SPATIAL AND DYNAMICAL CHAOS GENERATED BY REACTION-DIFFUSION EQUATIONS IN $\mathbb{R}^n$

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We study the longtime behavior of solutions of the following quasilinear reaction-diffusion system in an unbounded domain $\Omega = \mathbb{R}^n$:

$$\partial_t u = a\Delta_x u - (\vec{L}, \nabla_x)u - \lambda_0 u - f(u) + g(x), \quad u\big|_{t=0} = u_0, \tag{1}$$

where $u = (u^1, \cdots, u^k)$ is an unknown vector-valued function, a is a given diffusion matrix with the positive symmetric part $a + a^* > 0$, $(\vec{L}, \nabla_x)u := \sum_{i=1}^n L_i(x)\partial_{x_i} u$, $\vec{L} \in W^{1,\infty}(\mathbb{R}^n)$ is a given vector field in $\mathbb{R}^n$, which satisfies the assumption $\operatorname{div}(\vec{L}) \le \lambda_0/2$, $\lambda_0 > 0$ is a given positive constant and $f(u)$ and $g(x)$ are given nonlinear interaction function and external force respectively.

We assume that the nonlinear term $f \in C^3(\mathbb{R}^k, \mathbb{R}^k)$ satisfies the following dissipativity conditions:

$$1. \quad f(v).v \ge -C, \quad 2. \quad f'(v) \ge -K, \quad \forall v \in \mathbb{R}^k \tag{2}$$

(where by $u.v$ we denote the standard inner product in $\mathbb{R}^k$) for some fixed constants $C, K > 0$ and the following growth restrictions:

$$1. \quad |f(v)| \le C(1 + |v|^p), \quad 2. \quad |f'(v)|^{p/p-1} \le C(|f(v)| + |v| + 1) \tag{3}$$

where $1 < p < p_{max} := 1 + \frac{4}{n-4}$ for $n > 4$ and p may be arbitrarily large if $n \le 4$.

As usual we consider only bounded with respect to $|x| \to \infty$ solutions $u(t)$ of the problem (1) and consequently we assume that the initial data belongs to the uniformly local Sobolev space

$$\Phi_b := W_b^{2,q}(\mathbb{R}^n) = \{u_0 \in W_{loc}^{2,q}(\mathbb{R}^n), \ \|u_0\|_{\Phi_b} := \sup_{x_0 \in \mathbb{R}^n} \|u_0\|_{W^{2,q}(B_{x_0}^1)} < \infty\}$$

for some fixed $q > n + 1$. Here and below we denote by $B_{x_0}^R$ the R-ball in $\mathbb{R}^n$ centered in x_0. Finally, it also assumed that the external force $g \in L_b^q(\mathbb{R}^n)$.

As usual we describe the longtime behavior of solutions in terms of a global attractor for the semigroup associated with the equation (1).

Theorem 1. *Let the above assumptions hold. Then the equation (1) has a unique solution $u(t) \in \Phi_b$ for every $u_0 \in \Phi_b$ and thus the solving semigroup*

$$S_t : \Phi_b \to \Phi_b, \quad u(t) := S_t u_0 \text{ solves } (1) \tag{4}$$

is well defined. Moreover, this semigroup possesses a global locally-compact attractor $\mathcal{A} \subset \Phi_b$.

Recall that by definition $\mathcal{A}$ is a locally-compact attractor of the semigroup S_t if the following is true:

1. $\mathcal{A}$ is bounded in Φ_b and is compact in $W^{2,q}_{loc}(\mathbb{R}^n)$.
2. $\mathcal{A}$ is strictly invariant with respect to S_t, i.e. $S_t\mathcal{A} = \mathcal{A}$.
3. $\mathcal{A}$ attracts the images $S_t B$ of bounded subsets of $B \subset \Phi_b$ as $t \to \infty$ in the topology of $W^{2,q}_{loc}(\mathbb{R}^n)$.

Note, that in contrast to the case of bounded domains Ω, in unbounded domains the attractor $\mathcal{A}$ associated with the equation (1) is in general not compact in the phase space Φ_b but only its restrictions on every bounded subdomain $\Omega_1 \subset \Omega$ are compact in $W^{2,q}(\Omega_1)$, see e.g. [7]. Moreover, the dimension of the attractor $\mathcal{A}$ is usually infinite if the domain Ω is unbounded (see [1], [4], [5]), consequently (following to [2], [3], [5]) in order to obtain quantitative and qualitative information about the attractor it is natural to study it's Kolmogorov's ε-entropy.

Recall, that Kolmogorov ε-entropy is defined for every precompact set K in a metric space M and coincides with the logarithm of the minimal number of ε-balls in M which is sufficient to cover the set K. In particular, since the restrictions $\mathcal{A}\big|_{B^R_{x_0}}$ are compact in $W^{2,q}(B^R_{x_0})$ then their ε-entropies are well defined and finite for every $\varepsilon, R > 0$ and $x_0 \in \mathbb{R}^n$. The following theorem gives the upper bounds for these values, see also [6], [7].

Theorem 2. *Let the above assumptions hold. Then the Kolmogorov ε-entropy of $\mathcal{A}\big|_{B^R_{x_0}}$ possesses the following estimate:*

$$\mathbb{H}_\varepsilon\left(\mathcal{A}\big|_{B^R_{x_0}}, W^{2,q}_b(B^R_{x_0})\right) \le C\left(R + K \ln\frac{1}{\varepsilon}\right)^n \ln\frac{1}{\varepsilon}, \tag{5}$$

where the constants C and K are independent of R, x_0 and $\varepsilon > 0$.

