\partial_t u(s), \Omega\|_{0,2}^2 \, ds \to 0 \tag{13.20}$$

when $n \to \infty$ (since $t_n \to \infty$). Particularly the sequence $\partial_t \xi_u(t_n + \cdot)$ is uniformly bounded in $L^2([0, 1] \times \Omega)$ therefore without loss of generality we may assume that this sequence converges weakly in $L^2([0, 1] \times \Omega)$ to the function $\xi_{u_0}(\tau + \cdot)$ (the existence of weakly converging subsequence follows from the fact that the ball in L^2 is weakly compact and the explicit value of this weak limit can be found using the convergence (13.19).

Thus,

$$\int_{\tau}^{\tau+1} \|\partial_t u_0(s),\Omega\|_{0,2}^2\,ds = \int_0^1 \|\partial_t u_0(\tau+s),\Omega\|_{0,2}^2\,ds \le$$
$$\le \lim_{n\to\infty}\int_0^1 \|\partial_t u(t_n+s),\Omega\|_{0,2}^2\,ds = 0$$

Recall that $\tau\in\mathbb{R}$ is arbitrary therefore $\partial_t u_0(\tau)\equiv 0$ and consequently u_0 is an equilibria point of (3.1). The embedding (13.16) is proved.

Note now that the set $\omega(\xi_u(0))$ should be connected (see [16]) but the set $\mathbb{R}$ is discrete according to our assumptions. Consequently, (13.16) implies that $\omega(\xi_u(0))$ consists of one point u_0^+.

Thus we have proved that $\xi_u(t)$ stabilizes to u_0^+ when $t\to+\infty$. The case $t\to-\infty$ can be considered analogously. As usual the supposition $u_0^- = u_0^+$ contradicts (13.13). Lemma 13.2 is proved.

Recall now the definition of the unstable set of the equilibria point u_0.

Definition 13.1. *Let u_0 be an equilibria point of* (3.1). *Then the unstable set $\mathcal{M}^+(u_0)$ can be defined by the following expression:*

(13.21) $$\mathcal{M}^+(u_0) := \{\xi\in\mathcal{A} : \exists \text{ a complete bounded trajectory } \xi_u(t),\ t\in\mathbb{R} \text{ such that } \xi_u(0)=\xi \text{ and } \lim_{t\to-\infty}\xi_u(t) = (u_0,0)\}$$

Corollary 13.3. *Let the assumptions of Lemma 13.2 hold. Then the attractor $\mathcal{A}$ of the equation* (3.1) *can be represented by* (13.9).

Thus, in order to complete the proof of the Theorem we should show that $\mathcal{M}^+(u_0)$ is a smooth manifold if the equilibria point u_0 is hyperbolic.

Lemma 13.3. *Let u_0 be the equilibria point of* (3.1) *such that* 0 *is not an eigenvalue of the operator $D_u\mathbb{F}(u_0)$. Then the unstable set $\mathcal{M}^+(u_0)$ is C^1-submanifold of $E_b(\Omega)$ which is diffeomorphed to $\mathbb{R}^\kappa$ and the dimension κ can be found by* (13.8) *and* (13.10).

Proof. The proof of this assertion is more or less standard (see e.g [2]) since we give below only the scheme of the proof.

As in Section 11 we consider firstly the linearization of (3.1) near u_0:

(13.22) $$\partial_t^2 v + \gamma\partial_t v + \lambda_0 v - \Delta_x v + f'(u_0)v = 0$$

which can be rewritten as the first order system

(13.23) $$\partial_t\xi_v(t) = L_{u_0}\xi_v(t), \text{ where } L_{u_0} := \begin{pmatrix} 0 & ; & 1 \\ \Delta_x - \lambda_0 - f'(u_0) & ; & -\gamma \end{pmatrix}$$

Simple computations show that the spectrum of L_{u_0} has the following structure

$$\sigma(L_{u_0}) = \{\lambda_\pm := \frac{-\gamma\pm\sqrt{\gamma^2+4\xi}}{2};\ \ \xi\in\sigma(\Delta_x - \lambda_0 + f'(u_0)) \equiv \sigma(D_u\mathbb{F}(u_0))\}$$

