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ATTRACTORS FOR DISSIPATIVE PARTIAL DIFFERENTIAL EQUATIONS IN BOUNDED AND UNBOUNDED DOMAINS

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Contents

1. Introduction

The study of the asymptotic behavior of dynamical systems arising from mechanics and physics is a capital issue, as it is essential, for practical applications, to be able to understand, and even predict, the long time behavior of the solutions of such systems.

A dynamical system is a (deterministic) system which evolves with respect to the time. Such a time evolution can be continuous or discrete (i.e., one only measures the state of the system at given times, e.g., every hour or every day). We will essentially consider continuous dynamical systems in this survey.

In many situations, the evolution of the system can be described by a system of ordinary differential equations (ODEs) of the form

$$y' = f(t, y), \ y = (y_1, ..., y_N), \tag{0.1}$$

together with the initial condition

$$y(\tau) = y_\tau, \ \tau \in \mathbb{R}. \tag{0.2}$$

Assuming that the above Cauchy problem is well-posed, we can define a family of solving operators $U(t, \tau), \ t \geq \tau, \ \tau \in \mathbb{R}$, acting on some subset Φ of $\mathbb{R}^N$ (called the phase space), i.e.,

$$\begin{aligned} U(t, \tau) \ : \ & \Phi \to \Phi \\ & y_\tau \mapsto y(t), \end{aligned}$$

where $y(t)$ is the solution of (0.1)-(0.2) at time t. It is easy to see that this family of operators satisfies

$$U(\tau, \tau) = \mathrm{Id}, \ U(t, s) \circ U(s, \tau) = U(t, \tau), \ t \geq s \geq \tau, \ \tau \in \mathbb{R},$$

where Id denotes the identity operator. We say that this family of operators forms a process. When the function f does not depend explicitly on the time (in that case, we say that the system is autonomous), we can write

$$U(t, \tau) = S(t - \tau),$$

where the family of operators $S(t), \ t \geq 0$, satisfies

$$S(0) = \mathrm{Id}, \ S(t) \circ S(s) = S(t + s), \ t, \ s \geq 0.$$

We say that this family of solving operators $S(t), \ t \geq 0$, which maps the initial datum at $t = 0$ onto the solution at time t, forms a semigroup. Furthermore, we say that the pair $(S(t), \Phi)$ (or $(U(t, \tau), \Phi)$ for a nonautonomous system) is the dynamical system associated with our problem.

The qualitative study of such finite dimensional dynamical systems goes back to the pioneering works of Poincaré on the N-body problem in the beginning of the 20th century (see, e.g., [26] ; see also [66] and the references therein for the study of discrete dynamical systems in finite dimensions). In particular, it was discovered, at the very beginning of the theory, that even relatively simple systems of ODEs can generate very complicated (chaotic) behaviors. Furthermore, these systems are extremely sensitive to perturbations, in the sense that trajectories with close, but different, initial data may diverge exponentially. As a consequence, in spite of the deterministic nature of the system, its temporal evolution is unpredictable on time scales larger than some critical value which depends on the error of approximation and on the rate of divergence of close trajectories, and can show typical stochastic behaviors.

Such behaviors have first been observed and established for the pendulum equation perturbed by time periodic external forces, namely,

$$y''(t) + \sin(y(t))(1 + \epsilon\sin(\omega t)) = 0,$$

$\epsilon,\ \omega > 0$. Another, important, example is the Lorenz system, obtained by truncation of the Navier-Stokes equations (more precisely, one considers here a three-mode Galerkin approximation (one in velocity and two in temperature) of the Boussinesq equations),

$$\begin{aligned} x' &= \sigma(y - x), \\ y' &= -xy + rx - y, \\ z' &= xy - bz, \end{aligned}$$

where the positive constants σ, r, and b correspond to the Prandtl number, the Rayleigh number, and the aspect ratio, respectively ; in the original work of Lorenz (see [149]), these numbers take the values 10, 28, and $\frac{8}{3}$, respectively. This system gives an approximate description of a two-dimensional layer of fluid heated from below: the warmer fluid formed at the bottom tends to rise, creating convection currents, which is similar to what is observed in the atmosphere. For a sufficiently intense heating, the time evolution has a sensitive dependence on the initial conditions, thus representing a very irregular (chaotic) convection. This fact was used by Lorenz to justify the so-called "butterfly effect", a metaphor for the imprecision of weather forecast. Other well-known relatively simple systems which exhibit chaotic behaviors are the Minea system [173] and the Rössler system [205].

Very often, the trajectories of such chaotic systems are localized, up to some transient process, in some subset of the phase space having a very complicated geometric structure, e.g., locally homeomorphic to the Cartesian product of $\mathbb{R}^m$ and some Cantor set, which thus accumulates the nontrivial dynamics of the system, the so-called strange attractor (see, e.g., [28]). One noteworthy feature of a strange attractor is its dimension. First, in order for the sensitivity to initial conditions to be possible on the strange attractor, this dimension has to be strictly greater than 2, so that the dimension of the phase space has to be greater than 3 ; let us assume, for simplicity, that this dimension is equal to 3, as in the Lorenz system. Then the volume of the strange attractor must be equal to 0 ; indeed, in systems having a strange attractor, one observes a contraction of volumes in the phase space. Thus, the dimension of a strange attractor is noninteger, strictly between 2 and 3, and we need to use other dimensions than the Euclidean dimension to measure it. Several dimensions, which are not equivalent and yield different values of the dimension in concrete applications, can be used (roughly speaking, some notions of dimensions are related to the connectedness of the sets that one measures, others are related to the way that these sets are embedded into the ambient space, for instance). We will mainly consider in this article the box-counting (or entropy) dimension (see below ; see also [87]), which we will call the fractal dimension. Other possible notions of dimensions are the Hausdorff dimension or the Lyapunov dimension (see [87]). Thus, the main features of a strange attractor are

- the trajectories (at least those starting from a neighborhood) are attracted to it ;
- close, but different, trajectories may diverge ;
- it has a noninteger (fractal) dimension (for instance, for the Lorenz system, numerical investigations show that this dimension is close to, but greater than, 2, namely, 2.05..., which means that there is a "strong" contraction of volumes).

Now, for a distributed system whose initial state is described by functions depending on the spatial variable, the time evolution is usually governed by a system of partial differential equations (PDEs). In that case, the phase space Φ is (a subset of) an infinite dimensional function space ; typically, $\Phi = L^2(\Omega)$ or $L^\infty(\Omega)$, where Ω is some domain of $\mathbb{R}^N$. We will thus speak of infinite dimensional dynamical systems.

A first, important, difference, when compared with ODEs, is that the analytical structure of a PDE is much more complicated. In particular, we do not have a unique solvability result in general, or such a result can be very difficult to obtain. We can, for instance, mention the three-dimensional Navier-Stokes equations, for which a proper global well-posedness result is not known yet (see, e.g., [221]). Nevertheless, the global existence and uniqueness of solutions has been proven for a large class of PDEs arising from mechanics and physics, and it is therefore natural to investigate whether the features mentioned above for dynamical systems generated by systems of ODEs, and, in particular, the strange attractor, generalize to systems of PDEs.

Such behaviors can be observed in a large class of PDEs which exhibit some energy dissipation and are called dissipative PDEs. Roughly speaking, the highly complicated behaviors observed in such systems usually arise from the interaction of the following mechanisms:
• energy dissipation in the higher part of the Fourier spectrum ;
• external energy income in its lower part (in order to have nontrivial dynamics, the system has to also account for the energy income) ;
• energy flux from the lower to the higher Fourier modes, due to the nonlinear terms of the equations.

As already mentioned, this class of PDEs contains a large number of equations from mechanics and physics ; we can mention for instance reaction-diffusion equations, the incompressible Navier-Stokes equations, pattern formation equations (e.g., the Cahn-Hilliard equation in materials science and the Kuramoto-Sivashinsky equation in combustion), and damped wave equations.

It is worth emphasizing once more that the phase space is an infinite dimensional function space. However, experiments showed that, as in the case of finite dimensional dynamical systems, the trajectories are localized, up to some transient process, in a "thin" invariant subset of the phase space having a very complicated geometric structure, which thus accumulates all the essential dynamics of the system.

From a mathematical point of view, this led to the notion of a global attractor (see [23], [51], [53], [122], [139], [140], [200], [214], and [220] ; see also [16] and [198] for some historical comments). Assuming that the problem is well-posed and that the system is autonomous (i.e., that the time does not appear explicitly in the equations), we have, as in the finite dimensional case, the semigroup $S(t),\ t \geq 0$, acting on the phase space Φ, which maps the initial condition onto the solution at time t. Then we say that $\mathcal{A} \subset \Phi$ is the global attractor for $S(t)$ if
(i) it is compact in Φ ;
(ii) it is invariant, i.e., $S(t)\mathcal{A} = \mathcal{A},\ \forall t \geq 0$;
(iii) $\forall B \subset \Phi$ bounded,

$$\lim_{t\to+\infty} \mathrm{dist}(S(t)B, \mathcal{A}) = 0,$$

where dist denotes the Hausdorff semi-distance between sets (we assume that Φ is a metric space with distance d) defined by

$$\mathrm{dist}(A,B) := \sup_{a\in A}\inf_{b\in B} d(a,b).$$

This is equivalent to the following: $\forall B\subset\Phi$ bounded, $\forall\epsilon>0$, $\exists t_0=t_0(B,\epsilon)$ such that $t\ge t_0$ implies $S(t)B\subset\mathcal{U}_\epsilon$, where $\mathcal{U}_\epsilon$ is the ϵ-neighborhood of $\mathcal{A}$.

We note that it follows from (ii) and (iii) that the global attractor, if it exists, is unique. Furthermore, it follows from (i) that it is essentially thinner than the initial phase space Φ ; indeed, in infinite dimensions, a compact set cannot contain a ball and is nowhere dense. It is also not difficult to prove that the global attractor is the smallest (for the inclusion) closed set enjoying the attraction property (iii) ; it thus appears as a suitable object in view of the study of the long time behavior of the system. It is also the maximal bounded invariant set. We finally note that $\mathcal{A}$ attracts all the trajectories (uniformly with respect to bounded sets of initial data), and not just those starting from a neighborhood. The global attractor is sometimes called the maximal or the universal attractor (which is reasonable in view of the above considerations), although these denominations are less used nowadays.

It has also been early conjectured that the invariant attracting sets mentioned above, and, in particular, the global attractor, should be, in a proper sense, finite dimensional and that the dynamics, restricted to these sets, should be effectively described by a finite number of parameters. The notions of dimensions mentioned above, and, in particular, the fractal dimension, should again be appropriate to measure the dimension of these sets. So, when this conjecture is true, the effective dynamics, restricted to the global attractor, is finite dimensional, even though the initial phase space is infinite dimensional. This also suggests that such systems cannot produce any new dynamics which are not observed in finite dimensions, the infinite dimensionality only bringing (possibly essential) technical difficulties.

Starting from the pioneering works of Ladyzhenskaya (see, e.g., [138], [139], and the references therein), this finite dimensional reduction, based on the global attractor, has been given solid mathematical grounds in the past decades for dissipative systems in *bounded domains*. In particular, the existence of the finite dimensional global attractor has been proven for many classes of dissipative PDEs, including the examples mentioned above. We refer the reader to [23], [51], [122], [139], [140], [200], [214], and [220] for extensive reviews on this subject.

Now, the global attractor may present several defaults. Indeed, it may attract the trajectories at a slow rate. Furthermore, in general, it is very difficult, if not impossible, to express the convergence rate in terms of the physical parameters of the problem. This can be seen on the following real Ginzburg-Landau equation in one space dimension:

$$\partial_t u-\nu\partial_x^2 u+u^3-u=0,\ x\in[0,1],\ \nu>0,$$

$$u(0,t)=u(1,t)=-1,\ t\ge 0,$$

see Remark 2.25. A second drawback, which can also be seen as a consequence of the first one, is that the global attractor may be sensitive to perturbations ; a given system is only an approximation of reality and it is thus essential that the objects that we study are robust under small perturbations. Actually, in general, the global attractor is upper semicontinuous with respect to perturbations, i.e.,

$$\mathrm{dist}(\mathcal{A}_\epsilon, \mathcal{A}_0) \to 0 \text{ as } \epsilon \to 0^+,$$

where $\mathcal{A}_0$ is the global attractor associated with the nonperturbed system and $\mathcal{A}_\epsilon$ that associated with the perturbed one, $\epsilon > 0$ being the perturbation parameter. Very roughly speaking, this property means that the global attractor cannot explode under small perturbations. Now, the lower semicontinuity, i.e.,

$$\mathrm{dist}(\mathcal{A}_0, \mathcal{A}_\epsilon) \to 0 \text{ as } \epsilon \to 0^+,$$

which, roughly speaking, means that the global attractor cannot implode also, is much more difficult to prove (see, e.g., [198]). Furthermore, this property may not hold. This can already be seen in finite dimensions by considering the following ODE (see [198]):

$$x' = (1 - x^2)(1 - \lambda^2), \ \lambda \in [-1, 1].$$

Then, when $\lambda = 0$, $\mathcal{A}_\lambda = [0, 1]$, whereas $\mathcal{A}_\lambda = \{1\}$ for $\lambda < 0$ and $\mathcal{A}_\lambda = [-\sqrt{\lambda}, 1]$ for $\lambda > 0$. Thus, there is a bifurcation phenomenon at $\lambda = 0$ and the global attractor is not lower semicontinuous at $\lambda = 0$. It thus follows that the global attractor may change drastically under small perturbations. Furthermore, in many situations, the global attractor may not be observable in experiments or in numerical simulations. This can be due to the fact that it has a very complicated geometric structure, but not necessarily. Indeed, we can again consider the above Ginzburg-Landau equation. Then, due to the boundary conditions, $\mathcal{A} = \{-1\}$. Now, this problem possesses many metastable "almost stationary" equilibria which live up to a time $t_\star \sim e^{\nu^{-\frac{1}{2}}}$. Thus, for ν small, one will not see the global attractor in numerical simulations. Finally, in some situations, the global attractor may fail to capture important transient behaviors. This can be observed, e.g., on some models of one-dimensional Burgers equations with a weak dissipation term (see [29]). In that case, the global attractor is trivial, it is reduced to one exponentially attracting point, but the system presents very rich and important transient behaviors which resemble some modified version of the Kolmogorov law. We can also mention models of pattern formation equations in chemotaxis for which one observes important transient behaviors, i.e., patterns, which are not contained in the global attractor (see [218]).

It is thus also important to construct and study larger objects which contain the global attractor, are more robust under perturbations, attract the trajectories at a fast (typically, exponential) rate, and are still finite dimensional. Two such objects have been proposed, namely, an inertial manifold (see [98]) and an exponential attractor (see [68]). We will discuss these objects in more details in the next sections, with an emphasis on exponential attractors (which are as general as global attractors).

An interesting question is whether one has a similar reduction principle for nonautonomous dissipative PDEs (in bounded domains). A first difference, compared with autonomous systems, is that both the initial and final times play an important role ; assuming that the problem is well-posed, it defines a process $U(t, \tau)$, $t \geq \tau$, $\tau \in \mathbb{R}$, which maps the initial condition at time τ onto the solution at time t. For such systems, the notion of a global attractor is no longer adequate (in particular, we will not be able to construct proper time independent invariant sets), and one needs to consider other notions of attractors.

A first approach, initiated by Haraux (see [124]) and further studied and developed by Chepyzhov and Vishik (see, e.g., [47] and [51]), is based on the notion of a uniform attractor. Actually, in order to construct the uniform attractor, one considers, together with the initial equations, a whole family of equations. Then one proves the existence of the global attractor for a proper semigroup on an extended phase space, and, finally, projecting this global attractor onto the first component, one obtains the uniform attractor. The major drawback of this approach is that the extended dynamical system is essentially more complicated than the initial one, which leads, for general (translation compact, see Section 3 ; see also [47] and [51]) time dependences, to an artificial infinite dimensionality of the uniform attractor. This can already be seen on the following simple linear equation:

$$\partial_t u - \Delta_x u = h(t), \quad u\big|_{\partial\Omega} = 0,$$

in a bounded smooth domain Ω, whose dynamics is simple, namely, one has one exponentially attracting trajectory. However, for more or less general external forces h, the associated uniform attractor has infinite dimension and infinite topological entropy (see [51]).

Nevertheless, for periodic and quasiperiodic time dependences, one has in general finite dimensional uniform attractors (i.e., if the same is true for the global attractor of the corresponding autonomous system). Furthermore, one can derive sharp upper and lower bounds on the dimension of the uniform attractor, so that this approach is appropriate and relevant in those cases.

A second approach, which resembles the so-called kernel sections proposed by Chepyzhov and Vishik (see [46] and [51]), but was studied and developed independently, is based on the notion of a pullback attractor (see, e.g., [64], [132], and [210]). In that case, one has a time dependent attractor $\{\mathcal{A}(t),\ t \in \mathbb{R}\}$, contrary to the uniform attractor which is time independent. More precisely, a family $\{\mathcal{A}(t),\ t \in \mathbb{R}\}$ is a pullback attractor for the process $U(t,\tau)$ if

(i) the set $\mathcal{A}(t)$ is compact in Φ, $\forall t \in \mathbb{R}$;

(ii) it is invariant, i.e., $U(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t),\ \forall t \geq \tau,\ \tau \in \mathbb{R}$;

(iii) it satisfies the following pullback attraction property:

$$\forall B \subset \Phi \text{ bounded},\ \forall t \in \mathbb{R},\ \lim_{s \to +\infty} \operatorname{dist}(U(t,t-s)B, \mathcal{A}(t)) = 0.$$

One can prove that, in general, $\mathcal{A}(t)$ has finite dimension, $\forall t \in \mathbb{R}$, see, e.g., [40] and [142]. Now, the attraction property essentially means that, at time t, the attractor $\mathcal{A}(t)$ attracts the bounded sets of initial data coming from the past (i.e., from $-\infty$). However, in (iii), the rate of attraction is not uniform in t, so that the forward convergence does not hold in general (see nevertheless [37], [42], and [141] for cases where the forward convergence can be proven). We can illustrate this on the following nonautonomous ODE:

$$y' = f(t,y),$$

where $f(t,y) := -y$ if $t \leq 0$, $(-1+2t)y - ty^2$ if $t \in [0,1]$, and $y - y^2$ if $t \geq 1$. Then one has the existence of a pullback attractor $\{\mathcal{A}(t),\ t \in \mathbb{R}\}$, namely, $\mathcal{A}(t) = \{0\}$, $\forall t \in \mathbb{R}$. However, for $t \geq 1$, every trajectory, different from $\{0\}$, starting from a small neighborhood of 0, will leave this neighborhood, never to enter it again. This clearly contradicts our intuitive understanding of attractors.

So, these two theories of attractors for nonautonomous systems do not yield an entirely satisfactory finite dimensional reduction principle, contrary to the autonomous case, since we have either an artificial infinite dimensionality or no forward attraction in general. We will see below that the construction of exponential attractors allows to overcome the main drawback of pullback attractors, namely, the problem of the forward attraction, as proven in [76] ; indeed, one has an exponential uniform control on the rate of attraction. This yields a satisfactory reduction principle for nonautonomous dynamical systems associated with dissipative PDEs in bounded domains.

Now, while the theory of attractors for dissipative dynamical systems in bounded domains is rather well understood, the situation is different for systems in unbounded domains and such a theory has only recently been addressed (and is still progressing), starting from the pioneering works of Abergel [1] and Babin and Vishik [22]. The main difficulty in this theory is the fact that, in contrast to the case of bounded domains discussed above, the dynamics generated by dissipative PDEs in unbounded domains is (as a rule) *purely* infinite dimensional and does not possess any finite dimensional reduction principle. Furthermore, the additional spatial "unbounded" directions lead to the so-called *spatial* chaos and the interactions between spatial and temporal chaotic modes generate a *space-time* chaos which also has no analogue in finite dimensions.

As a consequence, most of the ideas and methods of the classical (finite dimensional) theory of dynamical systems does not work here (as such systems have infinite Lyapunov dimension, infinite topological entropy, ...). Thus, we are faced with dynamical phenomena with new levels of complexity which do not have analogues in the finite dimensional theory and we need to develop a new theory in order to describe such phenomena in an accurate way.

It is also interesting to note that, in the case of bounded domains, the dimension of the global attractor grows at least linearly with respect to the volume of the spatial domain and, thus, for sufficiently large domains, the reduced dynamical system may be too large for reasonable investigations. Furthermore, as shown in [17], the spatial complexity of the system (e.g., the number of topologically different equilibria) grows *exponentially* with respect to the volume of the spatial domain. Therefore, even in the case of relatively small dimensions, the reduced system can be out of reach of reasonable investigations, due to its extremely complicated structure. As a consequence, it seems more natural, at least from a physical point of view, to replace large bounded domains by their limit unbounded ones (e.g., the whole space or cylindrical domains), which, of course, requires a systematic study of dissipative dynamical systems associated with PDEs in unbounded domains.

We will discuss such (for most of them new) developments in Section 5 of this survey.

In a last section, we will briefly discuss extensions of the theory of attractors to ill-posed problems, with an emphasis on the so-called trajectory attractor, see, e.g., [48], [49], [51], and [213]. Indeed, for many interesting problems, including the three-dimensional Navier-Stokes equations, various types of damped hyperbolic equations (e.g., damped wave equations with supercritical nonlinearities), ..., the well-posedness of the solution operator $S(t)$ has not been proven yet or/and the proper choice of the phase space is not known. Furthermore, e.g., for dissipative systems with non-Lipschitz nonlinearities or for systems arising from the dynamical approach of elliptic boundary value problems in unbounded domains, nonuniqueness results and the ill-posedness of the associated solution operator are known.

2. The global attractor

2.1. **Main definitions.** Let E be a Banach space with norm $\|\cdot\|_E$ (actually, in most results, E can more generally be a complete metric space ; furthermore, in some cases, e.g., for the so-called trajectory attractors, see Section 6 (see also Theorem 2.20), even metric spaces may be inadequate). We consider a semigroup $S(t)$, $t \geq 0$, acting on E, i.e., we assume that the phase space Φ is the whole space E (it is not difficult to adapt the definitions when Φ is a subset of E),

$$S(t) \ : \ E \to E, \ \forall t \geq 0, \tag{2.1}$$

$$S(0) = \mathrm{Id}, \tag{2.2}$$

$$S(t+s) = S(t) \circ S(s), \ \forall t, \ s \geq 0, \tag{2.3}$$

where Id denotes the identity operator. We will also need some continuity property on $S(t)$, and we assume from now on that

$$S(t) \text{ is continuous from } E \text{ into itself}, \ \forall t \geq 0. \tag{2.4}$$

Remark 2.1. a) It was recently proven in [189] that condition (2.4) can be relaxed and that one can prove the existence of global attractors under the following, much weaker, condition:

$$\text{if } x_k \to x \text{ and } S(t)x_k \to y, \text{ then } y = S(t)x. \tag{2.5}$$

A semigroup satisfying (2.5) is called a closed semigroup (see also [247] for another type of condition, contained in (2.5)). Condition (2.5) is also important for concrete applications ; indeed, there are situations in which $x_k \to x$ only implies that $S(t)x_k \to S(t)x$ for the weak topology (this is the case, e.g., for the damped wave equation with a nonlinear damping, see [189]). However, in contrast to the usual continuous case, the global attractor may not be connected (see the next subsection) for closed semigroups (even if the initial absorbing set is connected) and some additional assumptions are necessary to guarantee this property.
b) In general, the operators $S(t)$, $t \geq 0$, are not one-to-one (this property is equivalent to the backward uniqueness property, see, e.g., [220]). When $S(t)$, $t > 0$, is one-to-one, we can define its inverse, which we denote by $S(-t)$. It is then easy to see that the family $S(t)$, $t \in \mathbb{R}$, enjoys properties (2.1)-(2.3), and we say that it forms a group acting on E. One new feature of the infinite dimensional theory, compared with the finite dimensional one, is that, in general, as already mentioned, the operators $S(t)$, $t < 0$, are not defined everywhere.

Definition 2.2. A set $X \subset E$ is invariant for $S(t)$ if

$$S(t)X = X, \ \forall t \geq 0.$$

If $S(t)X \subset X$, $\forall t \geq 0$, we say that X is positively invariant and, if $X \subset S(t)X$, $\forall t \geq 0$, we say that X is negatively invariant.

A first, simple, example of invariant sets is given by fixed points (also called stationary trajectories or solutions) or by sets of fixed points ($a \in E$ is a fixed point if $S(t)a = a,\ \forall t \geq 0$). A second example is given by complete trajectories or by sets of complete trajectories. Let u_0 belong to E. Then the forward, or positive, trajectory starting at u_0 is the set

$$\{S(t)u_0,\ t \geq 0\}.$$

A backward, or negative, trajectory ending at u_0, if it exists, is a set of points of the form

$$\cup_{t\leq 0} u(t),\ u(t) \in S(-t)^{-1}u_0,\ \forall t \geq 0$$

(we can note that a negative trajectory, if it exists, is not necessarily unique). Finally, a complete trajectory through u_0, if it exists, is the union of the positive and a negative trajectories. It is not difficult to show that the positive trajectory is positively invariant, a negative trajectory is negatively invariant, and a complete trajectory is invariant.

Another, more complicated, example of invariant sets is given by ω-limit sets ; these sets are also essential in view of the construction of global attractors.

Definition 2.3. Let u_0 belong to E. The ω-limit set of u_0 is the set

$$\omega(u_0) := \overline{\cap_{s\geq 0}\cup_{t\geq s} S(t)u_0},$$

where the closure is taken in E. Similarly, for $B \subset E$, the ω-limit set of B is the set

$$\omega(B) := \overline{\cap_{s\geq 0}\cup_{t\geq s} S(t)B}.$$

We have the following important characterization of ω-limit sets: $x \in \omega(B)$ if and only if there exist sequences $\{x_k,\ k \in \mathbb{N}\}$ and $\{t_k,\ k \in \mathbb{N}\}$, with $x_k \in B$, $\forall k \in \mathbb{N}$, and $t_k \to +\infty$ as $k \to +\infty$, such that $S(t_k)x_k \to x$ as $k \to +\infty$.

Remark 2.4. Similarly, we define the α-limit set of B, if it exists, by

$$\alpha(B) := \overline{\cap_{s\leq 0}\cup_{t\leq s} S(-t)^{-1}B}.$$

We then have the

Proposition 2.5. *We assume that $B \subset E$, $B \neq \emptyset$, and that there exists $t_0 \geq 0$ such that $\cup_{t\geq t_0} S(t)B$ is relatively compact in E. Then $\omega(B)$ is nonempty, compact, and invariant.*

We are now ready to formulate some *mathematical* concepts of dissipativity. To this end, we need to recall the notions of absorbing and attracting sets for the semigroup $S(t)$.

Definition 2.6. (i) A bounded set $\mathcal{B}_0 \subset E$ is a bounded absorbing set for $S(t)$ if, $\forall B \subset E$ bounded, $\exists t_0 = t_0(B)$ such that $t \geq t_0$ implies $S(t)B \subset \mathcal{B}_0$.
(ii) A set $K \subset E$ is attracting if, $\forall B \subset E$ bounded,

$$\lim_{t\to+\infty} \operatorname{dist}(S(t)B, K) = 0,$$

where dist (or $\operatorname{dist}_{\mathrm{E}}$ if it is necessary to precise the topology) is the Hausdorff semidistance between sets in E, defined by

$$\operatorname{dist}(A, B) := \sup_{a\in A}\ \inf_{b\in B}\ \|a - b\|_E$$

(note that $\operatorname{dist}(A, B) = 0$ only implies $A \subset \overline{B}$).

The existence of an absorbing set is often used as a mathematical definition of a dissipative system. Following this tradition, we give the following definition.

Definition 2.7. The semigroup $S(t)$ is *dissipative* in E if it possesses a bounded absorbing set $B \subset E$.

In applications, this property is usually verified by proving a so-called *dissipative estimate* of the form

$$\|S(t)u_0\|_E \leq Q(\|u_0\|_E)e^{-\alpha t} + C_*, \ t \geq 0, \tag{2.6}$$

where the monotonic function Q and the positive constants α and C_* are independent of $u_0 \in E$.

Remark 2.8. A different notion of an absorbing set is considered in [122]: the semigroup $S(t)$ is called point dissipative if there exists a bounded set $\mathcal{B}_0 \subset E$ such that, $\forall u_0 \in E$, $\exists t_0 = t_0(u_0)$ such that $t \geq t_0$ implies $S(t)u_0 \in \mathcal{B}_0$.

We however have to note that the above mathematical definition of dissipativity is not sufficient to capture the typical physical properties of dissipative systems (see the introduction). Indeed, let us consider the semigroup generated by the following equation in an infinite dimensional Hilbert space E:

$$y'(t) = y(t)(1 - \|y(t)\|_E^2), \ y(0) = y_0 \in E.$$

Then this semigroup obviously satisfies a dissipative estimate as above and is dissipative according to the above mathematical definition. However, we do not have here an energy dissipation in the higher Fourier modes (in fact, the energy increases or decreases *simultaneously* in all modes depending on whether or not $\|y(t)\|_E \geq 1$). Thus, it is difficult to consider this semigroup as a dissipative system from a physical point of view.

In order to avoid such a situation, some kind of *asymptotic compactness* of the semigroup (e.g., the existence of a *compact* absorbing/attracting set ; this naturally gives a decay in the higher part of the Fourier spectrum) should be postulated. This asymptotic compactness can naturally be expressed in terms of the so-called Kuratowski measure of noncompactness.

Definition 2.9. Let B be a bounded subset of E. The Kuratowski measure of noncompactness of B is the quantity

$$\kappa(B) := \inf \{d, \ B \text{ has a finite covering with balls of } E \text{ with diameter less than } d\}.$$

The Kuratowski measure of noncompactness enjoys the following properties (see [122]):

- $\kappa(B) = 0$ if and only if B is relatively compact in E ;
- $\kappa(B) = \kappa(\overline{B})$;
- $B_1 \subset B_2$ implies $\kappa(B_1) \leq \kappa(B_2)$;
- $\kappa(B_1 + B_2) \leq \kappa(B_1) + \kappa(B_2)$.

Definition 2.10. We say that the semigroup $S(t)$ is *asymptotically* compact if, for every bounded set $B \subset E$, the Kuratowski measure of noncompactness of the image $S(t)B$ tends to zero as $t \to +\infty$,

$$\lim_{t \to +\infty} \kappa(S(t)B) = 0, \ \forall B \text{ bounded in } E.$$

We are now ready to define the main object of this survey, namely, a global attractor.

Definition 2.11. A set $\mathcal{A} \subset E$ is a global attractor of the semigroup $S(t)$ on E if the following properties are satisfied:
(i) it is a compact subset of E ;
(ii) it is invariant, $S(t)\mathcal{A} = \mathcal{A}$, $\forall t \geq 0$;
(iii) it is an attracting set for $S(t)$ on E.

It follows from this definition that the dissipativity and asymptotic compactness of the associated semigroup are *necessary* for the existence of a global attractor. As we will see below, these conditions are also *sufficient*.