A number of examples of natural evolution PDEs for which the estimate (5) is sharp for every values of R, x_0 and ε is constructed in [6] and [7].

More detailed information about the attractor $\mathcal{A}$ can be obtained for the spatially homogeneous case: $\Omega = \mathbb{R}^n$, $\vec{L} = const$, $g = const$.

In this case the attractor of the equation (1) possesses an additional structure, namely, it is occurred to be invariant with respect to the group $\{T_h, h \in \mathbb{R}^n\}$ of spatial translations

$$T_h\mathcal{A} = \mathcal{A}, \quad (T_h u_0)(x) := u_0(x+h), \quad x, h \in \mathbb{R}^n, \quad u_0 \in \mathcal{A} \tag{6}$$

and consequently an extended $(n+1)$-parametrical semigroup $\{\mathbb{S}_{(t,h)}, t \in \mathbb{R}_+, \ h \in \mathbb{R}^n\}$, defined via $\mathbb{S}_{(t,h)} := S_t \circ T_h$, acts on the attractor

$$\mathbb{S}_{(t,h)}\mathcal{A} = \mathcal{A}, \quad t \in \mathbb{R}_+, \quad h \in \mathbb{R}^n \tag{7}$$

The semigroup $\mathbb{S}_{(t,h)}$ is interpreted in the sequel as a dynamical system with multidimensional 'time' acting on the attractor $\mathcal{A}$, and it's dynamical properties are investigated. Particularly, it is proved using the estimate (5), that the topological entropy of this dynamical system is finite

$$h_{top}(\mathbb{S}_{(t,h)}, \mathcal{A}) < \infty, \tag{8}$$

and the topological entropies of k-parametrical subsemigroups of the dynamical system (7) are occurred to be infinite in general if $k < n+1$.

In order to describe the spatio-temporal dynamics on the attractor the following multidimensional generalization of a Bernoulli scheme with continual number of symbols $\omega \in [0,1]$ is introduced: consider a compact metrisable space $\mathcal{M} = [0,1]^{\mathbb{Z}^n}$ endowed by the Tikhonov's topology (i.e $\mathcal{M}$ is a set of functions $v : \mathbb{Z}^n \to [0,1]$ with a topology of locally compact convergence). A model dynamical system $\{\mathcal{T}_l, l \in \mathbb{Z}^n\}$ acts on $\mathcal{M}$ in the following natural way:

$$(\mathcal{T}_l v)(m) := v(l+m), \quad l, m \in \mathbb{Z}^n, \quad v \in \mathcal{M} \tag{9}$$

The main result of the paper is the following theorem.

Theorem 3. *Let the equation (1) possess a spatially-homogeneous equilibrium z_0 (without loss of generality we may assume that, $z_0 = 0$ and $\vec{L} = (L, 0, \cdots, 0)$) and let this equilibrium be exponentially unstable:*

$$\sigma(a\Delta_x - (\vec{L}, \nabla_x) - \lambda_0 - f'(0)) \cap \{\mathrm{Re}\, z > 0\} \neq \varnothing \tag{10}$$

Then there is $L_{min} = L_{min}(a, f)$, such that for $L > L_{min}$ there are $\alpha > 0$ and a homeomorphic embedding

$$\tau : \mathcal{M} \hookrightarrow \mathcal{A}, \tag{11}$$

such that for every $v \in \mathcal{M}$ and every $m \in \mathbb{Z}_+$, $l \in \mathbb{Z}$

$$S_{\alpha m} \circ \tau(v) = \tau(\mathcal{T}^1_m v), \quad T^i_{\alpha l} \circ \tau(v) = \tau(\mathcal{T}^i_l v), \ i = 2, \cdots, n \tag{12}$$

where $T^i_h := T_{h\vec{e}_i}$, e_i is an ith coordinate vector in $\mathbb{R}^n$, and $\mathcal{T}^i_l$ can be defined analogously.

Particularly, in contrast to (8), the topological entropy as for the group T_h of spatial translations as for the dynamical system S_t, generated by the equation (1) are occurred to be infinite on the attractor

$$h_{top}(S_t, \mathcal{A}) = h_{top}(T_h, \mathcal{A}) = \infty \tag{13}$$

Moreover, under the assumptions of Theorem 1 every finite dimensional dynamics can be obtained up to a homeomorphism restricting the semigroup S_t to the corresponding invariant subset of the attractor.

References

[1] Babin A. and Vishik M., Attractors of partial differential evolution equations in an unbounded domain, Proc. Royal Soc. Edinburgh, 116A (1990), 221–243

[2] Chepyzhov V. and Vishik M., Kolmogorov's ε-entropy for the attractor of reaction-diffusion equation, Sb. Math., 189(2) (1998), 81–110

[3] Collet P. and Eckmann J.-P., Extensive Properties of the Complex Ginzburg-Landau equation, Comm. in Math. Physics, 200 (1999), 699–722

[4] Efendiev M. and Zelik S., The attractor for a nonlinear reaction-diffusion system in an unbounded domain, Comm. Pure Appl. Math., 54(6) (2001), 625–688

[5] Zelik S., The attractor for an nonlinear reaction-diffusion system in $\mathbb{R}^n$ and the estimation of it's ε-entropy, Math. Notes, 65 (6) (1999), 941–943

[6] Zelik S., The attractor for a nonlinear hyperbolic equation in an unbounded domain, Disc. Cont. Dyn Sys. Ser. A, 7(3) (2001), 593–641

[7] Zelik S., The attractor for a nonlinear reaction-diffusion system in an unbounded domain and Kolmogorov's epsilon-entropy, to appear in Math. Nachr.