Particularly, the operator L_{u_0} is hyperbolic (i.e. $\sigma(L_{u_0}) \cap \{\operatorname{Re} z = 0\} = \varnothing$) if and only if $0 \notin \sigma(D_u\mathbb{F}(u_0))$. Moreover the unstable index of the operator L_{u_0} can be calculated by

$$\operatorname{Ind}(L_{u_0}) := \#\{\lambda \in \sigma(L_{u_0}) : \operatorname{Re}\lambda > 0\} = \operatorname{Ind}_{u_0} \tag{13.24}$$

Assume now that u_0 is hyperbolic. Then L_{u_0} is also hyperbolic and using the implicit function theorem as in the proof of Theorem 11.1 (see also [10]) we derive that there exists a sufficiently small positive δ such that the set

$$\begin{aligned}\mathcal{M}^+_\delta(u_0) := \{\xi \in \mathcal{A} : \exists \text{ a complete bounded trajectory } \xi_u(t),\ t \in \mathbb{R}\\ \text{such that } \xi_u(0) = \xi,\ \|\xi_u(t)\|_{E_b(\Omega)} \le \delta \text{ for } t \le 0 \text{ and } \lim_{t\to-\infty} \xi_u(t) = (u_0, 0)\}\end{aligned} \tag{13.25}$$

is κ-dimensional C^1-manifold in $E_b(\Omega)$ which is diffeomorphed to $\mathbb{R}^\kappa$.

Since the hyperbolic flow S_t generated by the autonomous equation (3.1) is invertible then the sets $\mathcal{M}^+_n(u_0) := S_n\mathcal{M}^+_\delta(u_0)$ are κ-dimensional C^1-submanifolds of $E_b(\Omega)$ also. Moreover, evidently

$$\mathcal{M}^+(u_0) = \cup_{n\ge 0}\mathcal{M}^+_n(u_0) \text{ and } \mathcal{M}^+_n(u_0) \subset \mathcal{M}^+_m(u_0) \text{ if } n \le m \tag{13.26}$$

The representation (13.26) implies that $\mathcal{M}^+(u_0)$ is C^1-manifold diffeomorphed to $\mathbb{R}^k$ (see [17]) and the fact that S_t possesses the Liapunov function on the attractor $\mathcal{A}$ implies that $\mathcal{M}^+(u_0)$ is a *submanifold* of $E_b(\Omega)$ (see [2]). Lemma 13.3 is proved. Theorem 13.2 is proved.

Now we are in a position to prove the stabilization of solutions of (3.1) not only for $\xi_u(0) \in \mathcal{A}$ but for every $\xi_u(0) \in E_b(\Omega)$. Note that in contrast to the case of bounded domains in our situation we have neither the dissipation integral nor the Liapunov function for such solutions. Nevertheless the regular structure of the attractor, obtained above admits to prove the stabilization.

Theorem 13.3. *Assume that the set $\mathcal{R}_g = \{u_0^0, u_0^1, \cdots, u_0^N\}$ is discrete and that the values of the Liapunov functional $\Phi_1((u_0^i, 0) := \widehat{\Phi}(u_0^i - u_0, 0)$ where $u_0 := u_0^0$ and $\widehat{\Phi}$ is defined by* (13.12) *are different for $i = 0, \cdots, N$.*

$$\Phi_1(u_0^i, 0) \ne \Phi_1(u_0^j, 0) \text{ for } i \ne j \tag{13.27}$$

Then every solution $\xi_u(t)$ of (3.1) *with $\xi_u(0) \in E_b(\Omega)$ stabilizes to one of the equilibria $u_0^i \in \mathcal{R}_g$.*

Proof. Note that according to (3.17) the problem (3.1) possesses bounded absorbing set $\mathbb{B}$ in $E_b(\Omega)$. Moreover, changing $\mathbb{B}$ by $\cup_{t\ge 0}S_t\mathbb{B}$ if necessary we may assume that $S_t\mathbb{B} \subset \mathbb{B}$ for $t \ge 0$. Thus, it is sufficient to prove the theorem only for solutions from $\mathbb{B}$. To this end we need the following Lemma.