As already mentioned in the introduction, the global attractor, if it exists, is unique. Furthermore, it is the smallest closed set which attracts the bounded subsets of E and the maximal bounded invariant set. We also note that it attracts the trajectories starting from the whole phase space (uniformly with respect to bounded sets of initial data), and not just those starting from a neighborhood.

We now formulate a simple, but very useful, result on the structure of the global attractor. To do so, we first give the following definition.

Definition 2.12. The kernel (in the terminology of Chepyzhov and Vishik) $\mathcal{K} \in L^\infty(\mathbb{R}, E)$ of the semigroup $S(t)$ is the set of all bounded complete trajectories of the semigroup $S(t)$, i.e., the functions $u : \mathbb{R} \to E$ such that

$$S(t)u(s) = u(t+s) \text{ and } \|u(s)\|_E \leq C_u < +\infty, \ \forall s \in \mathbb{R}, \ t \in \mathbb{R}_+.$$

Then we have the following result, which follows from the invariance of $\mathcal{A}$, see, e.g., [23].

Theorem 2.13. *The global attractor $\mathcal{A}$ (if it exists) is generated by the set $\mathcal{K}$ of all bounded complete trajectories of $S(t)$,*

$$\mathcal{A} = \mathcal{K}(0) := \{u(0), \ u \in \mathcal{K}\}. \tag{2.7}$$

In other words, $u_0 \in \mathcal{A}$ if and only if there exists a bounded complete trajectory u such that $u(0) = u_0$. Furthermore, $\mathcal{A} = \mathcal{K}(s)$, for every $s \in \mathbb{R}$.

Remark 2.14. Together with the concept of a *global* attractor given above, the so-called *local* attractors are widely used in the theory of dynamical systems. Such an attractor only attracts the images of all bounded subsets of some *neighborhood* $\mathcal{U}$ ($\mathcal{A} \subset \mathcal{U}$). The largest neighborhood which satisfies this property is then called the *basin* of attraction of the attractor $\mathcal{A}$. Another weaker concept of an attractor can be obtained by relaxing the attraction property. To be more precise, instead of requiring that all trajectories starting from a bounded subset of the phase space have a *uniform* rate of attraction to the attractor (see Definition 2.6), one may allow every trajectory to have its own (nonuniform) rate of attraction. This leads to the so-called pointwise attractor which has been used, e.g., in the original works of Ladyzhenskaya, see [140] and the references therein.

In some situations, e.g., for equations in unbounded domains, the attraction holds in a weaker topology, defined by some topological space E_1, $E \subset E_1$. To describe such a situation, Babin and Vishik proposed the terminology (E, E_1)-attractor, see [23]. Roughly speaking, an (E, E_1)-attractor attracts the bounded subsets of E in the topology of the space E_1 (thus, the space E is used here only to determine the class of bounded sets). In particular, if E_1 corresponds to E endowed with the weak topology, then one speaks of weak attractors. Furthermore, it is sometimes more convenient (especially, in the theory of the so-called trajectory attractors, see Section 6 below) to use more general classes of "bounded" sets which are not generated by any Banach space E and can be fixed almost arbitrarily. The only property of "bounded" sets which seems to be important for the theory of attractors is the following one.

Definition 2.15. A class $\mathcal{B}$ of subsets of E is called a class of "bounded" sets if

$$B \in \mathcal{B} \text{ and } B_1 \subset B \text{ imply } B_1 \in \mathcal{B}. \tag{2.8}$$

Then, naturally, a set $B \in \mathcal{B}$ is an absorbing set for the semigroup $S(t)$ if it absorbs the images of all "bounded" sets (i.e., all sets belonging to $\mathcal{B}$), see [210].

2.2. **Existence of the global attractor.** As mentioned in the previous subsection, ω-limit sets play an important role in the construction of global attractors. Indeed, one has the following result, based on Proposition 2.5 (see, e.g., [23] and [220]).

Theorem 2.16. *We assume that the semigroup $S(t)$ is continuous and has a compact absorbing set $\mathcal{B}_0$. Then it possesses the global attractor $\mathcal{A}$ such that $\mathcal{A} = \omega(\mathcal{B}_0)$. Furthermore, $\mathcal{A}$ is connected.*

We note that, owing to Proposition 2.5, one only needs to prove the attraction property to have the existence of the global attractor ; this property follows from the fact that $\mathcal{B}_0$ is an absorbing set.

In concrete situations, the above result will apply to (most) parabolic systems in bounded domains, since one has some compact regularizing effect in finite time. For damped hyperbolic equations and for parabolic equations in unbounded domains, we need a more general result, since such a regularizing effect is not available. However, noting that one has, in some sense, some compact regularizing effect at infinity, the following existence result, due to Babin and Vishik (see, e.g., [23]), can be used in most situations.

Theorem 2.17. *We assume that the semigroup $S(t)$ is continuous and possesses a compact attracting set. Then it possesses the connected global attractor $\mathcal{A}$. Furthermore, if K is a compact attracting set, then $\mathcal{A} = \omega(K)$.*

Remark 2.18. In order to prove that the attractor $\mathcal{A}$ is connected, one only needs the existence of a connected bounded absorbing set. Since the balls in a Banach space are always connected, this property holds automatically if the phase space E is the whole Banach space. In a more general setting, i.e., when E is a metric, or even a topological, space, this assumption should be added in order to ensure the connectedness.

We give another attractor's existence result which exploits the Kuratowski measure of noncompactness (see [122]). Although it is formally equivalent to Theorem 2.17, in practice, it can be used in a more general setting, namely, when the existence of a compact attracting set is difficult to verify directly (however, the existence of such a set a posteriori follows from that of the global attractor), see, e.g., [188] and [201].

Theorem 2.19. *We assume that the semigroup $S(t)$ is continuous, dissipative (i.e., it possesses a bounded absorbing set $\mathcal{B}_0$), and asymptotically compact (in the sense of Definition 2.10). Then it possesses the connected global attractor $\mathcal{A}$ such that $\mathcal{A} = \omega(\mathcal{B}_0)$.*

We now discuss a general strategy to verify the conditions of the above attractor's existence theorems in applications.

The existence of a compact *absorbing* set (for Theorem 2.16) is typical of *parabolic* problems in bounded domains for which the semigroup $S(t)$ usually consists of *compact* operators for $t > 0$. In that case, one usually has a *smoothing* property of the form

$$\|S(t)\|_{E_1} \le t^{-\beta} Q(\|u_0\|_E), \ t \in (0, 1],$$

where E_1 is some stronger space (i.e., it is compactly embedded into E) and where the monotonic function Q and the positive constant β are independent of u_0, see [23], [122], [220], and the references therein. Then, together with the dissipative estimate (2.6), this smoothing property guarantees that a ball in E_1 with a sufficiently large radius R is a compact absorbing set for $S(t)$. According to Theorem 2.16, this yields the existence of the global attractor $\mathcal{A} \subset E_1$ and its boundedness in E_1.

However, for more general classes of dissipative systems (e.g., damped hyperbolic equations), the smoothing property in *finite* time does not hold and should be replaced by an *asymptotically* smoothing property,

$$S(t) = S_1(t) + S_2(t),\ S_i(t)\ :\ E \to E,\ i = 1,\ 2, \tag{2.9}$$

where the operators $S_2(t)$ are *compact* for every fixed $t \geq 0$ (i.e., $S_2(t)B$ is precompact in E for every bounded subset B of E and every $t \geq 0$) and the operators $S_1(t)$ tend to zero as $t \to +\infty$,

$$\lim_{t\to+\infty} \|S_1(t)B\|_E = 0,\ \text{for every } B \subset E \text{ bounded},$$

where $\|B\|_E := \sup_{x\in B}\|x\|_E,\ B \subset E$ (we emphasize here that only the maps $S(t)$ should be continuous in E, and no additional *continuity* assumption on $S_1(t)$ and $S_2(t)$ is required).

It is not difficult to see that decomposition (2.9) is formally equivalent to the asymptotic compactness (in the sense of the Kuratowski measure of noncompactness, see Definition 2.10) and, consequently, together with the dissipative estimate (2.6), this gives the existence of the global attractor, due to Theorem 2.19 (when the space E is a uniformly convex Banach space, this decomposition can be artificially reduced to that of *continuous* operators $S_1(t)$ and $S_2(t)$, see [119] and [220]).

Very often, in applications, $S_2(t)$ maps E into some stronger space E_1 (which is compactly embedded into E). If, in addition, the operators $S_2(t)$ are uniformly bounded in E_1 as $t \to +\infty$,

$$\sup_{t\in\mathbb{R}_+} \|S_2(t)B\|_{E_1} < +\infty,\ \text{for every } B \subset E \text{ bounded}, \tag{2.10}$$

decomposition (2.9), together with the dissipative estimate (2.6), guarantee that a ball in E_1 with a sufficiently large radius is a compact *attracting* set for $S(t)$ and one can apply Theorem 2.17 to prove the existence of the global attractor $\mathcal{A}$ and to verify, in addition, that $\mathcal{A}$ is *bounded* in E_1. This is the usual way to verify the further regularity of gobal attractors, see [23] and [220] for details.

However, it is sometimes very difficult, if not impossible, to verify the additional boundedness (2.10) and the operators $S_2(t)$ may a priori grow as $t \to +\infty$, see, e.g., [56]. In that case, decomposition (2.9) is not strong enough to construct a compact attracting set (at least in a direct way) and Theorem 2.17 is not applicable. Nevertheless, as already mentioned, the boundedness property (2.10) is not necessary to verify the asymptotic compactness and Theorem 2.19 gives the existence of the global attractor $\mathcal{A}$. The main drawback of this scheme is that we now only have the compactness of $\mathcal{A}$ in E and cannot conclude that $\mathcal{A}$ belongs to a more regular space E_1. So, Theorem 2.19 cannot be used to prove further regularity results on the attractors.

We also recall another equivalent definition of asymptotic compactness, namely, in the terminology of Ladyzhenskaya (see, e.g., [140]), $S(t)$ is asymptotically compact if

$$\text{for every } \{x_k,\ k \in \mathbb{N}\} \text{ bounded and } \{t_k,\ k \in \mathbb{N}\} \text{ such that } t_k \to +\infty, \\ \{S(t_k)x_k,\ k \in \mathbb{N}\} \text{ is relatively compact in } E. \tag{2.11}$$

Ball proposed in [25] a general method in order to verify (2.11), based on energy functionals. Roughly speaking, this method is based on the simple observation that a *weakly* convergent sequence in a Hilbert (and, more generally, a reflexive Banach) space converges *strongly* if the corresponding sequence of norms converges to the norm of the limit function. Then, in order to verify (2.11), one first extracts a weakly convergent subsequence from $\{S(t_k)x_k\}$ by using the dissipativity and the fact that bounded subsets are precompact in the weak topology and then verifies the convergence of the norms by passing to the limit in the associated energy equality.

This method was applied with success to many equations, both in bounded and unbounded domains, see [25], [36], [65], [115], [118], [119], [120], [128], [129], [151], [183], [184], [185], [201], [202], [220], [229], [230], [231], and [232].

To conclude, we give a result related to global attractors for abstract classes of "bounded" sets (see [210]) which generalizes the concept of (E, E_1)-attractors (in the terminology of Babin and Vishik) and is very useful, e.g., in the theory of attractors in unbounded domains, for ill-posed dissipative systems, and for attractors in weak topologies.

Theorem 2.20. *Let E be a topological space and $S(t)$ be a semigroup acting on E. Assume also that a class of "bounded" subsets $\mathcal{B}$ of E satisfying* (2.8) *is given. Let finally $S(t)$ possess a "bounded", compact (in E), and metrizable absorbing set $B_0 \in \mathcal{B}$ and be continuous on B_0, for every fixed $t \geq 0$. Then there exists a compact and "bounded" global attractor $\mathcal{A} \subset B_0$ which is generated by all "bounded" complete trajectories of $S(t)$ in E.*

Remark 2.21. We refer the reader to [122], [138], [153], [189], and [247] for other existence results for the global attractor.

2.3. Attractors for semigroups having a global Lyapunov function.

Definition 2.22. Let X be a subset of E and $L : X \to \mathbb{R}$ be a continuous function. The function L is a global Lyapunov function for $S(t)$ on X if
(i) $\forall u_0 \in X$, the function $t \mapsto L(S(t)u_0)$ is decreasing (i.e., L is decreasing along the trajectories) ;
(ii) if $L(S(t)u_0) = L(u_0)$ for some $t > 0$, then u_0 is a fixed point of $S(t)$ (i.e., L is strictly decreasing along the trajectories which are not reduced to fixed points).

Let $\mathcal{N}$ be the set of all fixed points of $S(t)$,

$$\mathcal{N} := \{z \in E,\ S(t)z = z,\ \forall t \geq 0\}.$$

Let $z \in \mathcal{N}$. The unstable set $\mathcal{M}^{\mathrm{un}}(z)$ of z is the set of all points $u \in E$ such that $S(t)u$ is defined for all $t \leq 0$ and $\lim_{t\to-\infty} S(t)u = z$. Similarly, the stable set $\mathcal{M}^{\mathrm{s}}(z)$ of z is the set of all points $u \in E$ such that $\lim_{t\to+\infty} S(t)u = z$. More generally, let X be an invariant subset of E. Then the unstable set of X is the (possibly empty) set

$$\mathcal{M}^{\mathrm{un}}(X) := \{u_\star \in E,\ u_\star \text{ belongs to a complete trajectory } u(t),\ t \in \mathbb{R}, \\ \text{and } \lim_{t\to-\infty} \mathrm{dist}(u(t), X) = 0\}.$$

Similarly, the stable set of X is the (possibly empty) set

$$\mathcal{M}^{\rm s}(X) := \{u_\star \in E,\ u_\star \text{ belongs to a complete trajectory } u(t),\ t \in \mathbb{R}, \text{ and } \lim_{t\to+\infty}\operatorname{dist}(u(t),X) = 0\}.$$

Remark 2.23. We assume that $S(t)$ possesses the global attractor $\mathcal{A}$. We can note that $\mathcal{N} \subset \mathcal{A}$. Furthermore, it is not difficult to show that $\mathcal{M}^{\rm un}(z) \subset \mathcal{A}$, $\forall z \in \mathcal{N}$; we also note that $\mathcal{M}^{\rm un}(z)$ and $\mathcal{M}^{\rm s}(z)$ are invariant by $S(t)$. Finally, if X is an invariant set, then $\mathcal{M}^{\rm un}(X) \subset \mathcal{A}$, and $\mathcal{M}^{\rm un}(\mathcal{A}) = \mathcal{A}$.

We have the following result on the structure of the global attractor for a semigroup having a global Lyapunov function.

Theorem 2.24. *We assume that the semigroup $S(t)$ possesses a continuous global Lyapunov function. Then*

$$\mathcal{A} = \mathcal{M}^{\rm un}(\mathcal{N}).$$

If, furthermore, $\mathcal{N}$ is finite, $\mathcal{N} = \{z_1, ..., z_m\}$, and $t \mapsto S(t)x$ is continuous, $\forall x \in E$, then

$$\mathcal{A} = \cup_{i=1}^m \mathcal{M}^{\rm un}(z_i)$$

and every trajectory $u(t)$, $t \in \mathbb{R}$, lying on $\mathcal{A}$ satisfies

$$\lim_{t\to-\infty} u(t) = z_i,\ \ \lim_{t\to+\infty} u(t) = z_j,\ \ z_i \neq z_j,\ \ z_i,\ z_j \in \mathcal{N}.$$

Remark 2.25. We further assume that $S(t)$ is differentiable in E (to be more precise, $S(t) \in \mathcal{C}^{1+\delta}(E,E)$, $\delta > 0$), $\forall t \in \mathbb{R}_+$. A fixed point z is hyperbolic if the spectrum of $S'(t)z$ does not intersect the unit circle, $t > 0$. In that case, the unstable set of z, $\mathcal{M}^{\rm un}(z)$, is a k-dimensional submanifold of E, where k is the stability index of z (see [23] for more details). Therefore, if $\mathcal{N}$ is finite and all the fixed points are hyperbolic, the global attractor $\mathcal{A}$ of a semigroup having a continuous global Lyapunov function is a finite union of smooth finite dimensional submanifolds of the phase space. Such global attractors are called regular attractors by Babin and Vishik (see, e.g., [23]). They also possess several additional "good" properties and, to the best of our knowledge, it is the only general class of attractors for which a more or less complete description of their structure is available. In particular, regular attractors are automatically *exponential*, i.e., for every bounded subset $B \subset E$, the following estimate holds:

$$\operatorname{dist}(S(t)B,\mathcal{A}) \le Q(\|B\|_E)e^{-\alpha t},\ \ t \ge 0, \tag{2.12}$$

where the positive constant α and the monotonic function Q are independent of B. Furthermore, regular attractors are preserved under general sufficiently regular perturbations (the perturbed system may not have a Lyapunov function and may even be nonautonomous, see [23], [80], [117], [228], and the references therein). Finally, for one-dimensional scalar parabolic equations, it is even possible to find *explicitly* the so-called permutation matrix of the attractor (which shows whether or not two equilibria are connected by a heteroclinic trajectory) and, on some occasions, to describe the topological structure of the attractor in terms of the physical parameters of the problem, see [32], [94], and [95] for details. We however note that, although the finiteness of the set of fixed points and the hyperbolicity of these fixed points are, in some proper sense, generic properties, see [23], they are very difficult, if not impossible, to prove for concrete examples and given values of the physical parameters of the problem, except for scalar parabolic equations in one space dimension. Furthermore, even if the regularity of the attractor can be proven, one usually *cannot* compute explicitly the constant α and the function Q in the exponential attraction property (2.12) and these quantities can be extremely bad. Indeed, in the example

$$\partial_t u - \nu\partial_x^2 u + u^3 - u = 0,\ x \in [0,1],\ \nu > 0,\ u(0,t) = u(1,t) = -1,\ t \geq 0,$$

mentioned in the introduction, the global attractor $\mathcal{A} = \{-1\}$ is obviously regular and one can take $\alpha = 2$ in formula (2.12) (this is determined by the hyperbolicity constant of the equilibrium $u_0 = 1$). However, the function Q satisfies

$$Q(r) \geq e^{2e^{C\nu^{-1/2}}},\ r \geq 0.$$

Thus, even for a reasonably small ν, one will never "see" this regular attractor in numerical simulations. This phenomenon is related to the existence of *metastable* almost-equilibria with an extremely large lifetime in the phase space of this equation (it is also worth mentioning that they are situated far from the global attractor and have "nothing in common" with the properties of the global attractor). As we will see in the next section, this confusing drawback can be overcome by using the general concept of an *exponential* attractor, for which the constant α and the function Q can reasonably be found in terms of the physical parameters of the problem.

We conclude this subsection by the following result on the existence of the global attractor for a semigroup having a global Lyapunov function (see [61] ; see also [122] and [138]) which can be useful in applications (see, e.g., [61] and [177]).

Theorem 2.26. *We make the following assumptions:*
(i) $t \mapsto S(t)x$ is continuous, $\forall x \in E$;
(ii) $S(t)$ possesses a continuous global Lyapunov function L such that $L(x) \to +\infty$ if and only if $\|x\|_E \to +\infty$;
(iii) the set of fixed points of $S(t)$, $\mathcal{N}$, is bounded in E ;
(iv) $S(t)$ is asymptotically compact, i.e., $\forall B \subset E$ bounded,

$$\lim_{t\to+\infty} \kappa(S(t)B) = 0,$$

where κ is the Kuratowski measure of noncompactness.
Then $S(t)$ possesses the connected global attractor $\mathcal{A}$ such that $\mathcal{A} = \mathcal{M}^{\mathrm{un}}(\mathcal{N})$.

Remark 2.27. Theorem 2.26 can be useful, e.g., when the dissipative estimate (2.6) and the existence of a bounded absorbing set can be difficult to establish (see, e.g., [61] and [177]), although the existence of the global attractor implies the existence of a bounded absorbing set (it suffices to take any ϵ-neighborhood of the global attractor). Thus, the dissipativity can be obtained in an implicit way by using the Lyapunov function and the fact that the set of equilibria is bounded. Roughly speaking, the dissipativity is related to the fact that every trajectory converges to the set of equilibria (due to the Lyapunov function and the asymptotic compactness) and, since the set of equilibria is bounded, the energy of a "large" solution must decay (due to property (ii) of a Lyapunov function).

2.4. **Dimension of the global attractor.** As mentioned in the introduction, we will essentially consider the fractal (or box-counting) dimension here.

Definition 2.28. Let $X \subset E$ be a (relatively) compact set. For $\epsilon > 0$, let $N_\epsilon(X)$ be the minimal number of balls of radius ϵ which are necessary to cover X. Then the fractal dimension of X is the quantity

$$\dim_F X := \limsup_{\epsilon\to 0^+} \frac{\log_2 N_\epsilon(X)}{\log_2 \frac{1}{\epsilon}} \ (= \limsup_{\epsilon\to 0^+} \frac{\ln N_\epsilon(X)}{\ln \frac{1}{\epsilon}}) \tag{2.13}$$

(note that $\dim_F X \in [0, +\infty]$). Furthermore, the quantity $\mathcal{H}_\epsilon(X) := \log_2 N_\epsilon(X)$ is called the Kolmogorov ϵ-entropy of X.

The fractal dimension satisfies the following properties (see [87]):
- $\dim_F(X_1 \times X_2) \le \dim_F X_1 + \dim_F X_2$;
- if $f : X_1 \to X_2$ is Lipschitz, then $\dim_F X_2 \le \dim_F X_1$;
- if X is a smooth m-dimensional manifold, then $\dim_F X = m$.

It is important to note that, for sets which are not manifolds, the fractal dimension can be noninteger ; for instance, if X is the ternary Cantor set in $\mathbb{R}$, then

$$\dim_F X = \frac{\ln 2}{\ln 3} < 1$$

(see [87]). Furthermore, it follows from the definition that, if the minimal number of balls of radius ϵ which are necessary to cover X satisfies

$$N_\epsilon(X) \le c(\frac{1}{\epsilon})^d,$$

where c and d are independent of ϵ, then

$$\dim_F X \le d.$$

A strong interest, for considering the fractal dimension over other dimensions, is given by the (modified) Hölder-Mañé theorem (see [68], [97], and [125]). We start with the following definition.

Definition 2.29. [126] A Borel subset X of a Banach space E is prevalent if there exists a compactly supported probability measure μ such that $\mu(X+x) = 1$, $\forall x \in E$. A non-Borel set which contains a prevalent set is also prevalent.

Remark 2.30. Prevalence extends the notion of "Lebesgue almost every" from Euclidean spaces to infinite dimensional spaces (see [126] for a discussion on this subject).

Theorem 2.31. *(Modified Hölder-Mañé theorem,* [125]*) Let $X \subset E$ be compact and such that* $\dim_F X = d$ *and $N > 2d$ be an integer. Then almost every (in the sense of prevalence) bounded linear projector $P : E \to \mathbb{R}^N$ is one-to-one on X and has a Hölder continuous inverse.*

It follows from Theorem 2.31 that, if the global attractor has finite fractal dimension, then, fixing a projector P satisfying the assumptions of the theorem, the reduced dynamical system $(\overline{S}(t), \overline{\mathcal{A}})$, where $\overline{S}(t) := P \circ S(t) \circ P^{-1}$ and $\overline{\mathcal{A}} := P(\mathcal{A})$, is a finite dimensional dynamical system (i.e., in $\mathbb{R}^N$) which is Hölder continuous with respect to the initial data. This result, and the fractal dimension, thus play an important role in the finite dimensional reduction theory of infinite dimensional dynamical systems.

Remark 2.32. The Hausdorff dimension (see [87]) is also frequently used to measure the dimension of the global attractor (see, e.g., [23], [51], and [220]). However, Theorem 2.31 does not hold for the Hausdorff dimension.

The next result (see [237] ; see also [139]) gives a general method to prove the finite fractal dimensionality of a compact set.

Theorem 2.33. *Let X be a compact subset of E. We assume that there exist a Banach space E_1 such that E_1 is compactly embedded into E and a mapping $L : X \to X$ such that $L(X) = X$ and*

$$\|Lx_1 - Lx_2\|_{E_1} \leq c\|x_1 - x_2\|_E, \ \forall x_1, \ x_2 \in X. \tag{2.14}$$

Then the fractal dimension of X is finite and satisfies

$$\dim_F X \leq \mathcal{H}_{\frac{1}{4c}}(B_{E_1}(0,1)),$$

where c is the constant in (2.14) *and $B_{E_1}(0,1)$ is the unit ball in E_1 (note that it is relatively compact in E).*

In applications to parabolic systems in bounded domains, one usually proves that, for instance, (2.14) is satisfied for $L = S(1)$. Then, owing to the invariance property, one deduces from Theorem 2.33 that the global attractor has finite fractal dimension. We will come back to the "smoothing" property (2.14), and its generalizations (in particular, to damped hyperbolic systems), in the next section when discussing the construction of exponential attractors.

It is essential, in view of the finite dimensional reduction principle given by Theorem 2.31, to find sharp estimates on the dimension of the global attractor in terms of the physical parameters of the problem. In general, the best upper bounds are obtained by the so-called volume contraction method, which is based on the study of the evolution of infinitesimal k-dimensional volumes in the neighborhood of the attractor (see [23], [51], [200], and [220]) ; see however [70] for a sharp upper bound based on (2.14). One then proves that, if the dynamical system contracts the k-dimensional volumes, then the fractal dimension of $\mathcal{A}$ is less than k. This method requires some differentiability property of the semigroup $S(t)$.

Definition 2.34. A map $L : X \to X, \ X \subset E$, is uniformly quasidifferentiable on X if, for every $x \in X$, there exists a linear operator $L'(x)$ (called quasidifferential) such that

$$\|L(x+v) - L(x) - L'(x)v\|_E = o(\|v\|_E)$$

holds uniformly with respect to $x \in X, \ v \in X, \ x + v \in E$.

We now assume that E is a Hilbert space. We have the following result (see [43] ; see also [23], [51], and [220]).

Theorem 2.35. *We assume that X is an invariant subset of E and that $S(t)$ is uniformly quasidifferentiable on X, with $x \mapsto S'(t)x$ continuous, $\forall t \geq 0$, and that, for some $t_\star > 0$,*

$$\overline{\omega}_d(X) := \sup_{x \in X} \omega_d(S'(t_\star)x) < 1,$$

where, for a bounded linear operator $L : E \to E$,

$$\omega_d(L) := \sup_{B_d} \frac{\mathrm{Vol}_d(L(B_d))}{\mathrm{Vol}_d(B_d)},$$

Vol_d being the d-dimensional volume and the supremum being taken over all d-dimensional ellipsoids. Then

$$\dim_F X \leq d.$$

We can note that, when E is a Hilbert space, then, if E_d is a vector subspace of E of dimension d, a bounded linear operator L maps a d-dimensional ellipsoid $B_d \subset E_d$ onto the d-dimensional ellipsoid $L(B_d) \subset L(E_d)$. Furthermore, $\mathrm{Vol}_d(B_d)$ is well-defined. The quantity $\omega_d(L)$ measures the changes of d-dimensional volumes under the action of L.

Remark 2.36. Another powerful and useful method to prove the finite dimensionality of the global attractor is based on the so-called l-trajectories: one needs minimal regularity on the solutions in order to apply this method, see [33], [55], [56], [155], [156], [157], [176], [193], and [195]. In particular, this method allows to prove the finite dimensionality of the global attractor associated with generalized Navier-Stokes equations (see [137]) for which the smoothing property (2.14) and the quasidifferentiability are not known (see [155], [156], and [157]) ; the quasidifferentiability was however recently proven in [127] for some of these models in two space dimensions.

It is also essential to derive lower bounds on the dimension of the global attractor and to compare them with the known upper bounds. The derivation of lower bounds is based on the following observation: the global attractor always contains the unstable sets of equilibria. Thus, the stability index of a properly constructed (hyperbolic) equilibrium yields a lower bound on the dimension of the global attractor (see [23] for more details ; see also [147], [148], [174], [182], and [220] for examples).

2.5. **Robustness of the global attractor.** Very often, one needs to consider regular or singular perturbations of the system under study ; indeed, as mentioned in the introduction, a given system is only an approximation of reality. A natural question is how these perturbations will affect the asymptotic behavior of the system. One natural idea is to "compare" the global attractors of the perturbed and nonperturbed systems ; such results were first established in [121] for systems having a global Lyapunov function and then in [21] for general systems.

We thus consider a family of semigroups $\{S_\lambda(t),\ \lambda \in I\}$, $I \subset \mathbb{R}$ interval (more generally, I can be some topological space), acting on E such that $S_\lambda(t)$ possesses the global attractor $\mathcal{A}_\lambda$, $\forall \lambda \in I$.

Definition 2.37. (i) The attractors $\mathcal{A}_\lambda$ are upper semicontinuous at $\lambda_0 \in I$ if

$$\lim_{\lambda \in I \to \lambda_0} \mathrm{dist}(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) = 0.$$

(ii) The attractors $\mathcal{A}_\lambda$ are lower semicontinuous at $\lambda_0 \in I$ if

$$\lim_{\lambda \in I \to \lambda_0} \mathrm{dist}(\mathcal{A}_{\lambda_0}, \mathcal{A}_\lambda) = 0.$$

(iii) The attractors $\mathcal{A}_\lambda$ are continuous at $\lambda_0 \in I$ if they are both upper and lower semicontinuous at λ_0.

In general, global attractors are upper semicontinuous, i.e., we can prove the upper semicontinuity property under natural, and relatively easy to check in applications, conditions. We have, for instance, the following results (see [122]).

Theorem 2.38. *Let λ_0 belong to I. We assume that there exist $\delta > 0$, $t_0 > 0$, and a compact subset K of E such that*
(i) $\cup_{\lambda \in (\lambda_0-\delta, \lambda_0+\delta)\cap I} \mathcal{A}_\lambda \subset K$;
(ii) for every sequences $\{\lambda_k,\ k \in \mathbb{N}\}$ and $\{x_k,\ k \in \mathbb{N}\}$, $\lambda_k \in I$, $x_k \in \mathcal{A}_{\lambda_k}$, such that $\lambda_k \to \lambda_0$ and $x_k \to x_0$ as $k \to +\infty$, then

$$S_{\lambda_k}(t_0)x_k \to S_{\lambda_0}(t_0)x_0 \text{ as } k \to +\infty.$$

Then the attractors $\mathcal{A}_\lambda$ *are upper semicontinuous at* λ_0*.*

Theorem 2.39. *Let* λ_0 *belong to* I*. We make the following assumptions:*
(i) there exist $\delta > 0,\ t_0 > 0$*, and a bounded subset* B_0 *of* E *such that*

$$\cup_{\lambda\in(\lambda_0-\delta,\lambda_0+\delta)\cap I}\mathcal{A}_\lambda \subset B_0\ ;$$

(ii) $\forall\epsilon > 0,\ \forall t \geq t_0$*, there exists* $\theta = \theta(\epsilon, t),\ 0 < \theta < \delta$*, such that*

$$\|S_\lambda(t)x_\lambda - S_{\lambda_0}(t)x_\lambda\|_E \leq \epsilon,\ \forall x_\lambda \in \mathcal{A}_\lambda,\ \forall\lambda \in (\lambda_0 - \theta, \lambda_0 + \theta) \cap I.$$

Then the attractors $\mathcal{A}_\lambda$ *are upper semicontinuous at* λ_0*.*

Thus, roughly speaking, if the perturbation $S_\lambda(t)$ is continuous with respect to λ and the associated absorbing sets are uniformly bounded, the attractors $\mathcal{A}_\lambda$ are upper semicontinuous.

We also mention a simple theorem which follows in a straightforward way from the definition of upper semicontinuity and which, however, is especially useful for singular perturbations and, as a rule, gives the "simplest" way to establish the upper semicontinuity of attractors (see [23]).

Theorem 2.40. *Let the attractors* $\mathcal{A}_\lambda$ *possess the following property: for every sequences* $\{\lambda_k,\ k \in \mathbb{N}\}$ *and* $\{x_k,\ k \in \mathbb{N}\}$*,* $\lambda_k \in I,\ x_k \in \mathcal{A}_{\lambda_k}$*, such that* $\lambda_k \to \lambda_0 \in I$*, there exists a subsequence* x_{k_n} *which converges to some* $x_0 \in \mathcal{A}_{\lambda_0}$*. Then the attractors* $\mathcal{A}_\lambda$ *are upper semicontinuous at* λ_0*.*

In applications, the assumption of Theorem 2.40 is verified based on the fact that the global attractor is generated by bounded complete trajectories (see Theorem 2.13). Thus, there only remains to extract, from a sequence of complete trajectories $u_{\lambda_k} \in \mathcal{K}_{\lambda_k}$, a subsequence converging to some complete trajectory $u_{\lambda_0} \in \mathcal{K}_{\lambda_0}$ of the limit system. The advantage of this approach is that the semigroups $S_\lambda(t)$ (which, for singular perturbations, may have bad properties such as boundary layers, lack of regularity in finite time, ...) are not involved in the process and the result can be obtained by directly passing to the limit in the associated equations for u_{λ_k}, see [23] for details.

Remark 2.41. Although everything seems satisfactory as far as the upper semicontinuity of global attractors is concerned, the situation changes drastically if one is interested in *estimating* the distance between the perturbed and nonperturbed attractors in terms of the physical parameters of the problem. Indeed, this distance is naturally related to the rate of attraction to the limit attractor and, as already mentioned, this rate of attraction is, in general, impossible to find in terms of the physical parameters of the problem.

Now, the lower semicontinuity property is much more difficult to prove ; actually, as mentioned in the introduction, it may even not hold. We need, in order to prove this property, much more restrictive assumptions. For instance, we have the following result (see [23]).

Theorem 2.42. *Let* $\lambda_0 \in I$*, the attractors* $\mathcal{A}_\lambda$ *be uniformly bounded in* E*, i.e.,* $\mathcal{A}_\lambda \subset B_0$ *for every* $\lambda \in I$*, for some bounded subset* B_0 *of* E*, and the following uniform (with respect to* λ*) attraction property hold:*

$$\operatorname{dist}(S_\lambda(t)B_0, \mathcal{A}_\lambda) \leq \beta(t),\ t \geq t_0,\ \lambda \in I, \tag{2.15}$$

where $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ *is some monotonic function which tends to zero as* $t \to +\infty$. *Assume also that* S_λ *is continuous at* λ_0 *in the following sense: for every* $T \in \mathbb{R}_+$,

$$\sup_{t\in[0,T]}\sup_{x\in B_0} \|S_\lambda(t)x - S_{\lambda_0}(t)x\|_E \to 0 \text{ as } \lambda \to \lambda_0. \tag{2.16}$$

Then the attractors $\mathcal{A}_\lambda$ *are lower semicontinuous at* λ_0.

Remark 2.43. Under some natural additional assumptions, condition (2.15) of a uniform rate of attraction is *necessary and sufficient* to have the lower semicontinuity. However, it is completely unclear how to verify such a condition in applications (to the best of our knowledge, no general method to prove this uniform rate of attraction has been developed). An exception is again the case where the limit attractor $\mathcal{A}_{\lambda_0}$ possesses a global Lyapunov function and is *regular*, see Remark 2.25. Indeed, as already mentioned, regular attractors attract bounded subsets *exponentially*, see (2.12), and persist under sufficiently regular perturbations. Furthermore, the rate of attraction to the perturbed regular attractor $\mathcal{A}_\lambda$ remains exponential and uniform with respect to λ, for λ close to λ_0, i.e., (2.15) holds with $\beta(t) := Ce^{-\alpha t}$, $\alpha > 0$. This, in turn, gives the upper and lower semicontinuity, together with the estimate

$$\operatorname{dist}_{\text{sym}}(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) \le C|\lambda - \lambda_0|^\gamma, \tag{2.17}$$

where $\operatorname{dist}_{\text{sym}}$ denotes the symmetric Hausdorff distance between sets defined by

$$\operatorname{dist}_{\text{sym}}(A, B) := \max\left(\operatorname{dist}(A, B), \operatorname{dist}(B, A)\right)$$

and for some positive constants C and $0 < \gamma < 1$, see, e.g., [23] for details. In some cases, it is also possible to prove that the dynamical system considered is Morse-Smale, which means that the dynamics, restricted to the regular attractor, is also preserved, up to homeomorphisms, under perturbations (see [32], [122], and [198] for more details). Finally, for some one-dimensional scalar parabolic equations, the uniform rate of attraction is possible to establish even when the equilibria are nonhyperbolic (due to relatively simple and completely understood structures of degenerate equilibria, see [136]). However, as mentioned in Remark 2.25, even though regular attractors are, in some proper sense, generic, it is in general very difficult, if not impossible, to prove that the global attractor is regular for given values of the physical parameters of the problem. Furthermore, even when the regularity can be proven, it is also impossible, in general, to obtain *explicit* estimates on the rate of exponential attraction. So, the constants C and γ in (2.17) are also implicit.

Remark 2.44. We refer the reader to [123] for different approaches for the comparison of attractors under perturbations.

Remark 2.45. It follows from the above considerations that the existing perturbation theory of global attractors has a purely qualitative nature and no *quantitative* result (e.g., explicit estimates in terms of the physical parameters of the problem) is available in general. As we will see in the next section, this drawback can be overcome by using the so-called *exponential* attractors for which the analogues of estimates (2.12) and (2.17) hold without any assumption on the hyperbolicity of the equilibria and the existence of a Lyapunov function and all the constants can be computed explicitly.

3. Exponential attractors

3.1. **Inertial manifolds.** We established in Subsection 2.4 a finite dimensional reduction principle for infinite dimensional dynamical systems based on the finite fractal dimensionality of the global attractor, via the Hölder-Mañé theorem. However, even though it is very important, this finite dimensional reduction principle has essential drawbacks. Indeed, the reduced dynamical system $(\overline{S}(t), \overline{\mathcal{A}})$ given by the Hölder-Mañé theorem is only Hölder continuous and cannot thus be realized in a satisfactory way as a dynamical system generated by a system of ODEs, i.e., a system of ODEs which is well-posed. Furthermore, reasonable conditions on the global attractor which would guarantee that the Mañé projectors are Lipschitz are not known. A second drawback is that the complicated geometric structure of the attractors $\mathcal{A}$ and $\overline{\mathcal{A}}$ make the use of this finite dimensional reduction principle in computations hazardous: essentially, one only has a heuristic estimate on the number of unknowns which are necessary to capture all the dynamical effects in approximations.

It thus appears reasonable to embed the global attractor into a proper smooth finite dimensional manifold. The dynamics, reduced to this manifold, would then be realized as a (at least Lipschitz) system of ODEs which could be used in numerical simulations and would be a good approximation of the dynamics of the original system. This led Foias, Sell, and Temam to propose the notion of an inertial manifold in [98].

Definition 3.1. A Lipschitz finite dimensional manifold $\mathcal{M} \subset E$ is an inertial manifold for the semigroup $S(t)$ if
(i) it is positively invariant, i.e., $S(t)\mathcal{M} \subset \mathcal{M},\ \forall t \geq 0$;
(ii) it satisfies the following asymptotic completeness property:

$$\forall u_0 \in E,\ \exists v_0 \in \mathcal{M} \text{ such that } \|S(t)u_0 - S(t)v_0\|_E \leq Q(\|u_0\|_E)e^{-\alpha t},\ t \geq 0, \tag{3.1}$$

where the positive constant α and the monotonic function Q are independent of u_0.

It follows from this definition that an inertial manifold, if it exists, contains the global attractor and attracts the trajectories exponentially fast (and uniformly with respect to bounded sets of initial data).

Furthermore, the existence of such a set would confirm, in a perfect way, the heuristic conjecture on a finite dimensional reduction principle of infinite dimensional dissipative dynamical systems. Indeed, the dynamics, restricted to an inertial manifold, can be described by a system of ODEs which is Lipschitz continuous (and thus well-posed), called the inertial form of the system. Furthermore, the asymptotic completeness property gives, in a particularly strong form, the equivalence of the initial dynamical system $(S(t), E)$ with its inertial form $(S(t), \mathcal{M})$.

Remark 3.2. In turbulence, i.e., for the incompressible Navier-Stokes equations, the existence of an inertial manifold would also yield an exact interaction law between the small and the large structures of the flow (see, e.g., [96]).

Several methods have been proposed to construct inertial manifolds (by the Lyapunov-Perron method, by constructing converging sequences of approximate inertial manifolds, by the so-called graph-transform method, ...) ; we refer the interested reader to [60], [98], [200], [214], [220], and the references therein for more details.

However, all the known constructions of inertial manifolds make use of a restrictive condition, namely, the so-called spectral gap condition (see [98]), which requires arbitrarily large gaps in the spectrum of the linearization of the initial system (see [98] for more details). In general, this property can only be verified in one space dimension. Nevertheless, the existence of inertial manifolds has been proven for a large number of equations, essentially in one and two space dimensions ; we refer the reader to [60], [98], [200], [214], [220], and the numerous references therein. However, the existence of an inertial manifold is still an open problem for several physically important equations, such as the two-dimensional incompressible Navier-Stokes equations. Furthermore, nonexistence results have been proven for damped Sine-Gordon equations by Mora and Solà-Morales [186].

Remark 3.3. Notions of approximate inertial manifolds have been proposed when the existence of an (exact) inertial manifold is not known and, in particular, for the incompressible Navier-Stokes equations. We refer the reader to, e.g., [98], [99], [102], and [220] for more details.

3.2. **Construction of exponential attractors.** It follows from the previous subsection that it is not always possible to embed the global attractor into a proper smooth finite dimensional manifold. Nevertheless, and also in view of the possible defaults of the global attractor as discussed in the introduction, it can be useful to construct larger (not necessarily smooth) sets which contain the global attractor, are still finite dimensional, and attract the trajectories exponentially fast. This led Eden, Foias, Nicolaenko, and Temam to propose the notion of an exponential attractor (also sometimes called an inertial set) in [68].

Definition 3.4. A compact set $\mathcal{M} \subset E$ is an exponential attractor for $S(t)$ if
(i) it has finite fractal dimension, $\dim_F \mathcal{M} < +\infty$;
(ii) it is positively invariant, $S(t)\mathcal{M} \subset \mathcal{M},\ \forall t \geq 0$;
(iii) it attracts exponentially the bounded subsets of E in the following sense:

$$\forall B \subset E \text{ bounded},\ \operatorname{dist}(S(t)B, \mathcal{M}) \leq Q(\|B\|_E)e^{-\alpha t},\ t \geq 0,$$

where the positive constant α and the monotonic function Q are independent of B.

It follows from this definition that an exponential attractor, if it exists, contains the global attractor (actually, the existence of an exponential attractor $\mathcal{M}$ yields the existence of the global attractor $\mathcal{A} \subset \mathcal{M}$, since it is a compact attracting set, see Theorem 2.17 ; note that $S(t)$ is still assumed to satisfy the continuity assumption (2.4)).

Thus, an exponential attractor is still finite dimensional, like the global attractor (and one still has the finite dimensional reduction principle given by the Hölder-Mañé theorem) ; actually, proving the existence of an exponential attractor is also one way of proving that the global attractor has finite fractal dimension. Compared with an inertial manifold, an exponential attractor is not smooth in general, but one still has a uniform exponential control on the rate of attraction of the trajectories.

Now, the main drawback of exponential attractors is that the relaxation to positive invariance makes these objects nonunique ; actually, once we have the existence of an exponential attractor, we have the existence of a whole family of exponential attractors (see [68]). Therefore, the question of the best choice of an exponential attractor, if this makes sense, is a crucial one. One possibility, to overcome this drawback, is to find a "simple" algorithm which maps a semigroup $S(t)$ onto an exponential attractor $\mathcal{M}(S)$; by simple, we have in particular in mind the numerical realization of such an algorithm.

The first construction of exponential attractors, due to Eden, Foias, Nicolaenko, and Temam [68], was not constructible ; indeed, Zorn's lemma had to be used in order to construct exponential attractors. This construction consists in a way in constructing a "fractal expansion" of the global attractor $\mathcal{A}$. Very roughly speaking, one considers an iterative process in which one adds, at each step, a "cloud" of points around the global attractor. The difficulty is that, at each step, one needs to control the dimension of this new cloud of points around the global attractor, and also ensure that the new set remains positively invariant, without increasing its dimension. The key idea which allows to control the number of points added at each step is the so-called squeezing property which says, roughly speaking, that either the higher modes are dominated by the lower ones or that the flow is contracted exponentially: a mapping $S\ :\ X\to X$, where X is a compact subset of a Hilbert space E, enjoys the squeezing property on X if, for some $\delta\in(0,\frac{1}{4})$, there exists an orthogonal projector $P=P(\delta)$ with finite rank such that, for every $u,\ v\in X$, either

$$\|(I-P)(Su-Sv)\|_E\le\|P(Su-Sv)\|_E$$

or

$$\|Su-Sv\|_E\le\delta\|u-v\|_E.$$

We can note that this property makes an essential use of orthogonal projectors with finite rank, so that the corresponding construction is valid in Hilbert spaces only.

The construction of [68] essentially applies to semigroups which possess a compact absorbing set (although a construction valid for damped wave equations is also given in [68]). It was then improved by Babin and Nicolaenko (in the sense that one could also consider semigroups which possess a compact attracting set) in [19] (see also [69]). We have, based on the construction of [19], the following result (see [83], [84], and [175]).

Theorem 3.5. *Let E and E_1 be two Hilbert spaces such that E_1 is compactly embedded into E and $S(t)\ :\ X\to X$ be a semigroup acting on a closed subset X of E. We assume that*
(i) there exist orthogonal projectors $P_k\ :\ E\to E,\ k\in\mathbb{N}$, with finite rank such that

$$\|(I-P_k)y\|_E\le c(k)\|y\|_{E_1},\ \forall y\in E_1,\ c(k)\to 0\text{ as }k\to+\infty\ ;$$

(ii) $\forall x_1,\ x_2\in X,\ \forall t>0$,

$$\|S(t)x_1-S(t)x_2\|_{E_1}\le h(t)\|x_1-x_2\|_E,$$

where the function h is continuous ;
(iii) $(t,x)\mapsto S(t)x$ is Lipschitz on $[0,T]\times B,\ \forall T>0,\ \forall B\subset X$ bounded.
Then $S(t)$ possesses an exponential attractor $\mathcal{M}$ on X (i.e., $\mathcal{M}$ satisfies all the assertions of Definition 3.4 with E replaced by X).

Remark 3.6. a) Actually, (i) follows from the compact embedding $E_1\subset E$. Furthermore, it follows from (i) and (ii) that the squeezing property is satisfied for some $t_\star>0$.
b) Condition (ii) can be replaced by the more general condition

$$\forall x_1,\ x_2 \in X,\ \forall t \geq 0,\ S(t)x_1 - S(t)x_2 = S_1(t,x_1,x_2) + S_2(t,x_1,x_2),$$

where

$$\|S_1(t,x_1,x_2)\|_E \leq d(t)\|x_1 - x_2\|_E,\ d \text{ continuous},\ t \geq 0,\ d(t) \to 0 \text{ as } t \to +\infty,$$

and

$$\|S_2(t,x_1,x_2)\|_{E_1} \leq h(t)\|x_1 - x_2\|_E,\ t > 0,\ h \text{ continuous}.$$

This more general condition allows to construct exponential attractors for damped hyperbolic equations (see [85] and [101]).

c) One essential difficulty, when constructing exponential attractors for damped hyperbolic equations, is to prove that the exponential attractors attract the bounded subsets of the whole phase space, and not those starting from a subspace of the phase space only (typically, consisting of more regular functions), see [68]. This difficulty was overcome in [86] by proving the following transitivity property of the exponential attraction: let (E, d) be a metric space and $S(t)$ be a semigroup acting on E such that

$$d(S(t)x_1, S(t)x_2) \leq c_1 e^{\alpha_1 t} d(x_1,x_2),\ t \geq 0,\ x_1,\ x_2 \in E,$$

for some positive constants c_1 and α_1. We further assume that there exist three subsets M_1, M_2, and M_3 of E such that

$$\operatorname{dist}(S(t)M_1, M_2) \leq c_2 e^{-\alpha_2 t},\ t \geq 0,\ \alpha_2 > 0,$$

and

$$\operatorname{dist}(S(t)M_2, M_3) \leq c_3 e^{-\alpha_3 t},\ t \geq 0,\ \alpha_3 > 0.$$

Then

$$\operatorname{dist}(S(t)M_1, M_3) \leq c_4 e^{-\alpha_4 t},\ t \geq 0,$$

where $c_4 := c_1 c_2 + c_3$ and $\alpha_4 := \frac{\alpha_2\alpha_3}{\alpha_1+\alpha_2+\alpha_3}$.

We note that condition (ii) in Theorem 3.5 resembles the smoothing property (2.14) ; actually, in order to prove Theorem 3.5, one only needs to prove that $S(t_\star)$ satisfies (2.14) for a proper $t_\star$. Now, this smoothing property is sufficient in order to construct exponential attractors and one does not need the squeezing property (and, thus, one does not need orthogonal projectors with finite rank) ; therefore, exponential attractors can also be constructed in Banach spaces.

Let thus E and E_1 be two Banach spaces such that E_1 is compactly embedded into E and let X be a bounded subset of E. Let finally $S : X \to X$ be a (nonlinear) mapping. We then consider the discrete dynamical system (or semigroup) generated by S, i.e., we set

$$S(0) := \mathrm{Id},\ S(n) := S \circ ... \circ S\ (n \text{ times}),\ n \in \mathbb{N}.$$

It is easy to see that this family of operators satisfies (2.2)-(2.3), but for $t,\ s \in \mathbb{N}$. Then we say that $\mathcal{M} \subset X$ is an exponential attractor for this discrete semigroup on X if
(i) it is compact in E and has finite fractal dimension ;

(ii) it is positively invariant, i.e., $S\mathcal{M} \subset \mathcal{M}$;
(iii) $\operatorname{dist}(S(n)X, \mathcal{M}) \leq ce^{-\alpha n}$, $n \in \mathbb{N}$, where c and $\alpha > 0$ only depend on X.

We then have the

Theorem 3.7. [71] *We assume that the mapping S enjoys the smoothing property* (2.14) *on X, i.e.,*

$$\|Sx_1 - Sx_2\|_{E_1} \leq c\|x_1 - x_2\|_E, \ \forall x_1, \ x_2 \in E.$$

Then the discrete dynamical system generated by the iterations of S possesses an exponential attractor $\mathcal{M} \subset X$.

Let us now consider a continuous semigroup $S(t)$ acting on X, i.e.,

$$S(t) \ : \ X \to X, \ t \geq 0.$$

In order to construct an exponential attractor for $S(t)$ on X, we usually proceed as follows. We assume that $S(t_\star)$ satisfies the smoothing property (2.14) for some $t_\star > 0$. We then have, owing to Theorem 3.7, an exponential attractor $\mathcal{M}_\star$ for the discrete dynamical system generated by the mapping $S_\star := S(t_\star)$ and we set

$$\mathcal{M} := \cup_{t\in[0,t_\star]} S(t)\mathcal{M}_\star.$$

Finally, if $(t, x) \mapsto S(t)x$ is Lipschitz (or even Hölder) on $[0, t_\star] \times X$, we can prove that $\mathcal{M}$ is an exponential attractor for $S(t)$ on X (see [68]).

Remark 3.8. a) In applications to PDEs, it is in general not restrictive at all to consider a bounded invariant subset $X \subset E$ instead of the whole space E. Indeed, X usually is a positively invariant bounded absorbing set ; we note that, if $\mathcal{B}_0$ is a bounded absorbing set for $S(t)$, then $\mathcal{B}_1 := \overline{\cup_{t\geq t_0} S(t)\mathcal{B}_0}$, where t_0 is such that $t \geq t_0$ implies $S(t)\mathcal{B}_0 \subset \mathcal{B}_0$ and the closure is taken in E, is a positively invariant bounded absorbing set. Therefore, the exponential attractors still attract all the bounded subsets of E.

b) For applications to damped hyperbolic equations, we will need a weaker form of a smoothing property, and, more precisely, some asymptotically smoothing property (see Remark 3.6, b)). More precisely, the existence of an exponential attractor still holds if (2.14) is replaced by one of the following weaker conditions (see [71]):

$$\begin{aligned} &S = S_1 + S_2, \text{ where} \\ &\|S_1x_1 - S_1x_2\|_E \leq \alpha\|x_1 - x_2\|_E, \ \forall x_1, \ x_2 \in X, \ \alpha < \tfrac{1}{2}, \\ &\|S_2x_1 - S_2x_2\|_{E_1} \leq c\|x_1 - x_2\|_E, \ \forall x_1, \ x_2 \in X, \end{aligned} \tag{3.2}$$

or

$$\begin{aligned} &Sx_1 - Sx_2 = S_1(x_1, x_2) + S_2(x_1, x_2), \ \forall x_1, \ x_2 \in X, \text{ where} \\ &\|S_1(x_1, x_2)\|_E \leq \alpha\|x_1 - x_2\|_E, \ \alpha < \tfrac{1}{2}, \\ &\|S_2(x_1, x_2)\|_{E_1} \leq c\|x_1 - x_2\|_E. \end{aligned} \tag{3.3}$$

c) If E_1 and E_2 are Hilbert spaces, then we can prove that, if $\alpha < \frac{1}{8}$, (3.2) and (3.3) imply the squeezing property (see [70] ; see also Remark 3.6, a)).

d) Based on the above results, one has been able to prove the existence of exponential attractors in many situations, see [6], [7], [62], [63], [71], [73], [74], [77], [88], [103], [104], [105], [106], [108], [109], [110], [111], [112], [113], [114], [160], [178], [179], [180], [187], and [219]. Actually, exponential attractors are as general as global attractors: to the best

of our knowledge, exponential attractors exist indeed for all equations of mathematical physics for which we can prove the existence of the finite dimensional global attractor.
e) Another construction of exponential attractors in Banach spaces was proposed by Le Dung and Nicolaenko in [143]. This construction consists in adapting the original construction of [68] to a Banach setting. We can note that it is based on conditions which are contained in (and are more restrictive than) those given above. Furthermore, it is worth noting that the construction given in [71] is very simple, in particular, when compared to those of [68] and [143].
f) The method of l-trajectories is also very efficient to construct exponential attractors. In particular, this method allows to prove the smoothing property in a simple way. Furthermore, as already mentioned, it requires minimal regularity on the solutions. We refer the reader to [33], [81], [156], [176], [181], [194], [195], [206], and [209] for more details ; a necessary and sufficient condition on the existence of an exponential attractor is also given in [194].

3.3. **Robust families of exponential attractors.** As already mentioned in the introduction and Subsection 2.5, global attractors can be sensitive to perturbations ; more precisely, the lower semicontinuity property may not hold. Furthermore, even though this property is, in some proper sense, generic (see, e.g., [198]), it is in general very difficult, if not impossible, to prove it for given values of the physical parameters in applications. Similarly, regular attractors (see Remark 2.25) are robust (see [23]), and, in particular, lower semicontinuous, but, again, it is in general very difficult, if not impossible, to prove the existence of such sets for given values of the physical parameters.

It is also worth noting that inertial manifolds are robust under perturbations ; indeed, they are hyperbolic manifolds, see [203]. However, as mentioned in Subsection 3.1, the existence of such sets is not known for several important equations and may even not hold.

Now, since exponential attractors attract exponentially fast the trajectories, with a uniform control on the rate of attraction, it is reasonable to expect that these sets are robust under perturbations and that one should be able to construct robust families of exponential attractors, of course, up to the “best choice”, since they are not unique.

It is possible, based on the initial construction of [68], to construct families of exponential attractors which are upper and lower semicontinuous (see, e.g., [68], [85], and [101]). However, this continuity only holds up to some time shift, i.e., one has a result of the form

$$\lim_{\epsilon\to 0^+}\limsup_{t\to+\infty}\,[\operatorname{dist}(S_\epsilon(t)\mathcal{M}_\epsilon,\mathcal{M}_0)+\operatorname{dist}(S_0(t)\mathcal{M}_0,\mathcal{M}_\epsilon)]=0,$$

where $(S_\epsilon(t),\mathcal{M}_\epsilon)$ and $(S_0(t),\mathcal{M}_0)$ are the perturbed and nonperturbed dynamical systems, respectively, $\epsilon>0$ being the perturbation parameter. Consequently, we essentially have, as far as the lower semicontinuity is concerned,

$$\lim_{\epsilon\to 0^+}\operatorname{dist}(\mathcal{A}_0,\mathcal{M}_\epsilon)=0,$$

where $\mathcal{A}_0$ is the global attractor associated with the nonperturbed system, which is not satisfactory.

This result was improved in [73] (see also [77]) and one has the

Theorem 3.9. [73] *Let E and E_1 be two Banach spaces such that E_1 is compactly embedded into E and let X be a bounded subset of E. We assume that the family of operators $S_\epsilon : X \mapsto X$, $\epsilon \in [0, \epsilon_0]$, $\epsilon_0 > 0$, satisfies the following assumptions:*
(i) (Uniform, with respect to ϵ, smoothing property) $\forall \epsilon \in [0, \epsilon_0]$, $\forall x_1,\ x_2 \in X$,

$$\|S_\epsilon x_1 - S_\epsilon x_2\|_{E_1} \leq c_1 \|x_1 - x_2\|_E,$$

where c_1 is independent of ϵ.
(ii) (The trajectories of the perturbed system approach those of the nonperturbed one, uniformly with respect to ϵ, as ϵ tends to 0) $\forall \epsilon \in [0, \epsilon_0]$, $\forall i \in \mathbb{N}$, $\forall x \in X$,

$$\|S_\epsilon^i x - S_0^i x\|_E \leq c_2^i \epsilon,$$

where c_2 is independent of ϵ and, for a mapping L, $L^i := L \circ ... \circ L$ (i times).
Then, $\forall \epsilon \in [0, \epsilon_0]$, the discrete dynamical system generated by the iterations of S_ϵ possesses an exponential attractor $\mathcal{M}_\epsilon$ on X such that
1. the fractal dimension of $\mathcal{M}_\epsilon$ is bounded, uniformly with respect to ϵ,

$$\dim_F \mathcal{M}_\epsilon \leq c_3,$$

where c_3 is independent of ϵ ;
2. $\mathcal{M}_\epsilon$ attracts X, uniformly with respect to ϵ,

$$\mathrm{dist}(S_\epsilon^i X, \mathcal{M}_\epsilon) \leq c_4 e^{-c_5 i},\ c_5 > 0,\ i \in \mathbb{N},$$

where c_4 and c_5 are independent of ϵ ;
3. the family $\{\mathcal{M}_\epsilon,\ \epsilon \in [0, \epsilon_0]\}$ is continuous at 0,

$$\mathrm{dist}_{\mathrm{sym}}(\mathcal{M}_\epsilon, \mathcal{M}_0) \leq c_6 \epsilon^{c_7},$$

where c_6 and $c_7 \in (0, 1)$ are independent of ϵ and $\mathrm{dist}_{\mathrm{sym}}$ *denotes the symmetric Hausdorff distance between sets defined by*

$$\mathrm{dist}_{\mathrm{sym}}(A, B) := \max\ (\mathrm{dist}(A, B), \mathrm{dist}(B, A)).$$

Remark 3.10. a) The constants c_i, $i = 3,\ ...,\ 7$, can be computed explicitly in terms of the physical parameters of the problem in concrete situations. It is worth noting that this is not the case in general for the constants c_6 and c_7 in the estimate of the symmetric distance when such a result can be proven for global attractors, e.g., for regular attractors.
b) In [73], in order to construct this family of exponential attractors, one first constructs $\mathcal{M}_0$ and one then constructs $\mathcal{M}_\epsilon$, $\epsilon > 0$, based on $\mathcal{M}_0$. Therefore, $\mathcal{M}_\epsilon$ depends on S_ϵ, but also on S_0, and the continuity only holds at $\epsilon = 0$.
c) We also mention [7] for robustness results with respect to numerical approximations.

In applications to PDEs, Theorem 3.9 applies to parabolic systems (in bounded domains). In order to construct a robust family of exponential attractors $\mathcal{M}_\epsilon$ for the continuous semigroups $S_\epsilon(t)$, $\epsilon \in [0, \epsilon_0]$, associated with such systems, we usually first prove the existence of a uniform (with respect to ϵ) bounded absorbing set, i.e., a bounded subset $\mathcal{B}_0$ of E, independent of ϵ, such that, $\forall B \subset E$ bounded, $\exists T_0$ independent of ϵ such that

$$t \geq T_0 \text{ implies } S_\epsilon(t) B \subset \mathcal{B}_0,\ \forall \epsilon \in [0, \epsilon_0].$$

We then consider the discrete mappings $S_\epsilon^{T_0} := S_\epsilon(T_0)$, $\forall \epsilon \in [0, \epsilon_0]$ (possibly for a larger, but still independent of ϵ, T_0). We thus have $S_\epsilon^{T_0} : \mathcal{B}_0 \to \mathcal{B}_0$, $\forall \epsilon \in [0, \epsilon_0]$, and we then prove that the $S_\epsilon^{T_0}$, $\epsilon \in [0, \epsilon_0]$, satisfy the assumptions of Theorem 3.9, which yields the existence of a robust family of discrete exponential attractors $\mathcal{M}_\epsilon^{T_0}$, $\epsilon \in [0, \epsilon_0]$. Finally, we set

$$\mathcal{M}_\epsilon := \cup_{t \in [0,T_0]} S_\epsilon(t) \mathcal{M}_\epsilon^{T_0}.$$

Then, if $(t, x) \mapsto S_\epsilon(t)x$ is Lipschitz, or even Hölder, on $[0, T_0] \times \mathcal{B}_0$, the exponential attractors $\mathcal{M}_\epsilon$, $\epsilon \in [0, \epsilon_0]$, satisfy

• $\dim_F \mathcal{M}_\epsilon \le c_1'$, $\epsilon \in [0, \epsilon_0]$;

• $\forall B \subset E$ bounded,

$$\mathrm{dist}(S_\epsilon(t)B, \mathcal{M}_\epsilon) \le c_2' e^{-c_3' t}, \ t \ge 0, \ \epsilon \in [0, \epsilon_0], \ c_3' > 0 \ ;$$

• $\mathrm{dist}_{\mathrm{sym}}(\mathcal{M}_\epsilon, \mathcal{M}_0) \le c_4' \epsilon^{c_5'}$, $\epsilon \in [0, \epsilon_0]$, $c_5' \in (0, 1)$;

where the constants c_i', $i = 1, \ldots, 5$, are independent of ϵ and can be computed explicitly in terms of the physical parameters of the problem.