Lemma 13.4. *Let the above assumptions hold. Then for every $\delta > 0$ there exists $T_0 = T_0(\delta)$ such that for every solution $\xi_u(t) \in \mathbb{B}$ and every time moment $\tau \ge 0$*

$$\{\, \xi_u(t),\ t \in [\tau, \tau + T_0]\,\} \cap \mathcal{O}_\delta(\mathcal{R}_g, E_b) \ne \varnothing \tag{13.28}$$

where the δ-neighborhood $\mathcal{O}_\delta(\mathcal{R}_g, E_b) \equiv \cup_{i=0}^N O_\delta((u_0^i, 0), E_b)$.

Proof. Indeed, assume that the assertion of the lemma is wrong. Then there exist $\delta > 0$, a sequence of $T_n \to \infty$, a sequence of $\tau_n \ge 0$ and a sequence of solutions $\xi_{u_n}(t) \in \mathbb{B}$ such that

$$\{\, \xi_{u_n}(t),\ t \in [\tau_n, \tau_n + T_n] \,\} \subset \mathbb{B} \backslash \mathcal{O}_\delta(\mathcal{R}_g, E_b) \tag{13.29}$$

Let us consider now a sequence $\xi_{u_n}(\tau_n + T_n/2) \equiv S_{\tau_n + T_n/2}\xi_{u_n}(0)$. Since (3.1) possesses the globally compact attractor $\mathcal{A}$ in $E_b(\Omega)$ (see Theorem 6.1) and $\tau_n + T_n/2 \to \infty$ then the sequence $\xi_{u_n}(\tau + T_n/2)$ is precompact in $E_b(\Omega)$. Thus, without loss of generality we may assume that

$$\xi_{u_n}(\tau_n + T_n/2) \to \xi \in \mathcal{A} \tag{13.30}$$

Let $\widehat{u}(t) \in \mathcal{A}$, $t \ge 0$ be a solution of (3.1) with $\xi_{\widehat{u}}(0) = \xi$. Moreover, as in the proof of Lemma 13.2 we have the estimate

$$\|\xi_{u_n}(\tau_n + T_n/2 + s) - \xi_{\widehat{u}}(s)\|^2_{E_b(\Omega)} \le Ce^{Ks}\|\xi_{u_n}(\tau_n + T_n/2) - \xi\|^2_{E_b(\Omega)} \tag{13.31}$$

which implies that $\xi_{u_n}(\tau_n + T_n/2 + \cdot) \to \xi_{\widehat{u}}(\cdot)$ in the space $C_{loc}(\mathbb{R}_+, E_b(\Omega))$.

Passing to the limit $n \to \infty$ in (13.29) and taking into the account that $T_n \to \infty$ we derive that

$$\xi_{\widehat{u}}(t) \subset \mathbb{B} \backslash \mathcal{O}_\delta(\mathcal{R}_g, E_b) \ \text{ for every } t \ge 0 \tag{13.32}$$

Recall that $\xi_{\widehat{u}}$ belong to the attractor and consequently (13.32) contradicts the assertion of Theorem 13.2. Lemma 13.4 is proved.

Now we are in a position to complete the proof of the theorem. To this end we consider an arbitrary solution $\xi_u(t) \in \mathbb{B}$ and construct (as in the proof of Theorem 13.2) it's ω-limit set $\omega(\xi_u(0))$ in the space $E_b(\Omega)$. Evidently $\omega(\xi_u(0)) \subset \mathcal{A}$. We are going to prove that this set consists of the only point $u_0^i \in \mathcal{R}_g$. Indeed, assume that this assertion is wrong. Then this ω-limit set contains a nontrivial complete bounded trajectory of the equation (3.1) (since ω-limit set is connected and strictly invariant). But according to Theorem 13.2 every nontrivial complete bounded trajectory is a geterocline which connects two different equilibria points. Thus, $\omega(\xi_u(0))$ contains at least two different equilibria points.