For damped hyperbolic equations, we should replace (2.14) by some asymptotically smoothing property (see Remark 3.8, b)). More generally, we have, for singularly perturbed problems, the following result, proven in [86] (see also [107] for a reformulation of this result).

Theorem 3.11. *We consider two families of Banach spaces $E(\epsilon)$ and $E_1(\epsilon)$, $\epsilon \in [0, \epsilon_0]$ (which are embedded into a larger topological space V), such that, $\forall \epsilon \in [0, \epsilon_0]$, $E_1(\epsilon)$ is compactly embedded into $E(\epsilon)$. We further assume that these compact embeddings are uniform with respect to ϵ in the sense that*

$$\mathcal{H}_\delta(B_{E_1(\epsilon)}(0, 1), E(\epsilon)) \le c_1(\delta), \ \forall \delta > 0,$$

where $\mathcal{H}_\delta(., E(\epsilon))$ denotes the δ-Kolmogorov entropy in the topology of $E(\epsilon)$ and c_1 is a monotonic function which is independent of ϵ. We then consider a family of closed sets $B_\epsilon \subset E(\epsilon)$, with B_0 bounded in $E(0)$, and a family of maps $S_\epsilon : B_\epsilon \to B_\epsilon$, $\epsilon \in [0, \epsilon_0]$, such that

(i) $\forall \epsilon \in [0, \epsilon_0]$, $B_0 \subset E(\epsilon)$ and

$$\|b_0\|_{E(\epsilon)} \le c_2 \|b_0\|_{E(0)} + c_3 \epsilon, \ \forall b_0 \in B_0,$$

where c_2 and c_3 are independent of ϵ ;

(ii) $\forall \epsilon \in [0, \epsilon_0]$, $S_\epsilon = \mathcal{C}_\epsilon + \mathcal{K}_\epsilon$, where $\mathcal{C}_\epsilon$ and $\mathcal{K}_\epsilon$ map B_ϵ into $E(\epsilon)$ and, $\forall b_\epsilon^1$, $b_\epsilon^2 \in B_\epsilon$,

$$\|\mathcal{C}_\epsilon b_\epsilon^1 - \mathcal{C}_\epsilon b_\epsilon^2\|_{E(\epsilon)} \le c_4 \|b_\epsilon^1 - b_\epsilon^2\|_{E(\epsilon)},$$
$$\|\mathcal{K}_\epsilon b_\epsilon^1 - \mathcal{K}_\epsilon b_\epsilon^2\|_{E_1(\epsilon)} \le c_5 \|b_\epsilon^1 - b_\epsilon^2\|_{E(\epsilon)},$$

where $c_4 < \frac{1}{2}$ and c_5 are independent of ϵ ;

(iii) there exist nonlinear "projectors" $\Pi_\epsilon : B_\epsilon \to B_0$, $\epsilon \in [0, \epsilon_0]$, such that $\Pi_\epsilon B_\epsilon = B_0$ and

$$\|S_\epsilon b_\epsilon - S_0^k \Pi_\epsilon b_\epsilon\|_{E(\epsilon)} \le c_6 c_7^k \epsilon, \ \epsilon \in [0, \epsilon_0],$$

where c_6 and c_7 are independent of ϵ.

Then there exists a family of exponential attractors $\mathcal{M}_\epsilon \subset B_\epsilon$ *for the dynamical systems generated by the maps* S_ϵ, $\epsilon \in [0, \epsilon_0]$, *such that*
1. $\dim_F \mathcal{M}_\epsilon \le c_8$, $\epsilon \in [0, \epsilon_0]$ *;*
2. $\text{dist}_{E(\epsilon)}(S_\epsilon^k B_\epsilon, \mathcal{M}_\epsilon) \le c_9 e^{-c_{10}k}$, $\epsilon \in [0, \epsilon_0]$, $k \in \mathbb{N}$, $c_{10} > 0$ *;*
3. $\text{dist}_{\text{sym}}(\mathcal{M}_\epsilon, \mathcal{M}_0) \le c_{11}\epsilon^{c_{12}}$, $\epsilon \in [0, \epsilon_0]$, $c_{12} \in (0, 1)$ *;*
where the constants c_i, $i = 8$, ..., 12, *are independent of* ϵ *and can be computed explicitly.*

Remark 3.12. a) In order to construct a robust family of exponential attractors for continuous semigroups $S_\epsilon(t)$, $\epsilon \in [0, \epsilon_0]$, we essentially proceed as indicated above (see, e.g., [86]).
b) Condition (ii) in Theorem 3.11 can be replaced by the more general condition

$$S_\epsilon b_\epsilon^1 - S_\epsilon b_\epsilon^2 = \mathcal{C}_\epsilon(b_\epsilon^1, b_\epsilon^2) + \mathcal{K}_\epsilon(b_\epsilon^1, b_\epsilon^2),$$

where

$$\|\mathcal{C}_\epsilon(b_\epsilon^1, b_\epsilon^2)\|_{E(\epsilon)} \le c_4 \|b_\epsilon^1 - b_\epsilon^2\|_{E(\epsilon)}, \ c_4 < \tfrac{1}{2},$$
$$\|\mathcal{K}_\epsilon(b_\epsilon^1, b_\epsilon^2)\|_{E_1(\epsilon)} \le c_5 \|b_\epsilon^1 - b_\epsilon^2\|_{E(\epsilon)},$$

$\forall \epsilon \in [0, \epsilon_0]$, $\forall b_\epsilon^1$, $b_\epsilon^2 \in B_\epsilon$.
c) We refer the reader to [62], [63], [73], [77], [86], [104], [105], [106], [107], [108], [111], [112], [113], [114], [178], [179], and [180] for applications of Theorems 3.9 and 3.11 (or generalizations).
d) As in [73], the exponential attractors $\mathcal{M}_\epsilon$, $\epsilon > 0$, constructed in [86] depend both on S_ϵ and S_0. These constructions were improved in [76], where the following result was proven (we will come back to this construction, and its generalizations, in the next section when discussing nonautonomous systems). Let E and E_1 be two Banach spaces such that E_1 is compactly embedded into E. We then consider a mapping S which satisfies the following conditions:
• it maps the δ-neighborhood (for the topology of E) $\mathcal{O}_\delta(B)$ of a bounded subset B of E into B, for a proper constant $\delta > 0$;
• $\forall x_1$, $x_2 \in \mathcal{O}_\delta(B)$, one has the smoothing property (2.14),

$$\|Sx_1 - Sx_2\|_{E_1} \le K\|x_1 - x_2\|_E.$$

Then the discrete dynamical system generated by the iterations of S possesses an exponential attractor $\mathcal{M}(S) \subset B$ such that
• it is compact in E_1 and

$$\dim_F \mathcal{M}(S) \le c_1 \ ;$$

• $\text{dist}_{E_1}(S^k B, \mathcal{M}(S)) \le c_2 e^{-c_3 k}$, $k \in \mathbb{N}$, $c_3 > 0$;
• the map $S \mapsto \mathcal{M}(S)$ is Hölder continuous in the following sense: $\forall S_1$, S_2 satisfying the above conditions (for the same constants δ and K),

$$\text{dist}_{\text{sym},E_1}(\mathcal{M}(S_1), \mathcal{M}(S_2)) \le c_4 \|S_1 - S_2\|^{c_5}, \ c_5 > 0,$$

where

$$\|S\| := \sup_{h \in \mathcal{O}_\delta(B)} \|Sh\|_{E_1}.$$

Furthermore, all the constants c_i, $i = 1$, ..., 5, only depend on B, E, E_1, δ, and K (in particular, they are independent of the concrete choice of S) and can be computed explicitly.

We thus now have a mapping $S \mapsto \mathcal{M}(S)$ and, owing to the Hölder continuity of this mapping, we can now construct robust families of exponential attractors which are continuous at every point, and not just at $\epsilon = 0$ as in the previous constructions.

4. Nonautonomous systems

We now consider a system of the form

$$\frac{\partial u}{\partial t} = F(t, u), \ u|_{t=\tau} = u_\tau, \ \tau \in \mathbb{R},$$

in a Banach space E, i.e., we now assume that the time appears explicitly in the equations (e.g., in the forcing terms). Assuming that the problem is well-posed, we have the process $U(t, \tau), \ t \geq \tau, \ \tau \in \mathbb{R}$, acting on E,

$$\begin{aligned} U(t,\tau) \ : \ & E \to E \\ & u_\tau \mapsto u(t), \end{aligned}$$

which maps the initial datum at time τ onto the solution at time t.

For such a system, both the initial and final times are important, i.e., the trajectories are no longer (positively) invariant by time translations. Thus, the notion of a global attractor is no longer adequate and has to be adapted.

4.1. **Uniform attractors.**

We consider in this subsection an approach initiated by Haraux [124] and further developed by Chepyzhov and Vishik [47] and [51].

We rewrite the equations in the form

$$\frac{\partial u}{\partial t} = F_{\sigma_0(t)}(u), \tag{4.1}$$

where $\sigma_0(t)$ consists of all the time dependent terms of the equations and is called the symbol of the system. For instance, if $F(t, u) = \tilde{F}(u) + f(t)$, then $\sigma_0(t) := f(t)$.

The idea in the approach described here is to actually consider, together with (4.1), a whole family of equations. To do so, we assume that σ_0 belongs to some complete metric space Θ (e.g., $\Theta := \mathcal{C}_b(\mathbb{R}, M)$, where M is a complete metric space and $\mathcal{C}_b$ denotes the bounded continuous functions). We then consider the translations group $T(h), \ h \in \mathbb{R}$, defined by

$$T(h)f(s) := f(s + h), \ s, \ h \in \mathbb{R},$$

and we assume that $T(h)\Theta \subset \Theta$ and $T(h)$ is continuous on Θ, $\forall h \in \mathbb{R}$. We finally define the hull of σ_0 as the set

$$\mathcal{H}(\sigma_0) := \overline{\{T(h)\sigma_0, \ h \in \mathbb{R}\}},$$

where the closure is taken in Θ. We say that $\mathcal{H}(\sigma_0)$ is the symbol space and, for simplicity, we denote it as Σ (see also Remark 4.1 below). It is not difficult to see that Σ is invariant by the translations group, i.e.,

$$T(h)\Sigma = \Sigma,\ \forall h \in \mathbb{R}. \tag{4.2}$$

Remark 4.1. More generally, we can take any subset of Θ which is invariant by the translations group as symbol space Σ ; we will however restrict ourselves to the above symbol space in this subsection.

Now, together with equation (4.1), we consider the whole family of equations

$$\frac{\partial u}{\partial t} = F_{\sigma(t)}(u),\ \sigma \in \Sigma. \tag{4.3}$$

Assuming that (4.3) is well-posed, $\forall \sigma \in \Sigma$, we have the family of processes $\{U_\sigma(t,\tau),\ t \geq \tau,\ \tau \in \mathbb{R},\ \sigma \in \Sigma\}$ acting on E.

Definition 4.2. A set $\mathcal{A}_\Sigma \subset E$ is a uniform (with respect to σ) attractor for the family of processes $\{U_\sigma(t,\tau), t \geq \tau,\ \tau \in \mathbb{R},\ \sigma \in \Sigma\}$ if
(i) it is compact in E ;
(ii) it attracts the bounded subsets of E, uniformly with respect to σ, i.e.,

$$\forall B \subset E \text{ bounded},\ \lim_{t \to +\infty} \sup_{\sigma \in \Sigma} \operatorname{dist}(U_\sigma(t,\tau)B, \mathcal{A}_\Sigma) = 0\ ;$$

(iii) it is minimal among the closed sets which enjoy the attraction property (ii).

Remark 4.3. In general, the uniform attractor is not invariant (we say that $X \subset E$ is invariant if $U_\sigma(t,\tau)X = X,\ \forall t \geq \tau,\ \tau \in \mathbb{R},\ \forall \sigma \in \Sigma$) and, in some sense, the invariance is replaced by the minimality property (iii) ; in particular, it follows from (ii) and (iii) that the uniform attractor, if it exists, is unique.

We then have the following result which is the analogue of Theorem 2.17, see [47] and [51].

Theorem 4.4. *We assume that the family of processes* $\{U_\sigma(t,\tau),\ t \geq \tau,\ \tau \in \mathbb{R},\ \sigma \in \Sigma\}$ *possesses a compact uniformly (with respect to* σ*) attracting set, i.e., a compact subset* K *of* E *such that*

$$\forall B \subset E \text{ bounded},\ \lim_{t \to +\infty} \sup_{\sigma \in \Sigma} \operatorname{dist}(U_\sigma(t,\tau)B, K) = 0.$$

Then it possesses the uniform attractor $\mathcal{A}_\Sigma$.

Remark 4.5. a) It is easy to extend the other notions and definitions given for semigroups, e.g., bounded absorbing sets, to families of processes (see [47] and [51]).
b) Theorem 4.4 does not require any continuity assumption on the processes, contrary to Theorem 2.17. This is due to the fact that the invariance property is replaced by the minimality property.

In applications, we need further assumptions on the symbol space in order to prove the existence of the uniform attractor, and we assume from now on that the initial symbol σ_0 is translation compact, i.e., that Σ is compact in Θ (see however [150], [152], and [154] in which the translation compactness is relaxed ; more precisely, one considers classes of time dependences which are translation bounded (i.e., Σ is bounded), but not translation compact).

A first example of translation compact symbols is given by quasiperiodic symbols. More precisely, σ_0 is quasiperiodic (with values in a metric space M) if it can be written in the form

$$\sigma_0(s) = \varphi(\alpha s),\ \alpha = (\alpha_1, ..., \alpha_k),\ k \in \mathbb{N},$$

where φ is 2π-periodic in each argument and $\alpha_1,\ ...,\ \alpha_k$ are rationally independent (for $k = 1$, the symbol is periodic). We further assume that $\varphi \in \mathcal{C}(\mathbb{T}^k, M)$, where $\mathbb{T}^k$ is the k-dimensional torus. Then the hull of σ_0 in $\mathcal{C}_b(\mathbb{R}, M)$ coincides with $\{\varphi(\alpha s+\omega),\ \omega \in \mathbb{T}^k\}$. Actually, in that case, we take the torus $\mathbb{T}^k$ as symbol space ; we can note that the mapping $\omega \mapsto \varphi(\alpha s + \omega)$ is continuous, but not necessarily one-to-one. Furthermore, the translations group $T(h),\ h \in \mathbb{R}$, acts on $\mathbb{T}^k$ by the relation

$$T(h)\omega = h(1, ..., 1) + \omega \ (\mathrm{mod}\ \mathbb{T}^k),\ \omega \in \mathbb{T}^k,\ h \in \mathbb{R}.$$

Other examples of translation compact symbols are given by almost periodic (in Bochner-Amerio sense) symbols in $\mathcal{C}_b(\mathbb{R}, M)$ (see [47] and [51] for more details and other examples of translation compact symbols).

One interesting feature of nonautonomous systems with translation compact symbols is that we can reduce the construction of the uniform attractor to that of the global attractor for a semigroup acting on a proper extended phase space ; this also yields further properties on the uniform attractor.

Noting that, owing to the well-posedness,

$$U_{T(h)\sigma}(t, \tau) = U_\sigma(t + h, \tau + h),\ \forall t \geq \tau,\ \tau \in \mathbb{R},\ \forall \sigma \in \Sigma,\ \forall h \in \mathbb{R},$$

it is not difficult to show that the family of operators

$$\begin{aligned} S(t)\ :\ & E \times \Sigma \to E \times \Sigma \\ & (u, \sigma) \mapsto (U_\sigma(t, 0)u, T(t)\sigma), \end{aligned} \tag{4.4}$$

$t \geq 0$, forms a semigroup on $E \times \Sigma$.

We further assume that, $\forall t \geq \tau,\ \tau \in \mathbb{R}$,

$$(u, \sigma) \mapsto U_\sigma(t, \tau)u \text{ is continuous from } E \times \Sigma \text{ into } E$$

(we say that the family of processes $\{U_\sigma(t, \tau),\ t \geq \tau,\ \tau \in \mathbb{R},\ \sigma \in \Sigma\}$ is $(E \times \Sigma, E)$-continuous). Then the semigroup $S(t)$ satisfies the continuity property (2.4) on $E \times \Sigma$.

We can now use the results of Subsection 2.2 to construct the global attractor $\mathcal{A}$ for $S(t)$ on the extended phase space $E \times \Sigma$. In particular, if the family of processes $\{U_\sigma(t, \tau),\ t \geq \tau,\ \tau \in \mathbb{R},\ \sigma \in \Sigma\}$ possesses a compact uniformly attracting set, then $S(t)$ possesses a compact attracting set (note that Σ is compact) and we have the following result.

Theorem 4.6. *We assume that the family of processes* $\{U_\sigma(t, \tau),\ t \geq \tau,\ \tau \in \mathbb{R},\ \sigma \in \Sigma\}$ *is* $(E \times \Sigma)$*-continuous and possesses a compact uniformly attracting set. Then the semigroup* $S(t)$ *defined in* (4.4) *possesses the connected global attractor* $\mathcal{A}$*. Furthermore, if* Π_1 *(resp.,* Π_2*) denotes the projector onto* E *(resp.,* Σ*), then*

$$\mathcal{A}_\Sigma = \Pi_1 \mathcal{A}$$

is the uniform attractor for the family of processes $\{U_\sigma(t, \tau),\ t \geq \tau,\ \tau \in \mathbb{R},\ \sigma \in \Sigma\}$ *and*

$$\Pi_2\mathcal{A} = \Sigma.$$

Remark 4.7. It follows from Theorem 4.6 that, under the assumptions of this theorem, the uniform attractor $\mathcal{A}_\Sigma$ is connected.

We say that $u(s),\ s \in \mathbb{R}$, is a complete trajectory for the process $U(t,\tau),\ t \geq \tau,\ \tau \in \mathbb{R}$, acting on E if

$$U(t,\tau)u(\tau) = u(t),\ \forall t \geq \tau,\ \tau \in \mathbb{R}$$

(as in Subsection 2.1, we can also define the forward and backward trajectories) and we define the kernel of this process as the set

$$\mathcal{K} := \{u\ :\ \mathbb{R} \to E,\ u \text{ is a complete trajectory of the process } U(t,\tau),\ \sup_{t\in\mathbb{R}}\ \|u(t)\|_E < +\infty\}.$$

We then have the

Theorem 4.8. *Under the assumptions of Theorem 4.6, the global attractor $\mathcal{A}$ associated with the semigroup $S(t)$ defined by* (4.4) *satisfies*

$$\mathcal{A} = \cup_{\sigma\in\Sigma}\mathcal{K}_\sigma(0) \times \{\sigma\},$$

where $\mathcal{K}_\sigma$ is the kernel of the process $U_\sigma(t,\tau)$. Furthermore, the uniform attractor $\mathcal{A}_\Sigma = \Pi_1\mathcal{A}$ satisfies

$$\mathcal{A}_\Sigma = \cup_{\sigma\in\Sigma}\mathcal{K}_\sigma(0).$$

Remark 4.9. a) It follows from the invariance of $\mathcal{A}$ that

$$\mathcal{A}_\Sigma = \cup_{\sigma\in\Sigma}\mathcal{K}_\sigma(s),\ \forall s \in \mathbb{R}.$$

b) It follows from the above results that, under the assumptions of Theorem 4.6, the process $U_\sigma(t,\tau)$ possesses at least one bounded complete trajectory, $\forall \sigma \in \Sigma$.

Remark 4.10. It is also possible to construct, for the initial process $U_{\sigma_0}(t,\tau)$, the uniform, now with respect to $\tau \in \mathbb{R}$, attractor. We refer the reader to [51] for more details and conditions which ensure that this attractor coincides with $\mathcal{A}_\Sigma$.

An important issue is whether the uniform attractor $\mathcal{A}_\Sigma$ has finite (fractal) dimension. A natural way to prove such a result would be to prove that the global attractor $\mathcal{A}$ for the semigroup $S(t)$ defined by (4.4) has finite (fractal) dimension. Then, since the projector Π_1 is Lipschitz, we would infer that $\mathcal{A}_\Sigma$ has also finite dimension. Unfortunately, as mentioned in the introduction, the dynamics of $S(t)$ is much more complicated than that of the initial system in general and $\mathcal{A}$ has infinite dimension in general ; we also saw that the uniform attractor can already be infinite dimensional for simple linear equations.

Thus, in general, the uniform attractor does not yield a finite dimensional reduction principle. Essentially, we are only able to prove the finite dimensionality of the uniform attractor for quasiperiodic processes (see [51] ; see however [215] for a finite dimensional result for asymptotically periodic processes).

Remark 4.11. A direct way to study the dimension of $\mathcal{A}_\Sigma$ consists in computing its Kolmogorov ϵ-entropy (see Definition 2.28 ; see also [51] for details). In particular, if the Kolmogorov ϵ-entropy of $\mathcal{A}_\Sigma$ satisfies

$$\mathcal{H}_\epsilon(\mathcal{A}_\Sigma) \le d \log_2 \frac{1}{\epsilon} + c,$$

where c and d are independent of ϵ, then

$$\dim_F \mathcal{A}_\Sigma \le d.$$

The use of the Kolmogorov entropy allows in particular to obtain sharp bounds on the dimension of $\mathcal{A}_\Sigma$ for quasiperiodic processes, see [51].

4.2. **Pullback attractors.** We saw in the previous subsection that the uniform attractor does not yield a satisfactory finite dimensional reduction principle in general, i.e., for a general translation compact process. Furthermore, even though the time appears explicitly in the equations, the uniform attractor is time independent.

In this subsection, we introduce a second notion of a nonautonomous attractor, now time dependent.

We consider a process $U(t,\tau),\ t \ge \tau,\ \tau \in \mathbb{R}$, acting on a Banach space E,

$$U(t,\tau)\ :\ E \to E,\ t \ge \tau,\ \tau \in \mathbb{R},$$

and we assume that

$$U(t,\tau) \text{ is continuous on } E,\ \forall t \ge \tau,\ \tau \in \mathbb{R}.$$

Definition 4.12. A family $\{\mathcal{A}(t),\ t \in \mathbb{R}\}$ is a pullback attractor for the process $U(t,\tau)$ if

(ii) $\mathcal{A}(t)$ is compact in E, $\forall t \in \mathbb{R}$;

(ii) it is invariant in the following sense:

$$U(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t),\ \forall t \ge \tau,\ \forall \tau \in \mathbb{R}\ ;$$

(iii) it satisfies the following attraction property, called pullback attraction:

$$\forall B \subset E \text{ bounded},\ \forall t \in \mathbb{R},\ \lim_{s \to +\infty} \operatorname{dist}(U(t,t-s)B, \mathcal{A}(t)) = 0. \tag{4.5}$$

Remark 4.13. The pullback attraction (4.5) essentially means that, at time t, the set $\mathcal{A}(t)$ attracts the bounded sets of initial data coming from $-\infty$.

Remark 4.14. a) We can note that Definition 4.12 is too general to have the uniqueness of a pullback attractor, if it exists. Indeed, let us consider the following simple dissipative *autonomous* ODE:

$$y' + y = 0.$$

Then it possesses the global attractor $\mathcal{A} = \{0\}$. However, *any* trajectory $y(t) = Ce^{-t},\ t \in \mathbb{R}$, generates a pullback attractor (e.g., $\mathcal{A}(t) = \{0, Ce^{-t}\},\ t \in \mathbb{R}$)! Thus, the uniqueness of a pullback attractor fails and additional conditions must be added in order to restore such a property (see [38] and [42]). For instance, the uniqueness holds if one has some "backward boundedness", e.g.,

$$\sup_{s \in \mathbb{R}_+} \|\mathcal{A}(t-s)\|_E (:= \sup_{s \in \mathbb{R}_+} \sup_{x \in \mathcal{A}(t-s)} \|x\|_E) \le C_t,\ t \in \mathbb{R}. \tag{4.6}$$

b) If the system is autonomous and we further assume that (4.6) holds, then we recover the global attractor. Indeed, in that case, we can write $U(t,\tau) = S(t-\tau)$, where $S(t)$ is a semigroup, and we have, in the pullback attraction property, $U(t, t-s) = S(s)$.

Remark 4.15. The above definition of a pullback attractor resembles that of the so-called kernel sections introduced by Chepyzhov and Vishik, see [46], [47], and [51]. Actually, in order to prove that these two objects are equivalent, i.e.,

$$\mathcal{A}(t) = \mathcal{K}(t) := \{u(t),\ u \in \mathcal{K}\},\ t \in \mathbb{R}, \tag{4.7}$$

where $\mathcal{K}$ is the kernel (i.e., the set of all bounded complete trajectories of the associated process), we need to make further assumptions. In particular, this equivalence holds if one has the backward boundedness (4.6), together with some forward dissipativity (e.g., the existence of a (forward) bounded uniformly absorbing set for the process). Furthermore, as proven, e.g., in [47] (see also [51]), the kernel sections (i.e., the pullback attractor here) have finite fractal dimension in E,

$$\dim_F \mathcal{A}(t) < +\infty,\ t \in \mathbb{R},$$

under assumptions which are very close to those given in Subsection 2.4 for autonomous systems, see [46], [47], and [51] for more details. However, pullback attractors have been introduced and further studied independently ; it is also worth noting that they have been extended to cocycles in the context of random dynamical systems as well, see, e.g., [64].

Definition 4.16. The family $\{K(t),\ t \in \mathbb{R}\}$ is pullback attracting for the process $U(t,\tau)$ if, $\forall t \in \mathbb{R}$, $\forall B \subset E$ bounded,

$$\lim_{s \to +\infty} \operatorname{dist}(U(t, t-s)B, K(t)) = 0.$$

The following result is the analogue of Theorem 2.17 for pullback attractors (see, e.g., [37]).

Theorem 4.17. *We assume that the process $U(t,\tau)$ possesses a compact pullback attracting family $\{K(t),\ t \in \mathbb{R}\}$ (i.e., $K(t)$ is compact, $\forall t \in \mathbb{R}$). Then it possesses a pullback attractor $\{\mathcal{A}(t),\ t \in \mathbb{R}\}$. Furthermore, if the compact pullback attracting family $\{K(t),\ t \in \mathbb{R}\}$ satisfies* (4.6)*, then a pullback attractor $\{\mathcal{A}(t) \subset K(t),\ t \in \mathbb{R}\}$ also satisfies* (4.6) *and is unique (in this class).*

Remark 4.18. a) As in the case of semigroups, one usually proves the existence of a compact pullback attracting family $\{K(t),\ t \in \mathbb{R}\}$ by introducing a proper decomposition $U(t,\tau) = U_1(t,\tau) + U_2(t,\tau)$, see [37].
b) Actually, all notions, definitions, and properties introduced for global attractors have a "pullback counterpart", see, e.g., [41], [217], and [233]. For instance, the pullback version of Theorem 2.19 is given in [217] (see also [41]).

An interesting feature of pullback attractors is that, in general, $\mathcal{A}(t)$ has finite fractal dimension, $\forall t \in \mathbb{R}$ (see, e.g., [40] and [142] ; see also Remark 4.15), so that the finite dimensional reduction principle given by the Hölder-Mañé theorem holds. Unfortunately, as mentioned in the introduction, the forward convergence, i.e.,

$$\lim_{t \to +\infty} \operatorname{dist}(U(t,\tau)B, \mathcal{A}(t)) = 0,\ \forall B \subset E \text{ bounded},\ \forall \tau \in \mathbb{R},$$

does not hold in general, due to the fact that the pullback attraction property (4.5) is not uniform with respect to $t \in \mathbb{R}$ (see however [37] and [42] for examples for which the forward attraction holds) ; we also gave an example of an equation for which the pullback attractor satisfying (4.6) (we recall that we have the uniqueness in this class) does not reflect the asymptotic behavior of the solutions of the system. Thus, again, this notion of a nonautonomous attractor does not yield a satisfactory finite dimensional reduction principle in general.

Remark 4.19. Nonautonomous inertial manifolds (also called integral manifolds) were studied, e.g., in [44] (see also [13], [133], [134], and [165]). In that case, under natural assumptions, the forward exponential convergence, and even the asymptotic completeness (i.e., a property similar to (3.1)) hold. However, we also have here the drawbacks mentioned in Subsection 3.1 and, in particular, very restrictive spectral gap conditions are necessary to prove the existence of such objects. We also mention that, when the process $U(t,\tau)$ is, in some proper sense, close to an autonomous semigroup $S(t)$ which possesses a global Lyapunov function and has a *regular* attractor, the associated pullback attractor $\mathcal{A}(t)$, $t \in \mathbb{R}$, is also regular (i.e., it is a finite union of finite dimensional submanifolds of E, of course, now depending on t) and uniformly (forward and pullback) exponentially attracting, see [44], [45], [80], [117], and [228] for details.

4.3. **Finite dimensional reduction of nonautonomous systems.** We saw in the two previous subsections that neither the uniform attractor nor a pullback attractor yield a satisfactory finite dimensional reduction principle in general. We noted however that the problem of the forward attraction for pullback attractors is due to the fact that the pullback attraction (4.5) may not be uniform with respect to t. Therefore, if we are able to construct (possibly) larger sets for which the pullback attraction is uniform with respect to t, then we will also obtain the forward attraction: the concept of an exponential attractor appears as a natural one to reach this goal and it is thus important to extend it to processes.

We first consider a discrete process $U(l,m),\ l,\ m \in \mathbb{Z},\ l \geq m$, acting on E, i.e.,

$$U(l,l) = \mathrm{Id},\ \forall l \in \mathbb{Z},\ U(l,m) \circ U(m,n) = U(l,n),\ \forall l \geq m \geq n,\ l,\ m,\ n \in \mathbb{Z}.$$

We set $\mathcal{U}(n) := U(n+1,n),\ n \in \mathbb{Z}$. It is then easy to see that the process $U(l,m)$ is uniquely determined by the family $\{\mathcal{U}(l),\ l \in \mathbb{Z}\}$; indeed,

$$U(n+k,n) = \mathcal{U}(n+k-1) \circ \mathcal{U}(n+k-2) \circ ... \circ \mathcal{U}(n),\ n \in \mathbb{Z},\ k \in \mathbb{N}.$$

We have the following result, which extends to (discrete) processes that given in Remark 3.12, d).

Theorem 4.20. [76] *We consider a second Banach space E_1 such that E_1 is compactly embedded into E and a bounded subset B of E_1. We make the following assumptions:*
(i) $\forall l \in \mathbb{Z}$, $\mathcal{U}(l)$ maps the δ-neighborhood (for the topology of E_1) $\mathcal{O}_\delta(B)$ of B onto B, where δ is independent of l ;
(ii) $\forall l \in \mathbb{Z}$, $\mathcal{U}(l)$ satisfies the smoothing property (2.14) *on $\mathcal{O}_\delta(B)$,*

$$\|\mathcal{U}(l)x_1 - \mathcal{U}(l)x_2\|_{E_1} \leq K\|x_1 - x_2\|_E,\ \forall x_1,\ x_2 \in \mathcal{O}_\delta(B),$$

where K is independent of $l,\ x_1$, and x_2.
Then the discrete process $U(l,m)$ possesses a nonautonomous exponential attractor

$$\{\mathcal{M}_U(n),\ n \in \mathbb{Z}\}$$

such that
1. $\forall n \in \mathbb{Z},\ \mathcal{M}_U(n) \subset B$ *and is compact in* E_1 *;*
2. $\forall n \in \mathbb{Z},\ \mathcal{M}_U(n)$ *has finite fractal dimension (in the topology of* E_1*),*

$$\dim_F \mathcal{M}_U(n) \le c_1,$$

where c_1 *is independent of* n *;*
3. it is positively invariant in the following sense:

$$U(l,m)\mathcal{M}_U(m) \subset \mathcal{M}_U(l),\ l \ge m,\ l,\ m \in \mathbb{Z}\ ;$$

4. it satisfies the following exponential attraction property:

$$\mathrm{dist}_{E_1}(U(l+m,l)B, \mathcal{M}_U(l+m)) \le c_2 e^{-c_3 m},\ l \in \mathbb{Z},\ m \in \mathbb{N}, \tag{4.8}$$

where c_2 *and* c_3 *are independent of* l *and* m *;*
5. the map $U \mapsto \{\mathcal{M}_U(n),\ n \in \mathbb{Z}\}$ *is uniformly Hölder continuous in the following sense: for every processes* $U_1(l,m)$ *and* $U_2(l,m)$ *such that* $\mathcal{U}_i(l),\ i = 1,\ 2,$ *satisfy (i) and (ii),* $\forall l \in \mathbb{Z}$ *(for the same constants* δ *and* K*), there holds*

$$\mathrm{dist}_{\mathrm{sym},E_1}(\mathcal{M}_{U_1}(l), \mathcal{M}_{U_2}(l)) \le c_4 \sup_{m \in (-\infty,l)\cap\mathbb{Z}} \{e^{-c_5(l-m)} \|\mathcal{U}_1(m) - \mathcal{U}_2(m)\|^{c_6}\}, \tag{4.9}$$

where $c_4,\ c_5 > 0,$ *and* $c_6 > 0$ *are independent of* $l,\ U_1,$ *and* U_2 *and*

$$\|S\| := \sup_{h \in \mathcal{O}_\delta(B)} \|Sh\|_{E_1}.$$

Furthermore, all the constants only depend on $B,\ E,\ E_1,\ \delta,$ *and* K *and can be computed explicitly in terms of the physical parameters of the problem.*

Remark 4.21. a) It follows from (4.8) that the pullback attraction holds, but one now has the forward attraction (and, even better, one has a uniform forward attraction). Since, $\forall l \in \mathbb{Z}$, $\mathcal{M}_U(l)$ has finite fractal dimension, this shows that the asymptotic behavior of (discrete) nonautonomous systems is also, in some proper sense, finite dimensional in general, as in the case of autonomous systems.
b) It also follows from (4.9) that the influence of the past decays exponentially, in agreement with our physical intuition.
c) We can also construct the exponential attractor $\{\mathcal{M}_U(n),\ n \in \mathbb{Z}\}$ such that the following cocycle identity holds:

$$\mathcal{M}_U(l+m) = \mathcal{M}_{T_m U}(l),\ l,\ m \in \mathbb{Z},$$

where $T_k U(l,m) := U(l+k, m+k),\ k,\ l,\ m \in \mathbb{Z},\ l \ge m.$
d) If $\mathcal{U}(l) \equiv S,\ \forall l \in \mathbb{Z}$, i.e., if the system is autonomous, we recover the exponential attractor constructed in Remark 3.12, d).
e) If the dependence of $\mathcal{U}(l)$ on l is periodic or quasiperiodic, then the same holds for the dependence of $\mathcal{M}_U(l)$ on l.

Remark 4.22. As mentioned several times, the smoothing property (2.14) is typical of parabolic systems and, e.g., for damped hyperbolic systems, it has to be generalized. In particular, if the second assumption of Theorem 4.20 is replaced by the following: $\forall l \in \mathbb{Z}$, $\forall x_1,\ x_2 \in \mathcal{O}_\delta(B)$, B being a proper closed subset of E_1,

$$\|\mathcal{U}(l)x_1 - \mathcal{U}(l)x_2\|_{E_1} \le (1-\epsilon)\|x_1 - x_2\|_{E_1} + K\|x_1 - x_2\|_E,$$

where $\epsilon \in (0,1)$ and K are independent of l, x_1, and x_2, then, assuming that B can be covered by a finite number of balls of radius δ (in the topology of E_1) with centers belonging to B, Theorem 4.20 also holds. We can obtain a similar result under the following more general (asymptotically) smoothing property: $\forall l \in \mathbb{Z}$, $\forall x_1,\ x_2 \in \mathcal{O}_\delta(B)$, $\mathcal{U}(l)x_1 - \mathcal{U}(l)x_2 = v_1 + v_2$, where

$$\|v_1\|_E \le (1-\epsilon)\|x_1 - x_2\|_E,$$
$$\|v_2\|_{E_1} \le K\|x_1 - x_2\|_E,$$

where $\epsilon \in (0,1)$ and K are independent of l, x_1, and x_2. However, in that case, all properties are obtained for the topology of E instead of that of E_1, see [76] for more details.

The next step is to extend such constructions to continuous processes $U(t,\tau)$, $t \ge \tau$, $\tau \in \mathbb{R}$.

For instance, for a parabolic system in a bounded domain, we usually proceed as follows. We first consider a uniform (with respect to $\tau \in \mathbb{R}$) bounded absorbing set B in E_1 (i.e., $\forall B_0 \subset E_1$ bounded, $\exists t_0 = t_0(B_0)$ such that $t \ge t_0$ implies $U(t+\tau,\tau)B_0 \subset B$, $\forall \tau \in \mathbb{R}$). We further assume that the map $U(T+\tau,\tau)$ satisfies the assumptions of Theorem 4.20, $\forall \tau \in \mathbb{R}$, for B as above and for some $T > 0$, $\delta > 0$, and K which are independent of τ (typically, in applications, we can take $\delta = 1$). Then, for every $\tau \in \mathbb{R}$, we consider the discrete process

$$U^\tau(l,m) := U(\tau + lT, \tau + mT),\ l,\ m \in \mathbb{Z},\ l \ge m.$$

Thus, owing to Theorem 4.20, we can construct, for every $\tau \in \mathbb{R}$, a discrete exponential attractor $\{\mathcal{M}_U(l,\tau),\ l \in \mathbb{Z}\}$ which satisfies all the assertions of this theorem. In addition, it satisfies the following properties:

$$\mathcal{M}_U(l,\tau) = \mathcal{M}_U(0, lT + \tau),\ l \in \mathbb{Z},\ \tau \in \mathbb{R}\ ;$$

$$\mathcal{M}_{T_s U}(l,\tau) = \mathcal{M}_U(l, \tau + s),\ l \in \mathbb{Z},\ s,\ \tau \in \mathbb{R},$$

where $T_s U(t,\tau) := U(t+s, \tau+s)$, $t \ge \tau$, s, $\tau \in \mathbb{R}$. We finally set

$$\mathcal{M}_U(t) := \cup_{s \in [0,T]} U(t, t-T-s)\mathcal{M}_U(0, t-T-s),\ t \in \mathbb{R}.$$

Then, assuming that $L\ :\ (t,\tau,x) \mapsto U(t,\tau)x$ is Lipschitz with respect to the x-variable and satisfies proper Hölder type properties with respect to t and τ, typically,

$$\|U(\tau+s+t,\tau)x - U(\tau+t,\tau)x\|_E \le c|s|^{\frac{1}{2}},$$

where c is independent of $t \ge 0$, $\tau \in \mathbb{R}$, $s \ge 0$, and $x \in B$, and

$$\|U(t+\tau+s,\tau+s)x - U(t+\tau,\tau)x\|_E \le ce^{c't}|s|^\gamma,\ t \ge T,\ s \in [0, \frac{T}{2}],\ \gamma > 0,$$

where c is independent of t, τ, s, and $x \in B$ (see [76] for more details), we can prove the following result.

Theorem 4.23. [76] *The family $\{\mathcal{M}_U(t),\ t \in \mathbb{R}\}$ is a nonautonomous exponential attractor for the process $U(t,\tau)$ in E_1 which satisfies the following properties:*
1. $\forall t \in \mathbb{R}$, $\mathcal{M}_U(t)$ is compact in E_1 and has finite fractal dimension,

$$\dim_F \mathcal{M}_U(t) \le c_1',\ \ \forall t \in \mathbb{R},$$

where c_1' is independent of t ;
2. it is positively invariant,

$$U(t,\tau)\mathcal{M}_U(\tau) \subset \mathcal{M}_U(t),\ \ t \ge \tau,\ \ \tau \in \mathbb{R}\ ;$$

3. it satisfies the following exponential attraction property:

$$\operatorname{dist}_{E_1}(U(t+\tau,\tau)B, \mathcal{M}_U(t+\tau)) \le c_2' e^{-c_3' t},\ \ \tau \in \mathbb{R},\ \ t \ge 0,$$

where c_2' and $c_3' > 0$ are independent of t and τ and where B is the bounded absorbing set introduced above ;
4. it satisfies the following Hölder continuity property: for every processes $U_1(t,\tau)$ and $U_2(t,\tau)$ such that $U_i(t+T,t)$, $i = 1,\ 2$, satisfy the assumptions of Theorem 4.20 (for the constants δ and K introduced above), $\forall t \in \mathbb{R}$, then

$$\operatorname{dist}_{\mathrm{sym},E_1}(\mathcal{M}_{U_1}(t), \mathcal{M}_{U_2}(t)) \le c_4' \sup_{s\ge 0}\ \{e^{-c_5' s}\|U_1(t,t-s) - U_2(t,t-s)\|^{c_6'}\},$$

where c_4', $c_5' > 0$, and $c_6' > 0$ are independent of $t \in \mathbb{R}$.
Furthermore, all the constants can be computed explicitly.

Remark 4.24. a) We can give a more precise Hölder continuity result in concrete applications, see [76].
b) We also have the following Hölder continuity with respect to the time:

$$\operatorname{dist}_{\mathrm{sym},E_1}(\mathcal{M}_U(t+s), \mathcal{M}_U(t)) \le c_7' |s|^{c_8'},\ \ t \in \mathbb{R},\ \ s \ge 0,$$

where c_7' and $c_8' > 0$ are independent of t and s.

Remark 4.25. a) We again have properties which are similar to those listed in Remark 4.21. In particular, we have the (uniform) forward attraction and, since $\mathcal{M}_U(t)$ has finite fractal dimension, $\forall t \in \mathbb{R}$, we obtain a satisfactory finite dimensional reduction principle for nonautonomous systems in bounded domains.
b) Such exponential attractors were constructed for nonautonomous reaction-diffusion equations (in bounded domains) in [76]. However, this construction has a universal nature and should be applicable to most equations (in bounded domains) for which the finite dimensionality of pullback attractors can be proven (e.g., the two-dimensional incompressible Navier-Stokes equations, the Cahn-Hilliard equation, damped hyperbolic equations, ...).

Remark 4.26. It follows from the Hölder continuity that we can construct robust families of nonautonomous exponential attractors which are continuous at every point, as in the autonomous case.

5. Dissipative PDEs in unbounded domains

As mentioned in the introduction, the study of the dynamics of dissipative systems in large and unbounded domains necessitates to develop new ideas and methods, when compared with the above sections, devoted to systems in bounded domains. Indeed, we are faced here with new phenomena which do not have analogues in the finite dimensional theory.

Our aim in this section is to give a short survey of the recent progress in this direction, including the so-called entropy theory, the description of the space-time chaos via Bernoulli schemes with an infinite number of symbols and its relations with the Kotelnikov formula, the Sinai-Bunimovich space-time chaos for continuous media, ...

We start by introducing and discussing the appropriate class of weighted and uniformly local Sobolev spaces, which is one of the main technical tools in the theory.

5.1. Weighted and uniformly local phase spaces: basic dissipative estimates.

We first note that, in contrast to the case of bounded domains, many physically relevant and interesting solutions of PDEs in unbounded domains (such as spatially periodic patterns, traveling waves, wave trains, spiral waves, ...) are not spatially localized and, thus, usually have *infinite* energy. Therefore, the typical, for bounded domains, choice of the phase space as $\Phi = L^2(\Omega)$ or $W^{l,p}(\Omega)$ does not seem to be reasonable here. On the other hand, all the above mentioned structures are bounded as $|x| \to +\infty$ and, therefore, belong to the phase space $\Phi = L^\infty(\Omega)$. However, the analytical properties of PDEs in L^∞-spaces are very bad (there is no maximal regularity, no analytic semigroups, ...), so that this choice of a phase space only works for relatively simple equations (for which the maximum principle holds).

Instead, it was suggested in [1], [22], and [168], to use *weighted* and so-called *uniformly local* Sobolev spaces which, on the one hand, contain all the sufficiently regular spatially bounded solutions and, on the other hand, enjoy regularity, embedding, and interpolation properties which are very similar to those of usual Sobolev spaces in bounded domains.

In order to introduce these spaces, we first need to define the appropriate class of admissible weight functions (see [78] and [240] for details).

Definition 5.1. Let $\mu > 0$ be arbitrary. A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is a weight function with an exponential growth μ if there holds

$$\phi(x+y) \le C_\phi e^{\mu|y|}\phi(x), \quad \phi(x) > 0, \tag{5.1}$$

for every $x,\ y \in \mathbb{R}^n$.

The most important examples of such weight functions are the following ones:

$$\phi_{\varepsilon,x_0}(x) := e^{-\varepsilon|x-x_0|}, \quad \varepsilon \in \mathbb{R}, \quad x_0 \in \mathbb{R}^n, \tag{5.2}$$

or their smooth analogues,

$$\varphi_{\varepsilon,x_0}(x) := e^{-\varepsilon\sqrt{1+|x-x_0|^2}}, \quad \varepsilon \in \mathbb{R}, \quad x_0 \in \mathbb{R}^n. \tag{5.3}$$

Obviously, these weight functions are weight functions with an exponential growth $|\varepsilon|$ and the constant C_ϕ is independent of x_0. Another important class of weight functions consists of the so-called polynomial weights,

$$\theta_{N,x_0}(x) := (1+|x-x_0|^2)^{-N/2}, \quad N \in \mathbb{R}, \quad x_0 \in \mathbb{R}^n, \tag{5.4}$$

which are also sometimes useful. Obviously, these weight functions have an exponential growth μ, for every $\mu > 0$.

We are now ready to introduce the proper classes of Sobolev spaces.

Definition 5.2. Let Ω be a sufficiently regular unbounded domain, ϕ be a weight function with an exponential growth, and $1 \le p \le +\infty$. Then the associated weighted spaces $L^p_\phi(\Omega)$ and weighted uniformly local spaces $L^p_{b,\phi}(\Omega)$ are defined by the following norms:

$$\begin{aligned} \|u\|^p_{L^p_\phi} &:= \int_\Omega \phi^p(x)|u(x)|^p\,dx, \\ \|u\|_{L^p_{b,\phi}} &:= \sup_{x_0\in\Omega}\{\phi(x_0)\|u\|_{L^p(\Omega\cap B^1_{x_0})}\}, \end{aligned} \tag{5.5}$$

where $B^R_{x_0}$ denotes the R-ball in the space $\mathbb{R}^n$ centered at x_0. For simplicity, we will write $L^p_b(\Omega)$ instead of $L^p_{b,1}(\Omega)$ and we naturally define the Sobolev spaces $W^{l,p}_\phi(\Omega)$ (resp., $W^{l,p}_{b,\phi}(\Omega)$) as the spaces of distributions whose derivatives up to the order l belong to $L^p_\phi(\Omega)$ (resp., $L^p_{b,\phi}(\Omega)$).

We note that $L^\infty(\Omega) \subset L^2_b(\Omega)$ and, consequently, all the dissipative structures mentioned above indeed belong to the uniformly local phase space $\Phi := L^2_b(\Omega)$. Furthermore, we also have the embedding $L^\infty(\Omega) \subset L^2_\phi(\Omega)$ if the weight function ϕ is integrable (i.e., $\phi \in L^1(\Omega)$). The important relations between weighted and uniformly local spaces are collected in the following proposition (see [240]).

Proposition 5.3. *Let Ω be a sufficiently regular unbounded domain and let ϕ be a weight function with an exponential growth ε. Then, for every $\mu > \varepsilon$, the following norms are equivalent:*

$$\begin{aligned} \|u\|^p_{L^p_\phi} &\sim \int_\Omega \phi^p(x_0)\int_\Omega e^{-\mu|x-x_0|}|u(x)|^p\,dx\,dx_0, \\ \|u\|^p_{L^p_{b,\phi}} &\sim \sup_{x_0\in\Omega}\left\{\phi^p(x_0)\int_\Omega e^{-\mu|x-x_0|}|u(x)|^p\,dx\right\}, \end{aligned} \tag{5.6}$$

with constants which depend on ε, μ, and C_ϕ, but are independent of the concrete form of ϕ.

Remark 5.4. Relations (5.6) allow to reduce the calculation and estimation of any weighted norm (for a weight function with an exponential growth) to those of the special exponential weight functions $\phi_{\mu,x_0}(x)$. In particular, relations of this type allow to reduce most results concerning embeddings and interpolation estimates for the weighted and uniformly local spaces, together with the associated regularity results for linear elliptic and parabolic operators, to the corresponding ones for the weight functions $\phi_{\varepsilon,x_0}(x)$ or $\varphi_{\varepsilon,x_0}(x)$ (and, thanks to the natural change of function $\tilde{u} = u\varphi_{-\varepsilon,x_0}$, to the classical spaces without weight). Furthermore, all the constants in such estimates only depend on the weight exponent and C_ϕ (and on some regularity constants of the boundary) and are independent of the concrete choice of ϕ and the shape of Ω, see [78], [168], [240], [245], and the references therein. This also explains why the linear theory of PDEs in uniformly local spaces is very similar to that in the unweighted spaces.

We are now ready to return to the main issues of this subsection, namely, the definition of the proper phase spaces Φ for dissipative PDEs in unbounded domains and the derivation of the basic dissipative estimate

$$\|S(t)u_0\|_\Phi \le Q(\|u_0\|_\Phi)e^{-\alpha t} + C,\ u_0 \in \Phi,\ t \ge 0,\ \alpha > 0. \tag{5.7}$$

A "general" answer to these questions can be formulated as follows:
1) use the *uniformly local* Sobolev spaces $W_b^{l,p}(\Omega)$ or $L_b^p(\Omega)$ as phase spaces, e.g., in a Hilbert setting, i.e., $p = 2$;
2) use the so-called weighted energy estimates and weighted regularity theory to obtain a dissipative estimate in the spaces $W^{l,p}_{\phi_{\varepsilon,x_0}}(\Omega)$;
3) pass from the weighted to the uniformly local spaces by using the second estimate of (5.6) with $\phi = 1$.

This machinery has been successfully applied to many physically relevant PDEs in unbounded domains, including various types of reaction-diffusion equations (see [9], [78], [79], [168], and [240]), damped wave equations (see [90] and [238]), elliptic equations in unbounded domains (see [169] and [227]), and even the Navier-Stokes equations in a strip (see [245]).

For the reader's convenience, we illustrate below such a scheme on the relatively simple example of a reaction-diffusion system in $\Omega = \mathbb{R}^3$ (see [240] for more details):

$$\partial_t u = a\Delta_x u - \lambda u - f(u) + g,\quad u\big|_{t=0} = u_0. \tag{5.8}$$

Here, $u = (u^1, ..., u^k)$ is an unknown vector-valued function, a is a constant diffusion matrix satisfying the standard assumption $a + a^* > 0$, $\lambda > 0$ is a fixed constant, g corresponds to the external forces and belongs to $L_b^2(\mathbb{R}^3)^k$, and f is a given nonlinear interaction function satisfying the following standard dissipativity assumptions:

$$\begin{cases} 1.\ f \in C^1(\mathbb{R}^k, \mathbb{R}^k)\ ; \\ 2.\ f(u).u \ge -C\ ; \\ 3.\ f'(u) \ge -K\ ; \\ 4.\ |f(u)| \le C(1 + |u|^p)\ ; \end{cases} \tag{5.9}$$

$u \in \mathbb{R}^k$, $C,\ K \ge 0$, where $u.v$ denotes the usual inner product in $\mathbb{R}^k$ and $p \ge 0$ is arbitrary.

Theorem 5.5. *Let the above assumptions hold. Then, for every* $u_0 \in \Phi_b := L_b^2(\mathbb{R}^3)^k$, *problem* (5.8) *possesses a unique solution* $u(t) \in \Phi_b$ *and the following dissipative estimate holds:*

$$\|u(t)\|_{L_b^2} \le C\|u_0\|_{L_b^2}e^{-\alpha t} + C(1 + \|g\|_{L_b^2}),\ t \ge 0, \tag{5.10}$$

where the positive constants α *and* C *are independent of* u_0.

Proof. We only give a formal derivation of the dissipative and uniqueness estimates. The remaining details can be found in [240]. We multiply equation (5.8) by $u\phi^2$, where $\phi(x) = \phi_{\varepsilon,x_0}(x) := e^{-\varepsilon|x-x_0|}$, for a sufficiently small ε which will be fixed below, and integrate with respect to $x \in \mathbb{R}^n$. Then we have

$$\begin{aligned} 1/2\partial_t\|u(t)\|^2_{L^2_\phi} + (a\nabla_x u(t), \nabla_x[\phi^2 u(t)]) + \lambda\|u(t)\|^2_{L^2_\phi} = \\ = -(f(u(t)).u(t), \phi^2) + (\phi^2 u(t), g) \end{aligned} \tag{5.11}$$

(here and below, $(\cdot,\cdot)$ denotes the scalar products in $L^2(\mathbb{R}^3)$, $L^2(\mathbb{R}^3)^k$, and $L^2(\mathbb{R}^3)^{3k}$). According to the dissipativity assumption (5.9) 2., we see that

$$-(\phi^2, f(u).u) \le C\|\phi\|^2_{L^2} = C\varepsilon^{-3} \tag{5.12}$$

and, thus, the nonlinear term can be controlled. Furthermore, thanks to the obvious inequality

$$|\nabla_x\phi_{\varepsilon,x_0}(x)|\le C\varepsilon\phi_{\varepsilon,x_0}(x), \tag{5.13}$$

together with the positivity of a, we conclude that, if $\varepsilon>0$ is small enough, the following estimate holds:

$$(a\nabla_x u,\nabla_x[\phi^2 u])+\lambda\|u\|^2_{L^2_\phi}\ge \\ \ge 2\alpha(\|\nabla_x u\|^2_{L^2_\phi}+\|u\|^2_{L^2_\phi})-C\varepsilon(|u|,|\nabla_x u|\phi^2)\ge\alpha(\|\nabla_x u\|^2_{L^2_\phi}+\|u\|^2_{L^2_\phi}), \tag{5.14}$$

for some positive constant α which is independent of x_0. Inserting these estimates into (5.11), we deduce that

$$\partial_t\|u(t)\|^2_{L^2_\phi}+2\alpha\|u(t)\|^2_{W^{1,2}_\phi}\le C(1+\|g\|^2_{L^2_\phi})$$

and the Gronwall inequality gives

$$\|u(T)\|^2_{L^2_{\phi_{\varepsilon,x_0}}}+\int_T^{T+1}\|u(t)\|^2_{W^{1,2}_{\phi_{\varepsilon,x_0}}}\,dt\le C\|u(0)\|^2_{L^2_{\phi_{\varepsilon,x_0}}}e^{-2\alpha T}+C(1+\|g\|^2_{L^2_{\phi_{\varepsilon,x_0}}}). \tag{5.15}$$

It is crucial here that the constants C and α in this inequality are independent of x_0. Therefore, taking the supremum over $x_0\in\mathbb{R}^3$ and using the second relation of (5.6) with $\phi=1$, we deduce the required dissipative estimate (5.10) in the uniformly local phase space $\Phi_b=L^2_b(\mathbb{R}^3)^k$.

We now verify the uniqueness. Let $u_1(t)$ and $u_2(t)$ be two solutions of (5.8) and set $v(t):=u_1(t)-u_2(t)$. Then this function solves the linear equation

$$\partial_t v=a\Delta_x v-\lambda v-l(t)v,\quad l(t):=\int_0^1 f'(su_1(t)+(1-s)u_2(t))\,ds. \tag{5.16}$$

We note that, due to the third (quasimonotonicity) assumption of (5.9), we have $l(t)\ge -K$. Multiplying now equation (5.16) by $v\phi^2_{\varepsilon,x_0}$, using the last inequality, and arguing exactly as in the derivation of the dissipative estimate, we obtain

$$\|v(t)\|^2_{L^2_{\phi_{\varepsilon,x_0}}}\le Ce^{Kt}\|v(0)\|^2_{L^2_{\phi_{\varepsilon,x_0}}}, \tag{5.17}$$

for some positive constant C which is independent of x_0. This estimate gives the uniqueness and finishes the proof of the theorem. ☐

Remark 5.6. We see that the growth restriction (5.9) 4. has not been used in the proof of uniqueness and of the derivation of the dissipative estimate. However, this assumption is necessary in order to show that the associated solution satisfies equation (5.8) in the sense of distributions. Furthermore, as shown in [240], $f(u(t))$ and $\Delta_x u(t)$ belong to $L^2_b(\mathbb{R}^3)^k$, for every $t>0$, so that the equation can be understood as an equality in $L^2_b(\mathbb{R}^3)^k$.

Remark 5.7. Estimates (5.15) and (5.17) show that the reaction-diffusion problem (5.8) is well-defined not only in the uniformly local phase space $\Phi_b=L^2_b(\mathbb{R}^3)^k$, but also in the larger phase space $\Phi_\varepsilon:=L^2_{e^{-\varepsilon|x|}}(\mathbb{R}^3)^k$, provided that $\varepsilon>0$ is small enough. Roughly speaking, this space contains not only all functions which are bounded as $|x|\to+\infty$, but also functions which grow at most like $e^{\varepsilon|x|}$ at infinity. Thus, the alternative choice of the *weighted* phase space Φ_ε (or the choice of weighted spaces with polynomial weights as in the first articles on this subject, see [22]) is also possible here, see also [19], [78], and [230]. However, such a choice has essential drawbacks related to the addition of the above spatially unbounded solutions. Indeed, on the one hand, all the dissipative structures mentioned above are *bounded* as $|x| \to +\infty$, so that the class of bounded (uniformly local) solutions seems physically natural and sufficient. On the other hand, the analytical properties of the equations in spaces of spatially unbounded functions are essentially more complicated (in particular, even in the case that we consider, there is no differentiability with respect to the initial data in Φ_ε). Furthermore, even the uniqueness in such classes is *strongly* related to the restrictive quasimonotonicity assumption (5.9) 3. and can be violated if it is not satisfied, see [78]. Thus, the choice of the *uniformly local* phase spaces seems more general and preferable.

Remark 5.8. We note that, in contrast to bounded domains, the space $\mathcal{C}^\infty(\Omega)$ *is not dense* in the uniformly local space $L^2_b(\Omega)$. As a consequence, even the linear equation (5.8) with $f = g = 0$ does not generate an analytic semigroup in $L^2_b(\mathbb{R}^3)^k$ and, in particular, the solution $u(t)$ is not continuous at $t = 0$ for generic u_0 (i.e., $u \notin \mathcal{C}([0,T], L^2_b(\mathbb{R}^3)^k)$). However,

$$u \in \mathcal{C}([0,T], L^2_\phi(\mathbb{R}^3)^k),$$

for every $\phi \in L^1(\mathbb{R}^3)$, see, e.g., [239]. This inconvenience can be overcome by introducing a more restrictive uniformly local space $\widetilde{L}^2_b(\Omega)$ as follows:

$$\widetilde{L}^2_b(\Omega) := [\mathcal{C}^\infty(\Omega)]_{L^2_b(\Omega)},$$

where $[\cdot]_V$ denotes the closure in the space V. Roughly speaking, $u \in \widetilde{L}^2_b(\Omega)$ means the boundedness of the L^2_b-norm, plus some kind of "translation compactness". Indeed, as proven in [239], at least for $\Omega = \mathbb{R}^n$, the space $\tilde{L}^2_b(\Omega)$ coincides with the space of the so-called translation compact functions introduced by Chepyzhov and Vishik for the theory of nonautonomous attractors, see [51]. We recall that the function $u \in L^2_b(\mathbb{R}^n)$ is translation compact if its hull,

$$\mathcal{H}(u) := [T_s u, s \in \mathbb{R}^n]_{L^2_{loc}},\ T_s u(x) := u(x+s),\ s,\ x \in \mathbb{R}^n,$$

is compact in the local space $L^2_{loc}(\mathbb{R}^n)$. Under such a more restrictive choice of the phase space, the analytic semigroups theory works and the continuity of $u(t)$ also holds, see [9] and the references therein. We note however that, although it is crucial for the general analytic semigroups approach, the above density problem does not seem to be essential for the weighted energy methods considered here, since every $u \in L^2_b(\Omega)$ can obviously be approximated by smooth functions in the *local* topology of $L^2_{loc}(\Omega)$, and this is enough in order to establish the existence of a solution, see, e.g., [78] and [240]. Furthermore, verifying the artificial translation compactness requirement is an extremely difficult (unsolvable?) problem for more complicated equations (such as the two-dimensional and, especially, the three-dimensional Navier-Stokes equations in cylindrical domains, see [245] and [246]). Thus, we will no longer consider the space $\widetilde{L}^2_b(\Omega)$ in this survey.

5.2. **Attractors and locally compact attractors.** We now consider the theory of attractors in the uniformly local phase spaces. The first essential difference here is that, contrary to bounded domains, the embedding

$$W^{1,2}_b(\Omega) \subset L^2_b(\Omega) \tag{5.18}$$

is not compact. Thus, the usual smoothing (or asymptotically smoothing) properties are *not sufficient* to establish the existence of a compact attractor in the uniformly local phase spaces. As a consequence, the global attractor only exists in some exceptional cases (which will be considered in the next subsection) in the initial phase space and, in order to construct a general theory, the compactness assumption must be weakened. In particular, as shown in [240], already for the simple real Ginzburg-Landau equation in $\mathbb{R}$,

$$\partial_t u = \partial_x^2 u + u - u^3,$$

the associated set of equilibria is not compact in $L^2_b(\mathbb{R})$. Thus, this equation cannot have a compact global attractor in the phase space $L^2_b(\mathbb{R})$.

This difficulty is overcome by a systematic use of the local topology of $L^2_{loc}(\Omega)$ and the related locally compact global attractors. To be more precise, the set $\mathcal{A}$ is the locally compact global attractor for the semigroup $S(t)$ acting on the uniformly local phase space $\Phi_b := W^{l,p}_b(\Omega)$ if:
(i) it is bounded in Φ_b and compact in the local topology of $\Phi_{loc} := W^{l,p}_{\mathrm{loc}}(\overline{\Omega})$;
(ii) it is invariant, $S(t)\mathcal{A} = \mathcal{A}$, $\forall t \geq 0$;
(iii) it attracts the bounded subsets of the phase space Φ_b in the local topology of Φ_{loc}. This means that, for every bounded subset B of Φ_b and every bounded subdomain Ω_1 of Ω,

$$\lim_{t\to+\infty} \mathrm{dist}_{W^{l,p}(\Omega_1)}(S(t)B\big|_{\Omega_1}, \mathcal{A}\big|_{\Omega_1}) = 0,$$

where $u\big|_{\Omega_1}$ denotes the restriction of the function u (defined in Ω) to the subdomain Ω_1.