Since the Liapunov function Φ_1 is continuous on the attractor $\mathcal{A}$ in the topology of $E_b(\Omega)$ then (13.27) implies that there exists $\varepsilon > 0$ such that

$$\Phi_1(\mathcal{A} \cap \mathcal{O}_\varepsilon((u_0^i, 0), E_b)) \cap \Phi_1(\mathcal{A} \cap \mathcal{O}_\varepsilon((u_0^j, 0), E_b)) = \varnothing \text{ for } i \ne j \tag{13.33}$$

Fixing now $\delta = \varepsilon/2$ and $T_0 = T_0(\delta)$ the same as in Lemma 13.4. Since $\omega(\xi_u(0))$ is the ω-limit set then

$$\lim_{t\to\infty} \operatorname{dist}(\xi_u(t), \omega(\xi_u(0))) = 0$$

and consequently for every $\mu > 0$ there exists $T = T(\mu)$ such that

$$\operatorname{dist}(\xi(t), \omega(\xi_u(0))) \le \mu \text{ for } t \ge T \tag{13.34}$$

Recall now that according to Theorems 2.1 and 3.1 we have the uniform estimate

$$\|\xi_{u_1}(\tau+s)-\xi_{u_2}(\tau+s)\|_{E_b(\Omega)} \le Ce^{Ks}\|\xi_{u_1}(\tau)-\xi_{u_2}(\tau)\|_{E_b(\Omega)} \tag{13.35}$$

which holds for every $\xi_{u_1}, \xi_{u_2} \in \mathbb{B}$ and every $\tau > 0$. (Without loss of generality we may assume that $C, K \ge 1$).

Let us fix also μ small enough that $Ce^{KT_0}\mu < \varepsilon/4$. Without loss of generality we may assume that $T = T(\mu) = 0$.

Consider now the set $\mathcal{R}'$ of equilibria points which belongs to $\omega(\xi_u(0))$ and let $u_0^i \in \mathcal{R}'$ be the equilibria point with the maximal Liapunov function Φ_1 (on $\mathcal{R}'$).

By the definition of the ω-limit set the trajectory $\xi_u(t)$ should visit the $\varepsilon/2$-neighborhoods $\mathcal{O}_{\varepsilon/2}(u_0^k)$ of every point $u_0^k \in \mathcal{R}'$ infinitely many times.

Recall that according to our assumptions $\omega(\xi_u(0))$ contains at least two different equilibria points. Consequently (due to Lemma 13.4) there exists $\tau > 0$ and the equilibria point $u_0^j \in \mathcal{R}'$, $u_0^i \ne u_0^j$ such that

$$\xi_u(\tau) \in \mathcal{O}_{\varepsilon/2}((u_0^i,0)) \text{ and } \{\,\xi_u(t),\, t\in[\tau,\tau+T_0]\,\} \cap \mathcal{O}_{\varepsilon/2}((u_0^j,0)) \ne \varnothing \tag{13.36}$$

According to (13.34) there exists $\xi_v(t) \in \omega(\xi_u(0)) \subset \mathcal{A}$ such that

$$\|\xi_u(\tau)-\xi_v(\tau)\|_{E_b(\Omega)} \le 2\mu < \varepsilon/2 \tag{13.37}$$

and consequently $\xi_v(\tau) \in \mathcal{O}_\varepsilon((u_0^i,0))$. The estimate (13.35) implies now that

$$\|\xi_u(\tau+s)-\xi_v(\tau+s)\|_{E_b(\Omega)} \le 2\mu Ce^{KT_0} < \varepsilon/2 \tag{13.38}$$

for every $s \in [0, T_0]$. The embedding (13.36) implies now that there exists $\tau_1 > \tau$ that $\xi_v(\tau_1) \in \mathcal{O}_\varepsilon((u_0^j,0))$.