Remark 5.9. It is not difficult to see that the attractor defined above is a (Φ_b, Φ_{loc})-attractor in the terminology of Babin and Vishik, and, consequently, its existence can be verified, e.g., by using the general attractor's existence Theorem 2.20. However, in contrast to the case of usual global attractors, the compactness assumption on the absorbing/attracting sets should now be verified in the *local* topology of Φ_{loc} only. Since the embedding

$$W^{l+\alpha,p}_b(\Omega) \subset W^{l,p}_{loc}(\overline{\Omega})$$

is compact for $\alpha > 0$, verifying such a compactness assumption can be reduced (exactly as in the case of bounded domains) to the derivation of an appropriate smoothing property for the equations under study.

For the reader's convenience, we illustrate the above theory on the reaction-diffusion system (5.8) (see [71], [78], [79], [164], and [168] for more general classes of reaction-diffusion equations, [54], [90], and [238] for damped wave equations, and [245] and [246] for the Navier-Stokes equations in unbounded domains).

Theorem 5.10. *Let the assumptions of Theorem 5.5 hold. Then the semigroup $S(t)$ possesses the locally compact global attractor $\mathcal{A}$ in the uniformly local phase space $\Phi_b = L^2_b(\mathbb{R}^3)^k$.*

Sketch of the proof. According to the abstract attractor's existence theorem mentioned above, we need to verify that the semigroup $S(t)$ is continuous in the local topology of $\Phi_{loc} := L^2_{loc}(\mathbb{R}^3)^k$ on every bounded subset of Φ_b and that there exists a compact (in the topology of Φ_{loc}) absorbing set (for bounded subsets of Φ_b).

The continuity follows in a standard way from estimate (5.17). Thus, we only need to construct a compact absorbing set.

As usual, the basic dissipative estimate (5.10) guarantees that the ball of radius R, $B_R := \{u, \|u\|_{L^2_b} \leq R\}$, in the phase space Φ_b is an absorbing set if R is large enough. However, this ball is obviously not compact in Φ_{loc}. For this reason, we construct a new absorbing set in the following standard way:

$$\mathcal{B} := [S(1)B_R]_{\Phi_{\rm loc}}.$$

Since the embedding $W^{1,2}_b(\mathbb{R}^3)^k \subset \Phi_{\rm loc}$ is compact, it is sufficient, in order to prove that the set $\mathcal{B}$ is compact in $\Phi_{\rm loc}$ (and, thus, finish the proof of the theorem), to prove a *smoothing* property on the solutions of problem (5.8) of the following form:

$$\|u(1)\|^2_{W^{1,2}_b} \le C(1 + \|u(0)\|^2_{L^2_b} + \|g\|^2_{L^2_b}), \tag{5.19}$$

where the constant C is independent of u.

In order to prove (5.19), we multiply equation (5.8) by the following expression:

$$t\sum_{i=1}^{3} \partial_{x_i}(\phi^2\partial_{x_i}u(t)) =: t.T_\phi u(t), \tag{5.20}$$

where $\phi(x) = \phi_{\varepsilon,x_0}(x) := e^{-\varepsilon|x-x_0|}$ and $\varepsilon > 0$ is small enough. Then, integrating with respect to x, we have

$$\begin{aligned}1/2\partial_t(t\|\nabla_x u(t)\|^2_{L^2_\phi}) + \lambda t\|\nabla_x u(t)\|^2_{L^2_\phi} + t(a\Delta_x u(t), T_\phi u(t)) =\\ = \|\nabla_x u(t)\|^2_{L^2_\phi} - t(\phi^2 f'(u(t))\nabla_x u(t), \nabla_x u(t)) + t(g, T_\phi u(t)).\end{aligned} \tag{5.21}$$

Using now the positivity of a and estimate (5.13), we note that

$$\begin{aligned}(a\Delta_x u, T_\phi u) \ge (a\Delta_x u, \phi^2\Delta_x u) - C\varepsilon(\phi^2|\Delta_x u|, |\nabla_x u|) \ge\\ \ge \alpha\|\Delta_x u\|^2_{L^2_\phi} - C\varepsilon^2\|\nabla_x u\|^2_{L^2_\phi},\end{aligned} \tag{5.22}$$

for some positive constant α. Using this estimate, together with the quasimonotonicity assumption $f'(u) \ge -K$, we deduce from (5.21) that

$$\partial_t(t\|\nabla_x u(t)\|^2_{L^2_\phi}) + \lambda t\|\nabla_x u(t)\|^2_{L^2_\phi} + t\|\Delta_x u(t)\|^2_{L^2_\phi} \le C(t+1)(\|g\|^2_{L^2_\phi} + \|\nabla_x u(t)\|^2_{L^2_\phi}).$$

Integrating this estimate with respect to $t \in [0, 1]$ and using (5.15) to estimate the time integral of $\nabla_x u$, we find

$$\|u(1)\|^2_{W^{1,2}_{\phi_{\varepsilon,x_0}}} \le C(1 + \|g\|^2_{L^2_{\phi_{\varepsilon,x_0}}} + \|u(0)\|^2_{L^2_{\phi_{\varepsilon,x_0}}}), \tag{5.23}$$

where the constant C is independent of x_0. Taking the supremum over $x_0 \in \mathbb{R}^3$ in both sides of this inequality and using (5.6), we obtain the required smoothing property (5.19), which finishes the proof of Theorem 5.10. □

Remark 5.11. The trick consisting in multiplying equation (5.8) by the expression $T_\phi u$ (suggested in [240]) allows to estimate the nonlinear term $f(u)$ in an optimal way by only using the quasimonotonicity assumption (exactly as in bounded domains). In contrast to this, the straightforward multiplication of the equation by $\phi^2\Delta_x u$ (as performed in [19] and [22]) gives, when integrating by parts in the nonlinear term, the additional "bad" term $(\nabla_x\phi^2 f(u)^T, \nabla_x u)$ and the extremely restrictive growth assumption $p \le 1 + \min\{4/n, 2/(n-2)\}$ in (5.9) 4. is necessary in order to handle it. Thus, in three space dimensions, this yields that $p < 7/3$, and even cubic nonlinearities cannot be treated. The above mentioned simple trick allows to avoid to impose a growth restriction to prove the existence of attractors.

5.3. **The finite dimensional case.** Before discussing the general infinite dimensional case in the next sections, we consider some rather exceptional cases in which the global attractor remains finite dimensional. As we will see below, in such cases, in spite of the fact that the underlying domain is unbounded, the attractor is localized (up to exponentially decaying terms) in some *bounded* domain (due to some special structural assumptions on the nonlinearity and the external forces). Thus, the corresponding theory is very similar to that in bounded domains and seems to be well-understood now (see [1], [19], [22], [71], [74], [78], [93], [163], [238], and the references therein).

As above, we consider, for simplicity, the reaction-diffusion system (5.8), although the approach described below has a general nature, see, e.g., [238] for nonlinear damped wave equations, [91] for degenerate parabolic equations, and [11] for the Navier-Stokes equations.

The most commonly used structural assumption on the nonlinearity f (suggested in [22]) is the following one:

$$f(u).u \geq 0, \quad \forall u \in \mathbb{R}^k \tag{5.24}$$

(compare with (5.9) 2.). In addition, some *decay* assumptions on the external forces g as $|x| \to +\infty$ are necessary. In order to formulate them, we need to introduce some more specific classes of uniformly local spaces.

Definition 5.12. Let Ω be a sufficiently smooth unbounded domain. The space $\dot{W}_b^{l,p}(\Omega)$ consists of all functions $u \in W_b^{l,p}(\Omega)$ which satisfy

$$\lim_{|x_0|\to+\infty} \|u\|_{W^{l,p}(\Omega\cap B^1_{x_0})} = 0. \tag{5.25}$$

Roughly speaking, the space $\dot{W}_b^{l,p}(\Omega)$ consists of all functions $u \in W_b^{l,p}(\Omega)$ which decay as $|x| \to +\infty$. In particular, obviously, $W^{l,p}(\Omega) \subset \dot{W}_b^{l,p}(\Omega)$.

Finally, following [71], we impose a decay assumption on the external forces g of the form

$$g \in \dot{L}^2_b(\mathbb{R}^3)^k. \tag{5.26}$$

The following simple lemma (see [74]) is a key technical tool in the theory.

Lemma 5.13. *Let $g \in \dot{L}^2_b(\Omega)^k$ and set*

$$R_g(x_0) := \|g\|^2_{L^2_{\phi_\varepsilon,x_0}},$$

for some positive ε. Then

$$\lim_{|x_0|\to+\infty} R_g(x_0) = 0. \tag{5.27}$$

Returning to the reaction-diffusion system (5.8) and to the weighted dissipative estimate (5.15), we see that, owing to the structural assumption (5.24) (instead of $f(u).u \geq -C$), the constant 1 disappears in the right-hand side of (5.15) and we have an homogeneous estimate,

$$\|u(t)\|^2_{L^2(B^1_{x_0})^k} \leq C_1\|u(t)\|^2_{L^2_{\phi_\varepsilon,x_0}} \leq C_2 e^{-\alpha t}\|u(0)\|^2_{L^2_{\phi_\varepsilon,x_0}} + C_2\|g\|^2_{L^2_{\phi_\varepsilon,x_0}}, \tag{5.28}$$

where the positive constants C_2 and α are independent of x_0 and u. In particular, the first term in the right-hand side vanishes on the attractor and we have

$$\|u\|^2_{L^2(B^1_{x_0})^k} \leq C_2 R_g(x_0), \quad \forall u \in \mathcal{A}. \tag{5.29}$$

Thus, owing to Lemma 5.13, $\mathcal{A} \subset \dot{L}^2_b(\mathbb{R}^3)^k$ and, besides, (5.29) gives a uniform "tail estimate" as $|x| \to +\infty$ with respect to all functions on the attractor. This tail estimate, together with the embedding $\mathcal{A} \subset W^{1,2}_b(\mathbb{R}^3)^k$ which follows from the smoothing property (5.19), guarantee the *compactness* of the (locally compact) attractor $\mathcal{A}$ on the initial topology of the phase space as well. Finally, a slightly more accurate analysis of estimate (5.28) allows to check the asymptotic compactness of the associated semigroup $S(t)$ in Φ_b. Thus, we have the following result (see [71] and [74] for a detailed proof).

Theorem 5.14. *Let the assumptions of Theorem 5.5 hold and let, in addition, the structural assumptions* (5.24) *and* (5.26) *be satisfied. Then the semigroup* $S(t)$ *associated with the reaction-diffusion system* (5.8) *possesses the compact global attractor* $\mathcal{A}$ *on the initial uniformly local phase space* Φ_b *(exactly as in bounded domains).*

Furthermore, exactly as in bounded domains, we have the finite dimensionality of the above global attractor in the phase space.

Theorem 5.15. *Let the assumptions of the previous theorem hold. Then the global attractor* $\mathcal{A}$ *has finite fractal dimension. Furthermore, the associated semigroup possesses a finite dimensional exponential attractor* $\mathcal{M}$ *in the phase space* Φ_b.

The proof of this theorem is also based on the uniform tail estimate (5.29) and can be found in [71] and [74].

Remark 5.16. In the original article [22], the uniform tail estimate on the global attractor was proven in an alternative and more complicated way. To be more precise, the equations were considered in the phase space $\Phi_\phi := L^2_\phi(\Omega)^k$, with *growing* weight functions of the form $\phi(x) := (1 + |x|^2)^N$, $N > 0$ (thus, Φ_ϕ consists of functions which *decay* sufficiently fast at infinity). Then the compactness of the global attractor in Φ_ϕ was deduced by proving the embedding

$$\mathcal{A} \subset L^2_{\phi^\alpha}(\Omega)^k \cap W^{1,2}_\phi(\Omega)^k,$$

for some $\alpha > 1$. This however requires the artificial restriction $g \in L^2_\phi(\Omega)^k$ and some additional assumptions on f. In particular, the Hilbert case $\phi = 1$ was not covered by this approach. This drawback was overcome in [230], in which a more accurate method to estimate the tails in the Hilbert case $\Phi = L^2(\Omega)^k$ was suggested and the compactness of the attractor for $\phi = 1$ was proven. An alternative very simple and effective way to handle the Hilbert case $\phi = 1$ is based on the so-called *energy* method, see [25], [201], and [220]. This approach is based on the elementary fact that a weakly convergent sequence in a Hilbert (reflexive) space converges strongly if the associated sequence of norms converges to the norm of the limit function. The convergence of the norms is then verified by passing to the limit in the energy equality. Thus, the asymptotic compactness of the semigroup can be verified *without* requiring to work on weighted spaces. This approach is especially helpful for complicated equations (such as the Navier-Stokes equations) for which estimates in weighted spaces are rather difficult to obtain, see [220]. A drawback of this approach is that it does not give any qualitative nor quantitative information on the spatial structure of the global attractor, which are available when using weighted spaces, and only works in the Hilbert case. However, it is worth noting that, as usual, the global attractor (if it exists) is independent of the choice of the admissible phase space, see [78], so that all cases mentioned above are actually contained in the general Theorems 5.14 and 5.15.

Remark 5.17. We finally mention that the constant λ in (5.8) can be replaced by x-dependent functions $\lambda(x)$ which are not necessarily positive everywhere, see [9] and [163] ; actually, it is sufficient to require that

$$(a\nabla_x v, \nabla_x v) + (\lambda v, v) \geq \lambda_0 \|v\|^2_{W^{1,2}},\ \forall v \in W^{1,2}(\mathbb{R}^3)^k,\ \lambda_0 > 0.$$

Indeed, all the estimates given above can be obtained by repeating word by word the corresponding proofs. Another slight generalization consists in considering functions f which depend on x and requiring that, instead of (5.24),

$$f(x,u).u \geq -C(x),\ x \in \mathbb{R}^3,\ u \in \mathbb{R}^k,$$

where C belongs to $\dot{L}^1_b(\mathbb{R}^3)$.

We now formulate, following essentially [238] (see also [74]), some natural generalizations of the structural assumption (5.24) and discuss the spatial asymptotics of the global attractor.

Assumption A. Let the nonlinearity f and the external forces $g \in L^2_b(\mathbb{R}^3)^k$ be such that there exists a solution $Z_0(x)$ of the associated elliptic equilibrium problem

$$a\Delta_x Z_0 - \lambda Z_0 - f(Z_0) + g = 0,\ x \in \mathbb{R}^3 \backslash B^R_0,\ Z_0 \in W^{2,2}_b(\mathbb{R}^3 \backslash B^R_0)^k, \tag{5.30}$$

outside a large ball B^R_0 of $\mathbb{R}^3$ which satisfies the following property:

$$[f(v + Z_0(x)) - f(Z_0(x))].v \geq 0,\ v \in \mathbb{R}^k,\ x \in \mathbb{R}^3 \backslash B^R_0. \tag{5.31}$$

The following generalization of Theorem 5.14 gives the spatial asymptotics of the global attractor up to exponentially small terms.

Theorem 5.18. *Let the assumptions of Theorem 5.5 hold and let Assumption A. be satisfied. Then the associated semigroup $S(t)$ possesses the global attractor $\mathcal{A}$ in the phase space Φ_b which satisfies the following estimate:*

$$\|u_0 - Z_0\|_{L^2(B^1_{x_0})^k} \leq Ce^{-\alpha|x_0|},\ |x_0| > R + 1,\ u_0 \in \mathcal{A}, \tag{5.32}$$

where the positive constants C and α are independent of $u_0 \in \mathcal{A}$ and x_0.

Sketch of the proof. Let $\tilde{Z}_0(x)$, $\tilde{Z}_0 \in W^{2,2}_b(\mathbb{R}^3)^k$, be some extension of $Z_0(x)$ inside the ball B^R_0. Then this function satisfies

$$a\Delta_x \tilde{Z}_0 - \lambda \tilde{Z}_0 - f(\tilde{Z}_0) + g = \tilde{g}, \tag{5.33}$$

where $\tilde{g} \in L^2_b(\mathbb{R}^3)^k$ and $\operatorname{supp} \tilde{g} \subset B^R_0$.

We now set $v(t) := u(t) - \tilde{Z}_0$. Then this function solves

$$\partial_t v = a\Delta_x v - \lambda v - [f(v + \tilde{Z}_0) - f(\tilde{Z}_0)] + \tilde{g}. \tag{5.34}$$

We recall that, owing to Assumption A., $[f(v + \tilde{Z}_0) - f(\tilde{Z}_0)].v \geq 0$, for $x \notin B^R_0$. Using the quasimonotonicity assumption $f'(v) \geq -K$ to estimate this term inside the ball B^R_0, we infer

$$\begin{aligned}[f(v(x) + \tilde{Z}_0(x)) - f(Z_0(x))].v(x) \geq -K|v(x)|^2 \chi_R(x) \geq \\ \geq -K(|u(x)|^2 + |\tilde{Z}_0(x)|^2)\chi_R(x),\ x \in \mathbb{R}^3,\end{aligned} \tag{5.35}$$

where $\chi_R(x)$ is the characteristic function of the ball B^R_0.

Multiplying now equation (5.34) by $\phi^2_{\varepsilon,x_0}v(t)$ and arguing exactly as in the derivation of (5.15) and (5.28), we conclude that

$$\|v\|^2_{L^2_{\phi_{\varepsilon,x_0}}} \le C(\|u\chi_R\|^2_{L^2_{\phi_{\varepsilon,x_0}}} + \|\tilde{Z}_0\chi_R\|^2_{L^2_{\phi_{\varepsilon,x_0}}} + \|\tilde{g}\chi_R\|^2_{L^2_{\phi_{\varepsilon,x_0}}}), \tag{5.36}$$

where the constant C is independent of x_0 and $v \in \mathcal{A} - \tilde{Z}_0$. Multiplying this inequality by the weight function

$$\phi(x_0) := \inf_{z\in B_0^R} e^{\alpha|x_0-z|},$$

with $\alpha < \varepsilon$ (which, obviously, is a weight function with an exponential growth α), taking the supremum over $x_0 \in \mathbb{R}^3$, and using the equivalence (5.6) 2., we finally find

$$\|v\|^2_{L^2_{b,\phi}} \le C(\|u\chi_R\|^2_{L^2_b} + \|\tilde{Z}_0\chi_R\|^2_{L^2_b} + \|\tilde{g}\|^2_{L^2_b}) \le C_1, \tag{5.37}$$

where we have implicitly used the fact that the L^2_b-norm of the attractor is bounded. There remains to note that (5.37) is equivalent to (5.32) to finish the proof of Theorem 5.18. □

Remark 5.19. If, in addition, the attractor $\mathcal{A}$ is bounded in $\mathcal{C}_b(\mathbb{R}^3)^k$ by some constant L, it is, obviously, sufficient to verify estimate (5.31) from Assumption A. only for $|v| \le 2L$. We also note that Theorem 5.18 shows, in particular, that the spatial asymptotics (5.32) holds with Z_0 replaced by any true equilibrium of the problem.

We conclude the section by giving two examples to illustrate the above theorem.

Example 5.20. Let the assumptions of Theorem 5.14 hold. We claim that Assumption A. is automatically satisfied here and, therefore, the global attractor $\mathcal{A}$ possesses the spatial asymptotics (5.32). Indeed, as proven in [240], $\mathcal{A}$ is globally bounded in $W^{2,2}_b(\mathbb{R}^3)^k$. This fact, together with a proper interpolation inequality and the tail estimate (5.29), yield

$$\|u\|_{\mathcal{C}(B^1_{x_0})^k} \le C[R_g(x_0)]^{1/4}. \tag{5.38}$$

Therefore, the global attractor also belongs to $\dot{\mathcal{C}}_b(\mathbb{R}^3)^k$ and is bounded in this space. In particular, any equilibrium $z_0(x)$ of this problem satisfies $\lim_{|x|\to+\infty} z_0(x) = 0$. Thus, in order to verify Assumption A., with $Z_0 = z_0$, it is sufficient to check that there exists $\varepsilon > 0$ such that

$$[f(v+z) - f(z)].v \ge -\lambda/2|v|^2, \tag{5.39}$$

for every $v,\ z \in \mathbb{R}^k$, $|v| \le 2L$ (L is the $\mathcal{C}$-diameter of the attractor) and $|z| \le \varepsilon$. Indeed, Assumption A. then holds with f replaced by $f(u)+\lambda/2u$, for a sufficiently large $R = R(\varepsilon)$. In order to verify inequality (5.39), we consider two cases, namely, $|v| \le \delta$ and $|v| > \delta$, where $\delta > 0$ is a sufficiently small number to be fixed. In the first case, both v and z are small, so that inequality (5.39) follows from the continuity of f' and the fact that, owing to assumption (5.24), $f(0) = 0$ and $f'(0) \ge 0$. We now consider the second case ($\delta > 0$ has been fixed at this stage). It is sufficient, in view of inequality (5.24) and the assumption $|v| > \delta$, to find $\varepsilon > 0$ such that

$$f(v+z).z + f(z).v \le \lambda\delta/2,$$

for every $|z| \le \varepsilon$ and $|v| \le 2L$. Since $f(0) = 0$, the existence of such an $\varepsilon = \varepsilon(\delta, L)$ is straightforward, see [238] for more details. Thus, Assumption A. is verified and Theorem 5.18 (together with the $W^{2,2}_b$-bound on the attractor) now gives

$$|u(x) - z_0(x)| \le Ce^{-\alpha|x|},\ \forall u \in \mathcal{A},\ x \in \mathbb{R}^3. \tag{5.40}$$

Remark 5.21. In particular, we see that, although the rate of convergence to zero of the external forces g determines that of any function belonging to the global attractor, the "thickness" of the attractor decays *exponentially*, no matter how slow this rate is. Thus, the attractor is, in fact, concentrated (up to exponentially small terms) in a bounded domain. This property clarifies the nature of the finite dimensionality of the attractor in that case. Furthermore, to the best of our knowledge, such an exponential localization holds for all examples for which the finite dimensionality is known.

Example 5.22. We consider the real Ginzburg-Landau equation in $\mathbb{R}^3$,

$$\partial_t u = \Delta_x u + u - u^3 + g. \tag{5.41}$$

We claim that Assumption A. is satisfied if $g \in L^2_b(\mathbb{R}^3)$ and

$$\liminf_{|x|\to+\infty} g(x) > \frac{2}{3\sqrt{3}}. \tag{5.42}$$

Indeed, an elementary analysis shows that

$$[f(v+z) - f(z)].v \geq 0,\ \forall v \in \mathbb{R},\ f(u) := u^3 - u,$$

if and only if $|z| > \frac{2}{\sqrt{3}}$. On the other hand, assumption (5.42) guarantees that the function $W_0(x) := \frac{2}{\sqrt{3}} + \varepsilon$ is a subsolution of (5.41) if ε is small enough and $|x|$ is large enough. Therefore, by the comparison principle, there exists a solution Z_0 of the equilibrium equation (5.30) outside a large ball which satisfies $Z_0(x) > \frac{2}{\sqrt{3}} + \varepsilon$ and Assumption A. is verified. Thus, we see that, under assumption (5.42), the global attractor is spatially localized (in the sense of estimate (5.40)) and, for this reason, it is compact in $L^2_b(\mathbb{R}^3)$ and finite dimensional. As already mentioned in the previous section, when $g = 0$, the associated global attractor is not compact in $L^2_b(\mathbb{R}^3)$ (and is infinite dimensional).

5.4. **The infinite dimensional case: entropy estimates.** Starting from this section, we consider the general case in which the dimension of the global attractor is infinite. Indeed, the simplest way to understand why this dimension must be infinite in general is to consider the real one-dimensional Ginzburg-Landau equation (5.41) with zero external forces ; we also consider the space periodic solutions with period $2L$. Then the associated dynamical system acting on the space $L^2_{\text{per}}([-L, L])$ of $2L$-periodic functions is dissipative and possesses the (finite dimensional) global attractor $\mathcal{A}_L$. Furthermore, we see that, by computing the dimension of the unstable set at $u = 0$,

$$\dim_F \mathcal{A}_L \geq \dim \mathcal{M}^{\text{un}}(0) \geq \frac{2L}{\pi}.$$

On the other hand, since the phase space $L^2_{\text{per}}([-L, L])$ is contained in the phase space $\Phi_b := L^2_b(\mathbb{R})$, we also have the embedding

$$\mathcal{A}_L \subset \mathcal{A},$$

where $\mathcal{A}$ is the (locally compact) global attractor of the equation in the whole space. Thus, since the dimension of $\mathcal{A}_L$ grows as $L \to +\infty$, the dimension of $\mathcal{A}$ cannot be finite.

This simple example shows that, in contrast to bounded domains, we cannot now expect any finite dimensional reduction in general and the dynamics reduced to the global attractor remains infinite dimensional. However, it is intuitively clear that the attractor $\mathcal{A}$ is essentially "thinner" than the initial phase space and, in some proper sense, the reduced dynamics can be described by less degrees of freedom here as well. Now, in order to make this observation rigorous, we need to be able to compare the "thickness" of infinite dimensional sets.

One possible approach to this problem (which is widespread in the approximation theory, see, e.g., [222]) consists in using the Kolmogorov ε-entropy, see Definition 2.28. Indeed, owing to the Hausdorff criterium, the entropy $\mathcal{H}_\varepsilon(X, M)$ is finite for every $\varepsilon > 0$ and every compact subset X of the metric space M. Then, according to formula (2.13), the set X is finite dimensional if and only if

$$\mathcal{H}_\varepsilon(X, M) \le d \log_2 \frac{1}{\varepsilon} + C,$$

for some constants C and d which are independent of $\varepsilon \to 0^+$. So, under this approach, the infinite dimensionality of X just means that the quantity $\mathcal{H}_\varepsilon(X)$ has another, more complicated, asymptotics as $\varepsilon \to 0^+$, which is to be found or estimated.

To the best of our knowledge, the idea of using the Kolmogorov ε-entropy in the theory of attractors was suggested by Chepyzhov and Vishik in [224] in order to study the infinite dimensional *uniform* attractors of nonautonomous dynamical systems in *bounded* domains. However, such an approach appears as especially adapted to the study of equations in unbounded domains and, starting from [58] and [236], the ε-entropy has become one of the most powerful technical tools in view of the study of the locally compact attractors in large and unbounded domains.

We start our considerations by giving several examples of asymptotics of the ε-entropy for some typical infinite dimensional function spaces.

Example 5.23. Let Ω be a regular bounded domain, $M := W^{l_1,p_1}(\Omega)$, and X be the unit ball of the space $W^{l_2,p_2}(\Omega)$, with

$$\frac{1}{p_1} - \frac{l_1}{n} > \frac{1}{p_2} - \frac{l_2}{n}.$$

Then it is well-known that X is (pre)compact in M, so that $\mathcal{H}_\varepsilon(X, M)$ is well-defined and satisfies

$$C_1 \left(\frac{1}{\varepsilon}\right)^{n/(l_2-l_1)} \le \mathcal{H}_\varepsilon(X, M) \le C_2 \left(\frac{1}{\varepsilon}\right)^{n/(l_2-l_1)}, \tag{5.43}$$

where the constants C_1 and C_2 are independent of ε, see, e.g., [222].

Thus, the typical asymptotics of the entropy of Sobolev spaces embeddings are *polynomial* with respect to ε^{-1}. The next example shows the typical behavior of the entropy for classes of *analytic* functions embeddings.

Example 5.24. Let K be the set of all analytic functions f in a ball B_R of radius R in $\mathbb{C}^n$ such that $\|f\|_{\mathcal{C}(B_R)} \le 1$ and let M be the space $\mathcal{C}(B^{\mathrm{Re}})$, where $B^{\mathrm{Re}} := \{z = (z_1, ..., z_n) \in \mathbb{C}^n,\ \mathrm{Im} z_i = 0,\ i = 1,\ ....,\ n,\ |z| \le 1\}$. Thus, K consists of all functions of $\mathcal{C}(B^{\mathrm{Re}})$ which can be holomorphically extended to the ball B_R and for which the $\mathcal{C}$-norm of this extension is less than one. Then

$$C_1 \left(\log_2 \frac{1}{\varepsilon}\right)^{n+1} \le \mathcal{H}_\varepsilon(K, M) \le C_2 \left(\log_2 \frac{1}{\varepsilon}\right)^{n+1}, \tag{5.44}$$

see [135].

In particular, the above asymptotics show, in a mathematically rigorous way, that the set of real analytic functions is indeed essentially smaller than that of functions with finite smoothness $\mathcal{C}^k$.

We now recall that, here, the global attractor is not compact, but only *locally compact*, in the phase space. In order to compare such types of sets, we need to introduce, following [135], the so-called entropy per unit volume or *mean* ε-entropy.

Definition 5.25. Let K be a locally compact set in some uniformly local space $\Phi_b := W_b^{l,p}(\mathbb{R}^n)$. Then, for every hypercube $[-R,R]^n$, the entropy $\mathcal{H}_\varepsilon(K\big|_{[-R,R]^n})$ of the restriction of K to this hypercube is well-defined. By definition, the mean ε-entropy of $\mathcal{K}$ is the following (finite or infinite) quantity:

$$\overline{\mathcal{H}}_\varepsilon(K,\Phi_b) := \limsup_{R\to+\infty} \frac{1}{(2R)^n}\mathcal{H}_\varepsilon(K\big|_{[-R,R]^n}). \tag{5.45}$$

As we will see below, the next example is crucial for the theory of attractors in unbounded domains.

Example 5.26. Let $\mathbb{B}_\sigma(\mathbb{R}^n)$, $\sigma\in\mathbb{R}_+$, be the subspace of $L^\infty(\mathbb{R}^n)$ consisting of all functions u whose Fourier transform $\widehat{u}$ has a compact support,

$$\operatorname{supp}\widehat{u}\subset B_0^\sigma := \{\xi\in\mathbb{R}^n,\ \|\xi\|\le\sigma\}.$$

It is well-known that the space $\mathbb{B}_\sigma(\mathbb{R}^n)$ consists of entire functions (i.e., functions which are analytic on the whole space $\mathbb{R}^n$) with an exponential growth. Furthermore, if $\mathcal{B}(\sigma)$ is the unit ball in this space (endowed with the usual L^∞-metric), then

$$C_1\log_2\frac{1}{\varepsilon} \le \overline{\mathcal{H}}_\varepsilon(\mathcal{B}(\sigma)) \le C_2\log_2\frac{1}{\varepsilon}, \tag{5.46}$$

where C_1 and C_2 depend on σ, but are independent of ε, see [135]. Moreover, we have, concerning the entropy of the restrictions $\mathcal{B}(\sigma)\big|_{[-R,R]^n}$,

$$C_1R^n\log_2\frac{1}{\varepsilon} \le \mathcal{H}_\varepsilon(\mathcal{B}(\sigma)\big|_{[-R,R]^n}) \le C_2(R+\log_2\frac{1}{\varepsilon})^n\log_2\frac{1}{\varepsilon}, \tag{5.47}$$

where C_1 and C_2 are independent of ε and R. We can note that these estimates are sharp for $R\gg\log_2\frac{1}{\varepsilon}$ and for $R\sim\log_2\frac{1}{\varepsilon}$, but, for $R\ll\log_2\frac{1}{\varepsilon}$, the lower bound is far from being optimal and can be corrected as follows:

$$\mathcal{H}_\varepsilon(\mathcal{B}(\sigma)\big|_{[-R,R]^n}) \ge C_R\left(\frac{\log_2\frac{1}{\varepsilon}}{(\log_2\log_2 1/\varepsilon)^n}\right)^{n+1}, \tag{5.48}$$

where C_R depends on R, but is independent of ε. The proof of estimates (5.47) and (5.48) can be found in [239].

Finally, we also mention the analogue of Example 5.23 for the uniformly local case.

Example 5.27. Let the exponents l_i and p_i, $i=1,\ 2$, be the same as in Example 5.23. Let also K be the unit ball in the space $W_b^{l_2,p_2}(\mathbb{R}^n)$ and set $M := W_b^{l_1,p_1}(\mathbb{R}^n)$. Then

$$C_1\left(\frac{1}{\varepsilon}\right)^{n/(l_2-l_1)} \le \overline{\mathcal{H}}_\varepsilon(K,M) \le C_2\left(\frac{1}{\varepsilon}\right)^{n/(l_2-l_1)}, \tag{5.49}$$

where the constants C_1 and C_2 are independent of ε. Actually, these estimates immediately follow from (5.43).

We are now ready to formulate the *universal* entropy estimates for the *uniformly local* attractors of dissipative systems in unbounded domains which, as we will see below, are natural generalizations of the fractal dimension estimates to systems in unbounded domains. These estimates have the following form:

$$\mathcal{H}_\varepsilon(\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}, \Phi_b(\Omega\cap B^R_{x_0})) \le C\,\mathrm{vol}(\Omega\cap B^{R+L\log_2 1/\varepsilon}_{x_0})\log_2\frac{1}{\varepsilon}, \tag{5.50}$$

where $\mathrm{vol}(\cdot)$ denotes the usual Lebesgue measure in $\mathbb{R}^n$ and the constants C and L are independent of R, x_0, and ε. Thus, (5.50) gives upper bounds on the entropy of the restrictions of the attractor $\mathcal{A}$ to all bounded subdomains $\Omega\cap B^R_{x_0}$ which depend on the three parameters R, x_0, and ε.

The above formula has a general nature, independent of the concrete class of dissipative systems considered, and has been verified for various classes of reaction-diffusion systems (see [79], [236], [239], and [240]), for damped wave equations (see [238]), and even for elliptic boundary value problems in unbounded domains (see [169]). Indeed, roughly speaking, it is sufficient, in order to prove such estimates, to verify a *weighted* analogue of the "parabolic" smoothing property (2.14),

$$\|S(1)u_1 - S(1)u_2\|_{W^{1,2}_{\phi_\mu,x_0}} \le L\|u_1-u_2\|_{L^2_{\phi_\mu,x_0}},\ u_1,\ u_2\in\mathcal{A}, \tag{5.51}$$

for some fixed μ and every x_0 in Ω (or its "hyperbolic" analogues (3.2) and (3.3)), see [238] and [239]. Thus, these entropy estimates are also based on rather simple and general (weighted) energy estimates and do not use any specific property of the dissipative system under study. This somehow clarifies the nature of their universality. We also mention that the upper entropy estimates are sharp with respect to the three parameters R, x_0, and R (appropriate examples of lower bounds will be given in the next subsections).

In order to further clarify these universal entropy estimates, we conclude this subsection by considering the most interesting particular cases and by comparing them with the typical asymptotics given above.

Example 5.28. Let Ω be a bounded domain. Then $\mathrm{vol}(\Omega\cap B^R_{x_0}) = \mathrm{vol}(\Omega)$ if R is large enough. Therefore, (5.50) gives

$$\mathcal{H}_\varepsilon(\mathcal{A}) \le C\,\mathrm{vol}(\Omega)\log_2\frac{1}{\varepsilon}.$$

Thus, in the case of bounded domains, the entropy formula allows to recover the standard result on the finite dimensionality of the global attractor and reflects in a correct way the typical dependence of the dimension on the size of the domain ($\dim_F\mathcal{A}\sim\mathrm{vol}(\Omega)$, see [23]). However, even in that case, the entropy estimate gives some additional information which may be important, especially for large bounded domains, namely, it allows to estimate the entropy of the restrictions $\mathcal{A}|_{B^1_{x_0}}$ and, thus, to study the "thickness" of the attractor with respect to the position inside the domain.

Example 5.29. We now assume that $\Omega=\mathbb{R}^n$. Then $\mathrm{vol}(\Omega\cap B^R_{x_0}) = cR^n$ and (5.50) reads

$$\mathcal{H}_\varepsilon(\mathcal{A}|_{B^R_{x_0}}) \le C(R+L\log_2\frac{1}{\varepsilon})^n\log_2\frac{1}{\varepsilon}. \tag{5.52}$$

We see that this estimate coincides with the upper bound (5.47) for the space $\mathbb{B}_\sigma(\mathbb{R}^n)$ of entire functions and, in particular, dividing (5.52) by R^n and passing to the limit $R\to+\infty$, we also obtain the analogue of (5.46),

$$\overline{\mathcal{H}}_\varepsilon(\mathcal{A}) \le C\log_2\frac{1}{\varepsilon} \tag{5.53}$$

(for the one-dimensional real Ginzburg-Landau and damped wave equations, this estimate was obtained independently in [57] and [59]). Thus, we see that the "thickness" of the attractor $\mathcal{A}$ is of the order of that of the class $\mathbb{B}_\sigma(\mathbb{R}^n)$ of entire functions and is essentially less than that of the class of finite smoothness, see Examples 5.23 and 5.27 (and, in particular, it is essentially less than the thickness of any absorbing set). However, even when all the terms in the equations are entire, the attractor is usually not entire (the simplest example is the real Ginzburg-Landau equation) and only the analyticity in a strip $\mathbb{R}_\mu := i[-\mu,\mu]^n \times \mathbb{R}^n$ takes place. The mean entropy for such classes of functions has an asymptotics of the form $(\log_2 \frac{1}{\varepsilon})^{1+p}$, for some $p > 0$, and is *worse* than (5.53). Therefore, even in the real analytic case, the nature of the universal entropy estimates cannot be explained by regularity arguments and reflects the *dynamical* reduction of the number of degrees of freedom by the dissipative dynamics. Furthermore, we emphasize here that the analyticity is *not necessary* for the validity of the entropy estimates. In particular, these estimates hold for the reaction-diffusion system (5.8) under the assumptions of Theorem 5.5, see [240]. In that case, the regularity of f and g only yields that $\mathcal{A} \subset W_b^{2,2}(\Omega)^k$, so that the best entropy estimates which can be extracted from this regularity is *polynomial* with respect to ε^{-1} ($\varepsilon^{-3/2}$ to be more precise).

Remark 5.30. Estimates (5.50) can be rewritten in the more compact equivalent form

$$\mathcal{H}_\varepsilon(\mathcal{A}, \Phi_{e^{-|x-x_0|}}) \le C(\log_2 \frac{1}{\varepsilon})^{n+1}, \tag{5.54}$$

i.e., the entropy of the attractor can be equivalently computed in *weighted* phase spaces with the *exponential* weight functions $e^{-|x-x_0|}$. In particular, in the spatially homogeneous case, the sole space $\Phi_{e^{-|x|}}$ with $x_0 = 0$ is sufficient. Indeed, using the simple "summation" properties of the Kolmogorov entropy, one can easily show that (5.54) implies (5.50). Actually, estimate (5.50) has first been obtained precisely in this form, see [236]. However, we prefer to use the more complicated formulation (5.50) in order to avoid artificial weight functions in the formulation and to prevent from the confusing feeling that $\log_2$ terms in the entropy estimates are related to the artificial choice of exponential weight functions and are, thus, also artificial.

5.5. **Infinite dimensional exponential attractors.** In this subsection, we discuss the theory of *exponential* attractors for systems in unbounded domains, following essentially [75]. Since even the global attractor (which is always contained in an exponential attractor) is now infinite dimensional, one cannot expect an exponential attractor to be finite dimensional. Thus, this assumption must be relaxed in Definition 3.4. On the other hand, this assumption cannot be simply omitted, since, otherwise, any compact absorbing set would be an exponential attractor, which does not make sense. In any case, one wants to make an exponential attractor as small as possible (i.e., to add a "minimal number" of new artificial points to the global attractor) and, therefore, it is natural to use the Kolmogorov entropy to control its "thickness" ; in particular, it is natural to look for an exponential attractor which satisfies the *universal* entropy estimates (5.50) known for global attractors (an analogous idea was also used in [72] for infinite dimensional exponential attractors for nonautonomous problems in bounded domains).

Another difference, when compared with bounded domains, is the fact that the locally compact global attractor only attracts the bounded sets in the *local* topology (counterexamples for the attraction in the uniform topology of the initial phase space can be easily constructed, see [239]). Thus, one would expect the same type of attraction for exponential attractors as well. However, as shown in [75], this drawback of the theory of global attractors can be overcome by constructing proper exponential attractors and one can obtain the (exponential) attraction in the topology of the initial phase space.

Thus, based on the above considerations, the following modifications of the concept of an exponential attractor are natural.

Definition 5.31. Let $S(t)$ be a dissipative semigroup in the uniformly local Sobolev space $\Phi_b := W^{l,p}_b(\Omega)$, for a regular unbounded domain Ω. A set $\mathcal{M}$ is an (infinite dimensional) exponential attractor for the semigroup $S(t)$ if the following conditions are satisfied:
(i) it is bounded in Φ_b and compact in Φ_{loc} ;
(ii) it is positively invariant, $S(t)\mathcal{M} \subset \mathcal{M}$, $t \geq 0$;
(iii) it attracts exponentially the bounded subsets of Φ_b in the *uniform* topology of Φ_b, i.e., there exist a monotonic function Q and a positive constant α such that, for every bounded subset $B \subset \Phi_b$, the following estimate:

$$\operatorname{dist}_{\Phi_b}(S(t)B, \mathcal{M}) \leq Q(\|B\|_{\Phi_b})e^{-\alpha t} \tag{5.55}$$

holds, for every $t \geq 0$;
(iv) it satisfies the universal entropy estimates (5.50), for some positive constants C and L which are independent of R, x_0, and ε.

The following theorem, proven in [75], gives the existence of such an object for the reaction-diffusion system (5.8).

Theorem 5.32. *Let the assumptions of Theorem 5.5 be satisfied. Then the associated semigroup $S(t)$ possesses an infinite dimensional exponential attractor $\mathcal{M}$ in the phase space $\Phi_b = L^2_b(\mathbb{R}^3)^k$ in the sense of the above definition.*

This result is, to the best of our knowledge, the only one on the existence of infinite dimensional exponential attractors in unbounded domains. However, its construction mainly exploits the smoothing estimate (5.51) on the difference of two solutions, but does not involve the specific properties of the reaction-diffusion system (5.8). Thus, we expect that the existence of such an exponential attractor is a general fact which can be established for all dissipative systems in unbounded domains for which the validity of the universal entropy estimates is satisfied (for the global attractor).

We conclude this subsection by considering the problem of the approximation of equations in an unbounded domain by appropriate equations in large bounded domains. It is well-known that the global attractor is not robust with respect to this singular limit and can change drastically. To illustrate this, we consider the one-dimensional real Ginzburg-Landau equation with a transport term,

$$\partial_t u = \partial_x^2 u - L\partial_x u + u - u^3,\ L > 2, \tag{5.56}$$

and approximate it by analogous equations in the bounded domains $\Omega_R := [-R, R]$, endowed with Dirichlet boundary conditions. Then, as shown in [75], the global attractor $\mathcal{A}_R$ for the approximate problem is trivial for every (finite) R, $\mathcal{A}_R = \{0\}$. However, the limit attractor for $R = +\infty$ is completely nontrivial and has infinite dimension and infinite topological entropy. Thus, this approximation problem seems to be very difficult as far as global attractors are concerned and, probably, cannot be solved in a reasonable way.

In contrast to this, as the following theorem (proven in [75]) shows, this approximation problem has a natural and adequate solution in terms of exponential attractors.

Theorem 5.33. *Let the reaction-diffusion system* (5.8) *in the unbounded domain* $\Omega = \mathbb{R}^3$ *satisfy the assumptions of Theorem 5.5 and let* $S_\infty(t)$ *be the associated dissipative semigroup acting on* $\Phi_b = L^2_b(\mathbb{R}^3)^k$. *We also consider the same problem in the large, but bounded, ball* $\Omega_R = B^R_0$ *in* $\mathbb{R}^3$ *with Dirichlet boundary conditions and we let* $S_R(t)$ *be the dissipative semigroup associated with this problem on* $\Phi_b(R) := L^2_b(B^R_0)^k$. *Then there exists a family of closed bounded sets* $\mathcal{M}_R$, $R \in [R_0, +\infty]$, *of* $\Phi_b(R)$ *such that, for every finite* R, $\mathcal{M}_R$ *is an exponential attractor for* $S_R(t)$ *in the usual sense and, for* $R = +\infty$, *the corresponding set is an infinite dimensional exponential attractor for* $S_\infty(t)$. *Furthermore, the following additional properties are satisfied:*
1) the sets $\mathcal{M}_R$ *are uniformly (with respect to* R*) bounded in* $\Phi_b(R)$ *;*
2) there exist a positive constant α *and a monotonic function* Q *such that, for every* R *and every bounded subset* B *of* $\Phi_b(R)$,

$$\operatorname{dist}_{\Phi_b(R)}(S_R(t)B, \mathcal{M}_R) \le Q(\|B\|_{\Phi_b(R)})e^{-\alpha t}$$

(uniform exponential attraction) ;
3) uniform entropy estimates:

$$\mathcal{H}_\varepsilon(\mathcal{M}_R\big|_{\Omega_R \cap B^r_{x_0}}) \le C \operatorname{vol}(\Omega_R \cap B^{r+L\log_2 1/\varepsilon}_{x_0}) \log_2 \frac{1}{\varepsilon},$$

where the constants C *and* L *are independent of* R, $r \le R$, x_0, *and* ε *;*
4) the attractors $\mathcal{M}_R$ *tend to* $\mathcal{M}_\infty$ *in the following sense:*

$$\operatorname{dist}_{\mathrm{sym},\Phi_b(r)}(\mathcal{M}_R\big|_{\Omega_r}, \mathcal{M}_\infty\big|_{\Omega_r}) \le Ce^{-\gamma(R-r)}, \tag{5.57}$$

where the positive constants C *and* γ *are independent of* R *and* $r \le R$.

In particular, estimate (5.57) shows that, if we want to approximate the attractor $\mathcal{M}_\infty$ with an accuracy ε inside the ball Ω_r, it is sufficient to construct the usual finite dimensional exponential attractor $\mathcal{M}_{R(\varepsilon)}$ for the reaction-diffusion problem in a ball of radius $R(\varepsilon) = r + L\log_2 \frac{1}{\varepsilon}$. We also note that one cannot expect that $\mathcal{M}_R$ approximates $\mathcal{M}_\infty$ in the whole ball Ω_R, since the additional boundary conditions on $\partial\Omega_R$ for the approximate problems should be satisfied. Nevertheless, estimate (5.57) also shows that the influence of the boundary and the boundary conditions decays exponentially with respect to the distance to the boundary (in agreement with our physical intuition).

5.6. **Complexity of space-time dynamics: entropy theory.** In the previous subsections, we gave sharp upper bounds on the Kolmogorov ε-entropy which characterize the "size" or "thickness" of the attractors. Starting from this subsection, we describe some general *dynamical* properties of a dissipative system in a large or an unbounded domain, restricted to its global attractor.

As already mentioned, contrary to bounded domains, the reduced dynamics now remains infinite dimensional and dynamical effects of essentially new higher levels of complexity (which are not observable in the classical finite dimensional theory of dynamical systems) may appear. In particular, the Lyapunov and topological entropy dimensions for such dynamics are usually infinite, see [242]. For this reason, most ideas and methods from the classical theory fail (at least in a straightforward way) to describe these new types of dynamics. Thus, a new theory, which is only developing now, is required.

Another essential difference from the classical theory is the fact that, in addition to complicated temporal dynamics, the solutions may have very irregular (chaotic) spatial structures, i.e., the so-called *spatial* chaos may appear. Furthermore, as a result of the chaotic temporal evolution of spatially chaotic structures, the so-called *space-time* chaos may appear.

The most studied case is that of spatial chaos which is already observable on the set of temporal equilibria of the dynamical system. Indeed, the equilibria satisfy some *elliptic* equation of the form

$$a\Delta_x u - f(u) + g = 0, \tag{5.58}$$

so that the number of independent variables is reduced by one, which is an essential simplification. So, in the particular case of one space variable, (5.58) becomes an ODE and the (spatially) chaotic behaviors of its solutions can be successfully studied by classical theories (homoclinic bifurcation analysis, variational methods for constructing complex solutions, ..., see [4], [130], [190], and the references therein). Furthermore, many interesting multi-dimensional problems in *cylindrical* domains can be reduced to this one-dimensional one by using the so-called spatial center manifold reduction, see [3], [31], [131], [166], [167], and the references therein. Also, direct generalizations of the techniques from ODEs to multi-dimensional elliptic PDEs of the form (5.58) (e.g., the shadowing lemma, variational methods, ...) are available, see, e.g., [8] and [197]. We finally mention a rather simple and very effective method to construct spatially chaotic patterns which are, in addition, stable with respect to the time developed in [2], [14], [15], and [17]. This method is based on the study of homotopy properties of the level sets of the nonlinear term f and related energy functionals and is somehow close to the variational methods, see the recent survey [16] for more details.

We however note that all the above mentioned methods give examples of spatial chaotic behaviors with *finite* topological entropy (usually related to the Bernoulli scheme $\mathbb{M} := \{0,1\}^{\mathbb{Z}}$ or $\mathbb{M}_n := \{0,1\}^{\mathbb{Z}^n}$ in the multi-dimensional case), which is typical of ODEs, but *does not* capture the "whole" complexity of the spatial dynamics, since its topological entropy is usually infinite, see [57], [169], [240], and [242]. In order to overcome this problem, an alternative method, related to the so-called infinite dimensional essentially unstable manifolds and the Kotelnikov formula, which gives a description of the spatial chaos in terms of the Bernoulli scheme $\mathbb{M}_n := [0,1]^{\mathbb{Z}^n}$ with a continuous number of symbols and an infinite topological entropy, has been suggested in [240]. This method will be discussed in more details in the next subsection.

Now, the case of full space-time dynamics is essentially less understood. However, even here, some reasonable progress related to the so-called Sinai-Bunimovich space-time for continuous media has recently been obtained. This topic will be discussed in a subsequent subsection.

In the remaining of this subsection, we discuss (following essentially [242] and [244]) topological and smooth invariants for the space-time dynamics which are strongly based on the universal entropy estimates on the global attractor and give useful "upper bounds" on the possible complexity of the dynamics. For simplicity, we restrict ourselves to $\Omega = \mathbb{R}^n$ and to spatially homogeneous dissipative systems (i.e., the coefficients and external forces do not depend explicitly on x ; this constitutes a natural analogue of "autonomous" systems for space-time dynamics). In that case, the attractor $\mathcal{A}$ possesses a very important additional structure, namely, the group $\{T_h,\ h \in \mathbb{R}^n\}$ of spatial shifts acts on it,

$$T_h\mathcal{A} = \mathcal{A},\ h \in \mathbb{R}^n,\ T_hu(x) := u(x+h),\ h,\ x \in \mathbb{R}^n. \tag{5.59}$$

Thus, in addition to the temporal evolution semigroup $S(t)$, we also have the action of the spatial shifts group T_h on the attractor which, obviously, commutes with $S(t)$. As a result, the extended $(n+1)$-parametric spatio-temporal semigroup $\mathbb{S}(t,h)$,

$$\mathbb{S}(t,h)\mathcal{A} = \mathcal{A},\ \mathbb{S}(t,h) := S(t)\circ T_h,\ t \geq 0,\ h \in \mathbb{R}^n, \tag{5.60}$$

acts on the attractor.

Following [240] and [242], we will treat this multi-parametric semigroup as a dynamical system with multi-dimensional "time" $(t,h) \in \mathbb{R}_+ \times \mathbb{R}^n$, which describes the space-time behavior of the dissipative system under study, and we will describe the space-time chaos by finding appropriate dynamical invariants of this action. In particular, under this approach, the spatial x and temporal t directions are treated in a unified way. Some justifications for such a unification will be given at the end of the next subsection when giving examples for which these directions are indeed equivalent (in spite of the fact that they seem essentially different from an intuitive point of view).

In order to introduce these invariants, we need to make some reasonable assumptions on the attractor $\mathcal{A}$, namely,
(i) it is locally compact on some uniformly local Sobolev phase space $\Phi_b = \Phi_b(\mathbb{R}^n)$ which is embedded into $L^\infty(\mathbb{R}^n)$;
(ii) the dissipative system is spatially homogeneous, i.e., the extended semigroup (5.60) acts on the attractor ;
(iii) the universal entropy estimates (5.52) hold ;
(iv) the evolution semigroup $S(t)$ is Lipschitz continuous in a weighted space $\Phi_{e^{-\varepsilon|x|}}(\mathbb{R}^n)$ on the attractor,

$$\|S(t)u_0 - S(t)u_1\|_{\Phi_{e^{-\varepsilon|x|}}} \leq Ce^{kt}\|u_0 - u_1\|_{\Phi_{e^{-\varepsilon|x|}}},\ u_0,\ u_1 \in \mathcal{A},\ t \geq 0, \tag{5.61}$$

for some fixed $\varepsilon > 0$ and positive constants C and L which are independent of t, u_0, and u_1.

We note that the assumption $\Phi_b \subset L^\infty(\mathbb{R}^n)$ is not essential and was introduced in [242] just to avoid additional technicalities.

The first, and most natural, dynamical invariant of the action of (5.60) is its topological entropy, see [130] for details.

Definition 5.34. We endow the attractor $\mathcal{A}$ with the topology of $L^\infty_{e^{-|x|}}(\mathbb{R}^n)$ and define, for every $R \in \mathbb{R}_+$, an equivalent metric d_R on $\mathcal{A}$ by

$$d_R(u_0,u_1) := \sup_{(t,h)\in R\cdot[0,1]^{n+1}} \|\mathbb{S}(t,h)u_0 - \mathbb{S}(t,h)u_1\|_{L^\infty_{e^{-|x|}}},\ u_0,\ u_1 \in L^\infty_{e^{-|x|}}(\mathbb{R}^n). \tag{5.62}$$

Since $\mathcal{A}$ is bounded in Φ_b and compact in Φ_{loc}, it is compact in $L^\infty_{e^{-|x|}}(\mathbb{R}^n)$ (thanks to the embedding $\Phi_b \subset L^\infty(\mathbb{R}^n)$) and, therefore, it is also compact in the metric of d_R and the Kolmogorov ε-entropy $\mathcal{H}_\varepsilon(\mathcal{A}, d_R)$ is well-defined. Then the topological entropy of the action of $\mathbb{S}(t,h)$ on $\mathcal{A}$ is the following quantity:

$$h_{\mathrm{top}}(\mathbb{S}(t,h),\mathcal{A}) := \lim_{\varepsilon\to 0^+} \limsup_{R\to+\infty} \frac{1}{R^{n+1}}\mathcal{H}_\varepsilon(\mathcal{A}, d_R). \tag{5.63}$$

Remark 5.35. Although this definition depends on the specific metric, it is well-known (see, e.g., [130]) that the topological entropy only depends on the *topology* and is independent of the choice of the equivalent metric on $\mathcal{A}$. Furthermore, it is also not difficult to show that the space $L^\infty(\mathbb{R}^n)$ in the definition of d_R can be replaced by $\Phi_{e^{-|x|}}(\mathbb{R}^n)$, see [242]. However, we define d_R by special exponentially weighted metrics keeping in mind other invariants which will depend on this choice.

We now recall that the topological entropy for one-parametric evolution semigroups $S(t)$ is usually finite in the classical theory of dynamical systems. The following theorem, proven in [242], can be considered as a generalization of this principle to spatially extended systems.

Theorem 5.36. *Let the attractor $\mathcal{A}$ satisfy the above conditions. Then the topological entropy of the action of the extended space-time semigroup $\mathbb{S}(t,h)$ is finite,*

$$h_{\text{top}}(\mathbb{S}(t,h),\mathcal{A}) < +\infty.$$

Furthermore, it coincides with the so-called topological entropy per unit volume (introduced by Collet and Eckmann, see [57] *and* [59]*) and can be computed by the following simplified formula:*

$$h_{\text{top}}(\mathbb{S}(t,h),\mathcal{A}) = \lim_{\varepsilon\to0^+}\overline{\mathcal{H}}_\varepsilon(\mathcal{K},L^\infty(\mathbb{R}^{n+1})) = \\ = \lim_{\varepsilon\to0^+}\lim_{R\to+\infty}\frac{1}{R^{n+1}}\mathcal{H}_\varepsilon(\mathcal{K}\big|_{R\cdot[0,1]^{n+1}},L^\infty(R\cdot[0,1]^{n+1})), \tag{5.64}$$

where $\mathcal{K}\subset L^\infty(\mathbb{R},\Phi_b)\subset L^\infty(\mathbb{R}^{n+1})$ is the set of all bounded trajectories of the dissipative system (the so-called kernel in the terminology of Chepyzhov and Vishik, see [51]*) and $\overline{\mathcal{H}}_\varepsilon(\mathcal{K})$ denotes its mean ε-entropy, see Definition 5.25.*

Remark 5.37. It can also be shown that any sufficiently regular bounded subdomain $V\subset\mathbb{R}^n$ can be chosen as a "window" instead of $[0,1]^{n+1}$ in (5.64), namely,

$$\overline{\mathcal{H}}_\varepsilon(\mathcal{K},L^\infty(\mathbb{R}^{n+1}) = \lim_{R\to+\infty}\frac{1}{\text{vol}(R\cdot V)}\mathcal{H}_\varepsilon(\mathcal{K}\big|_{R\cdot V},L^\infty(R\cdot V)).$$

We now note that the complexity of the dynamical behaviors of the extended system (5.60) may essentially differ in different directions. In particular, for the so-called extended gradient systems, see [100], [216], [241], and [242], the space-time topological entropy $h_{\text{top}}(\mathbb{S}(t,h),\mathcal{A})$ vanishes, due to the simpler *temporal* dynamics induced by the gradient structure, which however does not reduce the complexity of the *spatial* dynamics. In order to capture these directional dynamical effects, it seems natural to consider the k-parametric subsemigroups $\mathbb{S}^{V_k}(t,h)$ of the extended space-time dynamical system $\mathbb{S}(t,h)$ generated by the restrictions of the argument (t,h) to k-dimensional linear subspaces of the space-time $\mathbb{R}^{n+1}$,

$$\mathbb{S}^{V_k}(t,h) := \{\mathbb{S}(t,h),\ (t,h)\in V_k,\ t\ge0,\ h\in\mathbb{R}^n\}, \tag{5.65}$$

and to study their invariants with respect to the linear space V_k and its dimension k. For instance, the choice $V_1=\mathbb{R}_t$ gives the purely temporal dynamics, $\mathbb{S}^{V_1}(t,h)=S(t)$, the choice $V_n=\mathbb{R}^n_x$ gives the spatial dynamics and spatial chaos, $\mathbb{S}^{V_n}(t,h)=T_h$, and the intermediate choices of planes V_k describe the interactions between the temporal and the spatial chaotic modes, e.g., the complexity of profiles of traveling waves.

In particular, it seems natural to study the topological entropies $h_{\text{top}}(\mathbb{S}^{V_k}(t,h),\mathcal{A})$ of the action of these semigroups on the attractor (i.e., the directional topological entropies introduced by Milnor for cellular automata, see [172]). Now, in contrast to the cellular automata, these entropies are typically infinite for dissipative dynamics if $k < n+1$. In order to overcome this difficulty, it was suggested in [242] to modify the definition of the topological entropy by taking into account the typical rate of divergence of the mean entropy as $\varepsilon \to 0^+$.

Definition 5.38. Let V_k be a k-dimensional plane in $\mathbb{R}^{n+1}$ and let $[0,1]^k_{V_k}$ be its unit hypercube generated by some orthonormal basis in V_k. Analogously to (5.62), for every $R > 0$, we introduce a new metric $d_R^{V_k}$ by

$$d_R^{V_k}(u_0, u_1) := \sup_{(t,h)\in R\cdot[0,1]^k_{V_k}} \|\mathbb{S}(t,h)u_0 - \mathbb{S}(t,h)u_1\|_{L^\infty_{e^{-|x|}}}, \ u_0,\ u_1 \in L^\infty_{e^{-|x|}}(\mathbb{R}^n). \tag{5.66}$$

Then a modified topological entropy $\widehat{h}_{\text{top}}(\mathbb{S}^{V_k})$ of the action of the directional dynamical system $\mathbb{S}^{V_k}(t,h)$ on the attractor is given by the following quantity:

$$\widehat{h}_{\text{top}}(\mathbb{S}^{V_k}(t,h), \mathcal{A}) := \limsup_{\varepsilon\to 0^+} \left(\log_2 \frac{1}{\varepsilon}\right)^{k-n-1} \limsup_{R\to+\infty} \frac{1}{R^k}\mathcal{H}_\varepsilon(\mathcal{A}, d_R^{V_k}), \tag{5.67}$$

see [242] for details.

We see that the above definition differs from the classical one by the presence of a normalizing factor $(\log_2 1/\varepsilon)^{k-n-1}$ which guarantees that this quantity is finite. In particular, if the modified entropy is strictly positive (examples of such cases will be given in the next subsection), then the corresponding classical topological entropy must be infinite.

The next theorem from [242] establishes the finiteness of these modified quantities and gives some of their basic relations.

Theorem 5.39. *Let the assumptions of Theorem 5.36 hold. Then, for every k and every k-dimensional plane V_k, the associated modified entropy $\widehat{h}_{\text{top}}(\mathbb{S}^{V_k})$ is finite,*

$$\widehat{h}_{\text{top}}(\mathbb{S}^{V_k}(t,h), \mathcal{A}) < +\infty.$$

Furthermore, if $V_{k_1} \subset V_{k_2}$, then

$$\widehat{h}_{\text{top}}(\mathbb{S}^{V_{k_2}}(t,h), \mathcal{A}) \le L^{k_2-k_1}\widehat{h}_{\text{top}}(\mathbb{S}^{V_{k_1}}(t,h), \mathcal{A}), \tag{5.68}$$

where L is some constant which is independent of k_i and V_{k_i}, $i = 1, 2$.

Remark 5.40. Inequalities (5.68) can be considered as a natural generalization of the classical inequality relating the fractal dimension to the topological entropy to the spatially extended case. Indeed, in the case of an ODE (without spatial directions), we have $n = 0$ and, as it can easily be shown, $\widehat{h}_{\text{top}}(\mathbb{S}^{V_0})$ coincides with the fractal dimension of $\mathcal{A}$, $\widehat{h}_{\text{top}}(\mathbb{S}^{V_1})$ gives the classical topological entropy, and (5.68) reads

$$h_{\text{top}}(S(t), \mathcal{A}) \le L \dim_F \mathcal{A},$$

which coincides with a classical inequality, see [130]. Roughly speaking, the invariant $\widehat{h}_{\text{top}}(\mathbb{S}^{V_k})$ has the structure of a topological entropy in the directions of V_k and of a (generalized) fractal dimension in the orthogonal directions, see [242] and [244] for details.