Thus, we have constructed the solution $\xi_v(t) \in \mathcal{A}$ which comes from the ε-neighborhood of u_0^i to the ε-neighborhood of u_0^j when t growing from τ to τ_1. The equality (13.13) together with the assumption (13.33) implies now that

$$\Phi_1(\mathcal{O}_\varepsilon((u_0^i,0),E_b)) < \Phi_1(\mathcal{O}_\varepsilon((u_0^j,0),E_b)) \tag{13.39}$$

which contradicts the maximality of u_0^i in $\mathcal{R}'$. Thus, the ω-limit set $\omega(\xi_u(0))$ consists of the only equilibria point u_0^i. Theorem 13.3 is proved.

Remark 13.1. The result of Theorem 13.3 remains true without the assumption (13.27) but this technical assumption which holds for generic right-hand sides g essentially simplifies the proof of Theorem 13.3.

Remark 13.2. Arguing analogously to [2, Th. V.7.2] and using the result of Lemma 13.4 one can prove that under the assumptions of Theorem 13.2 the attractor $\mathcal{A}$ of (3.1) is exponential, i.e. there exists $\eta > 0$ such that for every bounded set $B \subset E_b(\Omega)$

$$\operatorname{dist}_{E_b(\Omega)}(S_tB, \mathcal{A}) \le C(B)e^{-\eta t} \tag{13.40}$$

The following theorem shows that the regular attractor, constructed in Theorem 13.2 may have an arbitrary large (but finite) dimension at least in the case $\Omega = \mathbb{R}^n$.

Theorem 13.4. *Let the assumptions of Theorem 13.2 hold, $\Omega = \mathbb{R}^n$ and let in addition the number $\xi \in \mathbb{R}$ such that $f'(\xi) + \lambda_0 < 0$ exist. Then for every $N \in \mathbb{N}$ there exists a right-hand side $g_N \in \dot{L}^s_b(\mathbb{R}^n)$ such that*

$$\dim_F(\mathcal{A}_{g_N}, E_b(\Omega)) \geq N \tag{13.41}$$

Proof. Indeed, according to Theorem 13.2,

$$\dim_F(\mathcal{A}_g) \geq \max_{u_0 \in \mathcal{R}_g} \#\{\lambda \in \sigma(\Delta_x - \lambda_0 - f'(u_0)) : \lambda > 0\} \tag{13.42}$$

Thus, it remains to construct for every fixed N the function $u_0 = u_0(N) \in \dot{W}^{2,s}_b(\mathbb{R}^n)$ in such a way that $\operatorname{Ind}_{u_0} > N$ (the appropriate right-hand side g can be calculated from the equation $\Delta_x u_0 - \lambda_0 u_0 - f(u_0) = g$).

The construction of such functions u_0 is given in [10] (under the assumption $f(0) = f'(0) = 0$ and $f'(\xi) + \lambda_0 > 0$). Moreover, all functions g_N thus constructed occurred to have a finite support in $\mathbb{R}^n$ and to be uniformly bounded with respect to $N \in \mathbb{N}$ in $L^s_b(\mathbb{R}^n)$. Theorem 13.4 is proved.

In conclusion we give an example of the equation of the view (3.1) which satisfies all assumptions of this Section.

Example 13.1. Let $\Omega = \mathbb{R}^2$. Then the equation

$$\partial_t^2 u + \gamma \partial_t u - \Delta_x u + 2u^3 - 8u^2 + 9u = g \tag{13.43}$$

with generic $g \in \dot{L}^2_b(\Omega)$ satisfies all assumptions of this Section. Indeed, all growth assumptions are evidently satisfied,

$$f(u).u = u(2u^3 - 8u^2 + 9u) = 2u^2(u-2)^2 + u^2 \geq u^2$$

and $f'(1) = -1 < 0$. Consequently the assertions of Theorems 13.1–13.4 hold for the equation (13.43)

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