Remark 5.41. Inequalities (5.68) are particularly useful to verify whether or not some modified directional entropy is strictly positive. In particular, the positivity of the full space-time entropy $h_{\rm top}(\mathbb{S}(t,h),\mathcal{A})$ (which, e.g., corresponds to the presence of the so-called Sinai-Bunimovich space-time chaos in the system, see the next subsections) implies that *all* the above modified entropies are strictly positive. On the contrary, if

$$\lim_{\varepsilon\to 0^+}\left(\log_2\frac{1}{\varepsilon}\right)^{-n-1}\mathcal{H}_\varepsilon(\mathcal{A},L^\infty_{e^{-|x|}}(\mathbb{R}^n))=0,$$

then all the above modified entropies automatically vanish.

Remark 5.42. Analogues of the simplified formulas (5.64) to compute the directional entropies are also deduced in [242]. In particular, for the spatial dynamics $V_n=\mathbb{R}^n_x$, we have a particularly simple formula,

$$\widehat{h}_{\rm sp}(\mathcal{A}):=\widehat{h}_{\rm top}(T_h,\mathcal{A})=\limsup_{\varepsilon\to 0^+}\left(\log_2\frac{1}{\varepsilon}\right)^{-1}\overline{\mathcal{H}}_\varepsilon(\mathcal{A}). \tag{5.69}$$

Thus, contrary to the usual Kolmogorov ε-entropy which measures the "thickness" of a set (of the attractor here), the mean ε-entropy is more related to the complexity of its spatial structure.

Remark 5.43. To conclude this subsection, it is worth noting that, in contrast to the full space-time topological entropy, the directional entropies introduced above are not *topological* invariants, but only Lipschitz continuous invariants (like the fractal dimension), due to the presence of the term $\log_2\frac{1}{\varepsilon}$ in the definition. Furthermore, it is possible to show that there is no topological invariant which is typically finite and strictly positive when $k<n$. When $k=n$ (e.g., for spatial dynamics and spatial chaos), such an invariant exists, namely, the so-called mean topological dimension introduced in [145] (for the Bernoulli scheme with a continuous number of symbols) which can be obtained as in Definition 5.38, but by taking the additional infimum with respect to all metrics which induce the local topology on $\mathcal{A}$, see [242] for details.

5.7. **Lower bounds on the entropy, the Kotelnikov formula, and spatial chaos.** In this subsection, we discuss, following essentially [240] and [242], the derivation of lower bounds on the Kolmogorov ε-entropy and related lower bounds on the complexity of the dynamics. We start by recalling that, in bounded domains, one usually estimates the dimension of the global attractor from below by finding a proper equilibrium with a large instability index and by constructing the associated unstable set. Since an unstable set always belongs to the global attractor, the instability index of this equilibrium then gives a lower bound on its dimension, see [23], [220], and the references therein.

Thus, it seems natural to try to extend this theory to unbounded domains and to obtain lower bounds on the ε-entropy from the existence of large (infinite dimensional) unstable sets for appropriate equilibria. However, the main difficulty here is that, contrary to bounded domains, the spectrum of an equilibrium usually consists of continuous curves (or continuous sets) and does not have reasonable spectral gaps in order to use the usual theory of unstable manifolds. As a consequence, the unstable set of an equilibrium is usually *not* a manifold and a straightforward extension fails.

This obstacle can be overcome by using (following [78], [240], and [242]) the so-called essentially unstable manifolds which consist of the initial data of the solutions which tend to an equilibrium as $t\to-\infty$ with a sufficiently fast exponential rate. As the following theorem (proven in [242]) shows, no spectral gap condition is required for the existence of such manifolds.

Theorem 5.44. *Let X be a Banach space and let $S : X \to X$ be a nonlinear map satisfying*

$$S(u) = S_0 u + K(u), \quad K \in \mathcal{C}^{1+\alpha}(X, X), \ K(0) = K'(0) = 0, \tag{5.70}$$

for some $0 < \alpha \le 1$ and some linear operator $S_0 \in \mathcal{L}(X, X)$. Let then the linearization S_0 of the operator S at zero be exponentially unstable, i.e.,

$$r(S_0) := \sup |\sigma(S_0, X)| > 1,$$

where $\sigma(L, V)$ denotes the spectrum of the operator L in the space V. We finally assume that there exists a closed invariant subspace X_+ of S_0 such that

$$\inf |\sigma(S_0\big|_{X_+}, X_+)| > \theta_0 > 1, \quad \theta_0^{1+\alpha} > r(S_0). \tag{5.71}$$

Then there exists a ball $\mathcal{B} := B_{X_+}(0, \rho)$ and a $\mathcal{C}^{1,\alpha}$ map $\mathbb{V} : \mathcal{B} \to X$ such that

$$\|\mathbb{V}(x_+) - x_+\|_X \le C\|x_+\|_X^{1+\alpha}, \quad x_+ \in \mathcal{B}.$$

Furthermore, for every $u_0 \in \mathbb{V}(\mathcal{B})$, there exists a backward trajectory $\{u(n)\}_{n \in \mathbb{Z}_-}$ such that

$$u(n+1) = S(u(n)), \ u(0) = u_0, \ \|u(n)\|_X \le C\theta_0^n, \ n \in \mathbb{Z}_-,$$

and, consequently, $\mathbb{V}(\mathcal{B})$ is an essentially unstable manifold of the equilibrium $u = 0$ of the map S.

We see that, in contrast to the usual theory of unstable manifolds, see, e.g., [23], neither the finite dimensionality of X_+ nor any spectral gap assumption (and nor even the existence of a complement to X_+ in X) are required.

We illustrate the application of this theorem to dissipative dynamical systems on the simple example of the real Ginzburg-Landau equation in $\mathbb{R}^n$,

$$\partial_t u = \Delta_x u + u - u^3, \tag{5.72}$$

and we consider the equilibrium $u = 0$. In that case, the first variation equation reads

$$\partial_t v = \Delta_x v + v. \tag{5.73}$$

Let $S(t)$ and $S_0(t)$ be the solution operators associated with equations (5.72) and (5.73) in $X := L^\infty(\mathbb{R}^n)$, respectively. Then condition (5.70) is obviously satisfied for $S = S(1)$, $S_0 = S_0(1)$, and $\alpha = 1$. In order to find the space X_+, it is sufficient to write the Fourier transform of S_0,

$$\widehat{S_0(t)u_0}(\xi) = e^{(1-|\xi|^2)t}\widehat{u}_0(\xi).$$

This shows that $r(S_0) = e$ and that the unstable part of the spectrum is related to the functions $\mathbb{B}_1(\mathbb{R}^n)$, the support of the Fourier transform of which belongs to the unit ball, see Example 5.26. Furthermore, condition (5.71) is satisfied if we take $X_+ := \mathbb{B}_\sigma(\mathbb{R}^n)$, with $\sigma < \frac{1}{\sqrt{2}}$.

Thus, thanks to Theorem 5.44 and to the fact that an unstable manifold always belongs to the global attractor, we have verified that the attractor $\mathcal{A}$ contains a smooth image of a ball $\mathcal{B}$ of the space $\mathbb{B}_\sigma(\mathbb{R}^n)$ (of entire functions with an exponential growth). Combining this embedding with the lower bounds on the ε-entropy of the spaces $\mathbb{B}_\sigma(\mathbb{R}^n)$ collected in Example 5.26, we obtain the following result.

Theorem 5.45. *The Kolmogorov ε-entropy of the global attractor $\mathcal{A}$ of the real Ginzburg-Landau equation has lower bounds which are analogous to estimates* (5.47) *and* (5.48) *and, consequently, the universal entropy estimates* (5.52) *are sharp.*

Of course, the approach based on infinite dimensional essentially unstable manifolds described above is not related to any specific property of the Ginzburg-Landau equation, but has a universal nature. Actually, only the existence of at least one spatially homogeneous exponentially unstable equilibrium is necessary to apply this method (and, as a consequence, to obtain sharp lower bounds on the entropy), see [78], [79], [240], and [242] for applications of this method to various types of reaction-diffusion systems and [238] for damped hyperbolic equations.

As a next step, we mention that the embedding $\mathbb{V} : \mathcal{B}(\sigma) \to \mathcal{A}$ of the unit ball $\mathcal{B}(\sigma) = B_{\mathbb{B}_\sigma}(0,1)$ in the space of entire functions into the attractor $\mathcal{A}$ gives much more than just estimates on the ε-entropy. Indeed, since the dissipative system and the equilibrium are spatially homogeneous, the unstable manifold map $\mathbb{V}$ commutes with the spatial shifts T_h,

$$T_h \circ \mathbb{V} = \mathbb{V} \circ T_h, \ h \in \mathbb{R}^n,$$

and, consequently, we have obtained a smooth embedding of the spatial dynamics on the space $\mathbb{B}_\sigma(\mathbb{R}^n)$ of entire functions into that on the attractor $\mathcal{A}$,

$$\mathbb{V} : (\mathcal{B}(\sigma), T_h) \to (\mathcal{A}, T_h), \tag{5.74}$$

see [240] for details. Thus, the shifts dynamics on the unit ball $\mathcal{B}(\sigma)$ gives a universal model for the spatial dynamics on the attractor.

In order to clarify the complexity of this model dynamics, we need to introduce a special type of Bernoulli shift dynamics.

Definition 5.46. Let $\mathbb{M}_n := [0,1]^{\mathbb{Z}^n}$ be endowed with the Tikhonov topology. We recall that $\mathbb{M}_n$ consists of all functions $v : \mathbb{Z}^n \to [0,1]$ and the Tikhonov topology can be generated by the following metric:

$$\|v_1 - v_2\|_\phi := \sup_{m\in\mathbb{Z}^n} \{\phi(m)|v_1(m) - v_2(m)|\}, \ v_1, \ v_2 \in \mathbb{M}_n, \tag{5.75}$$

where ϕ is an arbitrary weight function such that $\lim_{|m|\to+\infty} \phi(m) = 0$. We define the action of the group $\mathbb{Z}^n$ on $\mathbb{M}_n$ in the following standard way:

$$\mathcal{T}_l v(m) := v(l+m), \ v \in \mathbb{M}_n, \ l, \ m \in \mathbb{Z}^n,$$

and interpret the group $(\mathbb{M}_n, \mathcal{T}_l)$ as a multi-dimensional Bernoulli scheme with a continuum of symbols $\omega \in [0,1]$.

Our approach to the study of the dynamics generated by the shifts group $(\mathcal{B}(\sigma), T_h)$ is based on the following elementary observation: according to the classical Kotelnikov formula (see [30] and [135]), every function $w \in \mathbb{B}_\sigma(\mathbb{R}) \cap L^2(\mathbb{R})$ can be uniquely recovered from its values on the lattice $\rho\mathbb{Z}$, $\rho = \frac{\pi}{\sigma}$,

$$w(x) = \sum_{l=-\infty}^{+\infty} w(\rho l) \frac{\sin(\sigma x - \pi l)}{\sigma x - \pi l} \tag{5.76}$$

(see also the Whittaker-Shennon-Kotelnikov formula, e.g., in [30], which allows to recover an arbitrary function $w \in \mathbb{B}_\sigma(\mathbb{R}^n)$ from its values on a lattice). Given an arbitrary sequence $v = \{v_l\}_{l\in\mathbb{Z}} \in l^2$, formula (5.76) allows to construct a function $w \in \mathbb{B}_\sigma(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $w(\rho l) = v_l$, for every $l \in \mathbb{Z}$. Furthermore, the spatial shifts $T_{\rho l} w$ of this function obviously corresponds to the shifts $\mathcal{T}_l v$ of the sequence v. This leads to a description of the spatial dynamics on $\mathbb{B}_\sigma(\mathbb{R}) \cap L^2(\mathbb{R})$ in terms of the Bernoulli scheme introduced above (with the additional restriction $v \in l^2$). The extension of representation (5.76) in the spirit of the Whittaker-Shennon-Kotelnikov formula leads to the following result, see [242].

Lemma 5.47. *For every $\sigma > 0$, there exist $\rho = \rho(\sigma)$ and a map*

$$\mathbb{U} \ : \ \mathbb{M}_n \to \mathcal{B}(\sigma) \ \ \text{such that} \ \ T_{\rho l} \circ \mathbb{U} = \mathbb{U} \circ \mathcal{T}_l, \ l \in \mathbb{Z}^n. \tag{5.77}$$

Furthermore, for every polynomial weight $\theta = \theta_{N,x_0}$ (see (5.4) with $N > 0$), there holds

$$C_1^{-1}\|v_1 - v_2\|_\theta \le \|\mathbb{U}(v_1) - \mathbb{U}(v_2)\|_{L^\infty_\theta} \le C_1\|v_1 - v_2\|_\theta,$$

where C_1 depends on N, but is independent of $v_i \in \mathbb{M}_n$, $i = 1, 2$.

Combining this lemma with (5.74), we obtain the following result, see [240] for details.

Theorem 5.48. *Let $\mathcal{A}$ be the global attractor of the real Ginzburg-Landau equation (5.72). Then there exist a positive constant ρ and a map*

$$\mathcal{U} \ : \ \mathbb{M}_n \to \mathcal{A} \ \text{such that} \ \mathcal{U} \circ \mathcal{T}_l = T_{\rho l} \circ \mathcal{U}, \ l \in \mathbb{Z}^n. \tag{5.78}$$

Furthermore, $\mathcal{U}$ is continuous in the local topology (and even Lipschitz continuous in appropriate weighted spaces).

Thus, we see that the Bernoulli scheme $(\mathbb{M}_n, \mathcal{T}_l)$ can be considered as a universal model for the spatial dynamics on the attractor $\mathcal{A}$. Indeed, on the one hand, this model has infinite topological entropy and strictly positive modified entropy $\widehat{h}_{\text{sp}}(\mathbb{M}_n, \mathcal{T}_l)$ (see (5.69)),

$$1 = \widehat{h}_{\text{sp}}(\mathbb{M}_n, \mathcal{T}_l) = \rho^n \widehat{h}_{\text{sp}}(\mathcal{U}(\mathbb{M}_n), T_h) \le \widehat{h}_{\text{sp}}(\mathcal{A}, T_h) < +\infty, \tag{5.79}$$

and, therefore, this gives an example of spatial dynamics of "maximal" complexity (in the sense of the entropy theory). On the other hand, (5.78) holds under very weak assumptions on the dissipative system under study (namely, the existence of at least one spatially homogeneous exponentially unstable equilibrium, see [240] and [242]) and thus has a universal nature.

To conclude this section, we briefly discuss the possibility of extending such a complexity description from the spatial dynamics $V_n = \mathbb{R}^n_x$ to the dynamics of $\mathbb{S}^{V_n}(t, h)$, where V_n contains the *temporal* direction, e.g., $V_n = \text{span}\{e_t, e_{x_2}, \cdots, e_{x_n}\}$. As above, we restrict ourselves to the real Ginzburg-Landau equation, but now with a transport term along the x_1-axis,

$$\partial_t u = \Delta_x u - L\partial_{x_1} u + u - u^3, \tag{5.80}$$

although the result also holds for the general reaction-diffusion system (5.8) under the assumptions of Theorem 5.5, plus the spatial homogeneity and the exponential instability of the zero equilibrium, see [244].

The main idea here is to "change" the temporal t and spatial x_1 directions by considering x_1 as a new "time" and t as one of the "spatial" variables. Then, describing the spatial chaos in this new dissipative system by the scheme introduced above, we would automatically obtain the description of the n-directional space-time chaos in the plane V_n. In order to realize this strategy, we consider equation (5.80) in the half-space $x_1 > 0$, endow it with the following unusual "initial" condition:

$$\begin{cases} \partial_t u = \Delta_x u - L\partial_{x_1} u + u - u^3, \ t \in \mathbb{R}, \ (x_2, ..., x_n) \in \mathbb{R}^{n-1}, \ x_1 > 0, \\ u\big|_{x_1=0} = u^0 \in \Psi_b := L^\infty(\mathbb{R}^n), \end{cases} \tag{5.81}$$

and treat it as an "evolutionary" equation with respect to the time variable x_1 and the spatial variables t, x_2, ..., x_n. Clearly, this problem is ill-posed if $L = 0$ (as well as for small L). However, as proven in [242] and [244], it indeed generates a well-posed and smooth dissipative system in Ψ_b if L is large enough ($L > 2$ for the real Ginzburg-Landau equation). Furthermore, the zero equilibrium remains exponentially unstable for this new system, so that the theory of essentially unstable manifolds is applicable and gives the following result, see [242].

Theorem 5.49. *Let $\mathcal{A}$ be the global attractor of the real Ginzburg-Landau equation* (5.80) *with a sufficiently large transport term ($L > 2$). Then there exist a positive constant σ and a map $\mathcal{W} : \mathcal{B}(\sigma) \to \mathcal{A}$, which is continuous in the local topology, such that*

$$S(t) \circ \mathcal{W} = \mathcal{W} \circ T_{te_{x_1}}, \ T_{he_{x_i}} \circ \mathcal{W} = \mathcal{W} \circ T_{he_{x_i}}, \ i = 2, \ ..., \ n, \ h \in \mathbb{R}, \ t \geq 0, \tag{5.82}$$

where $T_{h\vec{e}}v(x) := v(x + h\vec{e}), \ h \in \mathbb{R}, \ x \in \mathbb{R}^n$.

A combination of this result and Lemma 5.47 gives the desired embedding of the Bernoulli scheme $(\mathbb{M}_n, \mathcal{T}_l)$ into the n-directional space-time dynamics of $\mathbb{S}^{V_n}(t, h)$. In particular, this embedding shows that the modified topological entropy of this dynamics is strictly positive,

$$\widehat{h}_{\text{top}}(\mathbb{S}^{V_n}(t, h), \mathcal{A}) > 0,$$

and, owing to inequalities (5.68), the modified entropy of the temporal evolution group $S(t)$ ($V_1 = \mathbb{R}_t$) is also strictly positive,

$$\widehat{h}_{\text{top}}(S(t), \mathcal{A}) > 0.$$

We also recall that the modified entropy for $S(t)$ differs from the classical topological entropy by the presence of a factor $(\log_2 \frac{1}{\varepsilon})^{-n}$ in the definition and, consequently, its positivity implies that the classical topological entropy is infinite,

$$h_{\text{top}}(S(t), \mathcal{A}) = +\infty. \tag{5.83}$$

To the best of our knowledge, this is the first example of a reasonable dissipative system with an infinite topological entropy.

Remark 5.50. To conclude, we note that, although the above method gives an adequate description of the n-directional complexity and the n-directional space-time chaos for an arbitrary plane V_n under weak assumptions on the system, it does not give reasonable information on the full $(n+1)$-directional space-time complexity, since one direction should be interpreted as the time and we should have exponential divergence in this direction. Furthermore, the $(n + 1)$-dimensional Bernoulli scheme $\mathbb{M}_{n+1}$ cannot be embedded into the global attractor, since its space-time entropy $h_{\text{top}}(\mathbb{S}(t, h), \mathcal{A})$ is finite, see Theorem 5.36.

5.8. **Sinai-Bunimovich space-time chaos in PDEs.** In this concluding subsection, we discuss very recent results concerning the full $(n + 1)$-directional space-time chaos and, in particular, we give examples of dissipative systems in unbounded domains with a strictly positive space-time topological entropy,

$$h_{\text{top}}(\mathbb{S}(t, h), \mathcal{A}) > 0, \tag{5.84}$$

which shows that space-time dynamics with a maximum level of complexity (from the point of view of the entropy theory) can indeed appear in dissipative systems generated by PDEs.

We first recall that, in spite of a huge amount of numerical and experimental data on various types of space-time irregular and turbulent behaviors in various physical systems, see, e.g., [116], [158], [159], [196], and the references therein, there are very few rigorous mathematical results on this topic and mathematically relevant models which describe such phenomena.

The simplest and most natural known model which exhibits such phenomena is the so-called Sinai-Bunimovich space-time chaos which was initially defined and found for discrete lattice dynamics, see [5], [35], [191], and [192]. We also recall that this model consists of a $\mathbb{Z}^n$-grid of temporally chaotic oscillators coupled by a weak interaction. Then, if a single chaotic oscillator of this grid is described by the Bernoulli scheme $\mathcal{M}^1 := \{0,1\}^{\mathbb{Z}}$ (now with only two symbols $\omega \in \{0,1\}$, in contrast to the previous subsection!), the uncoupled system naturally has an infinite dimensional hyperbolic set which is homeomorphic to the multi-dimensional Bernoulli scheme $\mathcal{M}^{n+1} := \{0,1\}^{\mathbb{Z}^{n+1}} = (\mathcal{M}^1)^{\mathbb{Z}^n}$. The temporal evolution operator is then conjugated to the shift in $\mathcal{M}^{n+1}$ along the first coordinate vector and the other n coordinate shifts are associated with the spatial shifts on the grid. Finally, owing to the stability of hyperbolic sets, the above structure survives under a sufficiently small coupling. Thus, according to this model, the space-time chaos can naturally be described in terms of the multi-dimensional Bernoulli scheme $\mathcal{M}^{n+1}$.

It is worth noting that, although this model is clearly not relevant to describe the space-time chaos in the so-called fully developed turbulence (since it does not reproduce the typical properties, such as energy cascades and the Kolmogorov laws, which are believed to be crucial for the understanding of this phenomenon), it can be useful and relevant to describe weak space-time chaos and weak turbulence (close to the threshold), where the generation and long-time survival of such global spatial patterns are still possible. Furthermore, to the best of our knowledge, it is the only mathematically rigorous model which gives positive space-time topological entropy and an associated space-time dynamics with maximal complexity.

Thus, the possibility of having $h_{\mathrm{top}}(\mathbb{S}(t,h))$ positive is clear for space-discrete lattice dissipative systems. However, verifying the existence of such space-time dynamics in continuous media described by PDEs is an extremely complicated problem. Furthermore, even the existence of a single PDE which possesses such an infinite dimensional Bernoulli scheme has been a long-standing open problem.

The first examples of reaction-diffusion systems in $\mathbb{R}^n$ with Sinai-Bunimovich space-time chaos were recently constructed in [170]. We describe below this construction in more details.

We consider the following special *space-time periodic* reaction-diffusion equation:

$$\partial_t u = \gamma \Delta_x u - f_\lambda(t,x,u) \text{ in } \mathbb{R}^n,\ \gamma > 0, \tag{5.85}$$

where the nonlinearity f_λ has the following structure: there exists a smooth bounded domain $\Omega_0 \Subset (0,1)^n$ such that, for every $x \in [0,1]^n$, there holds

$$f_\lambda(t,x,u) := \begin{cases} f(t,u) & \text{for } x \in \Omega_0, \\ \lambda u & \text{for } x \in [0,1]^n \backslash \Omega_0, \end{cases} \tag{5.86}$$

where $f(t,u)$ is a given function (which is assumed to be 1-periodic with respect to t) and $\lambda \gg 1$ is a large parameter. Then we extend (5.86) by space-periodicity from $[0,1]^n$ to the whole space $\mathbb{R}^n$. Thus, we have a periodic grid of "islands" $\Omega_l := l + \Omega_0$, $l \in \mathbb{Z}^n$, on which the nonlinearity f_λ coincides with $f(t,u)$ and can generate nontrivial dynamics. These islands are separated from each other by the "ocean" $\Omega_- := \mathbb{R}^n \backslash (\cup_{l\in\mathbb{Z}^n}\Omega_l)$, where we have the strong absorption provided by the nonlinearity $f_\lambda(t,x,u) \equiv \lambda u$.

It is intuitively clear that, for a sufficiently large absorption coefficient λ, the solutions of equation (5.85) should be small in the absorption domain Ω_- and, consequently, the interactions between the islands are also expected to be small, and the dynamics inside the islands are “almost-independent”. Thus, if the reaction-diffusion system in Ω_0,

$$\partial_t v = \gamma\Delta_x v - f(t, v) \text{ in } \Omega_0,\ v = 0 \text{ on } \partial\Omega_0, \tag{5.87}$$

which describes the limit dynamics inside one “island” as $\lambda = +\infty$, possesses a hyperbolic set Γ_0, then, according to the structural stability principle, the whole system (5.85) should have a hyperbolic set which is homeomorphic to $(\Gamma_0)^{\mathbb{Z}^n}$ if the absorption parameter λ is large enough. Furthermore, if, in addition, the initial hyperbolic set Γ_0 is homeomorphic to the Bernoulli scheme $\{0,1\}^{\mathbb{Z}}$, then (5.85) contains an $(n+1)$-dimensional Bernoulli scheme $\{0,1\}^{\mathbb{Z}^{n+1}} \sim (\{0,1\}^{\mathbb{Z}})^{\mathbb{Z}^n}$, in a complete analogy with the Sinai-Bunimovich lattice model.

These intuitive arguments were rigorously justified in [170], where the following result was obtained.

Theorem 5.51. *Let the limit equation* (5.87) *possess a hyperbolic set which is homeomorphic to the usual Bernoulli scheme* $\mathcal{M}^1 = \{0,1\}^{\mathbb{Z}}$ *and let some natural assumptions on* f *be satisfied. Then there exists* $\lambda_0 = \lambda_0(f, \mathcal{M}^1)$ *such that, for every* $\lambda > \lambda_0$, *problem* (5.85) *possesses an infinite dimensional hyperbolic set which is homeomorphic to* $\mathcal{M}^{n+1} = \{0,1\}^{\mathbb{Z}^{n+1}}$. *Furthermore, the action of the space-time dynamics on this set (restricted to* $(t, h) \in \mathbb{Z}^{n+1}$*) is conjugated to the Bernoulli shift on* $\mathcal{M}^{n+1}$.

Since the existence of a hyperbolic set which is homeomorphic to $\mathcal{M}^1$ for the reaction-diffusion system (5.87) in a *bounded* domain is well-known (the existence of a single transversal homoclinic trajectory is sufficient in order to have such a result, see [130] ; see also [170] for an explicit construction), the above theorem indeed provides examples for Sinai-Bunimovich space-time chaos in reaction-diffusion systems and, in particular, examples of reaction-diffusion systems with a strictly positive space-time topological entropy. Furthermore, owing to the stability of hyperbolic sets, the space discontinuous nonlinearity f_λ can then be replaced by close $\mathcal{C}^\infty$ ones and, finally, by embedding the space-time periodic system that we obtain into a larger autonomous one (one creates the space-time periodic modes by using the additional equations), examples of space-time autonomous reaction-diffusion systems of the form (5.8) were also constructed in [170].

Remark 5.52. We note that the spatial grid $\mathbb{Z}^n$ (which is crucial for the Sinai-Bunimovich model) is directly modulated by the special spatial structure of the nonlinearity f_λ in the continuous model (5.85) (see (5.86)) and, therefore, the above approach does not allow to find such phenomena in many physically relevant equations for which the structure of the nonlinearity is a priori given (such as the Navier-Stokes equations, the real and complex Ginzburg-Landau equations, ...). In order to overcome this drawback, an alternative, potentially more promising, approach was suggested in [171], where the spatial grid is obtained by using the so-called spatially-localized solutions (pulses, standing solitons, ...) initially situated in the nodes of the grid. Then, due to the "tail"-interaction between solitons, a weak temporal dynamics appears and this dynamics allows a center manifold reduction to a *lattice* system of ODEs (roughly speaking, this system describes the temporal evolution of the soliton centers, see [82], [171], and [207] for details). Finding then the Sinai-Bunimovich space-time chaos in these reduced lattice equations, one can lift it to the initial PDE. The advantage of this method is that the spatial grid is now modulated in an implicit way by the positions of localized solutions in space and the center manifold reduction, and the underlying dissipative system may be autonomous and spatially homogeneous. In particular, this approach was realized in [171] for the one-dimensional space-time periodically perturbed Swift-Hohenberg equation,

$$\partial_t u + (\partial_x^2 + 1)^2 u + \beta^2 u + u^3 + \kappa u^2 = h(t, x), \tag{5.88}$$

for values of β and κ for which the existence of a spatially localized soliton is known. To be more precise, for these values of β and κ, the existence of a hyperbolic set $\mathcal{M}^2$ for (5.88) is proven for special (rather artificial) space-time periodic external forces h with *arbitrary small* amplitudes. The presence of these external forces are unavoidable for the Swift-Hohenberg model, since it belongs to the class of the so-called extended gradient systems and, when $h = 0$, its space-time topological entropy vanishes, see [242], and the Sinai-Bunimovich space-time chaos is then impossible. Finally, we also mention a very recent result [223] in which the above method allowed to prove the existence of a Sinai-Bunimovich space-time chaos for the one-dimensional complex Ginzburg-Landau equation,

$$\partial_t u = (1 + i\beta)\partial_x^2 u + \gamma u - \delta u|u|^3 + \varepsilon,$$

where $\beta \in \mathbb{R}$, γ, $\delta \in \mathbb{C}$, and $\varepsilon \in \mathbb{R}$ is an arbitrary small real parameter. Contrary to the above examples, this equation is already space-time homogeneous and does not contain any artificial nonlinearity or external forces. This confirms that the Sinai-Bunimovich space-time chaos may appear in *natural* PDEs arising from mathematical physics.

6. Ill-posed dissipative systems and trajectory attractors

In this concluding subsection, we briefly discuss possible extensions of the theory of attractors to ill-posed problems. Indeed, in all the above results, we required the solution operator

$$S(t) \ : \ u_0 \mapsto u(t) \tag{6.1}$$

to be well-defined and continuous (in a proper phase space). However, as mentioned in the introduction, in several cases, such a result is not known or does not hold.

There exist two approaches to handle dissipative systems without uniqueness.

The first one allows the solution operator (6.1) to be *multi-valued* (set-valued) and then extends the theory of attractors to semigroups of multi-valued maps. Actually, all the results on the existence of the global attractor given in subsection 2.2 have their natural analogues in the multi-valued setting, see [12], [20], [24], [25], [52], [161], [162], [204], [211], [212], and the references therein ; see also [39], [144], and [234] for nonautonomous systems.

An alternative, more geometric, approach consists in changing the phase space of the problem and in passing to the so-called *trajectory* phase space and the associated trajectory dynamical system, which is single-valued, and, thus, the usual theory of attractors can be applied, see [48], [49], [50], [51], [92], [213], [214], [225], [226], [235], and the references therein. We illustrate this approach on the simple example of an ODE in $E = \mathbb{R}^n$, see [51] for details,

$$u' + f(u) = 0, \quad u(0) = u_0, \tag{6.2}$$

for some, at least, continuous nonlinearity f. We also assume that the system is *dissipative*, so that it is globally solvable for every $u_0 \in E$ and the following estimate holds:

$$\|u(t)\|_E \le Q(\|u(0)\|_E)e^{-\alpha t} + C_F, \quad t \ge 0, \tag{6.3}$$

for some positive constants α and C_F and monotonic function Q and for every solution u of (6.2). This holds, e.g., if f satisfies a dissipativity assumption of the form

$$f(u).u \ge -C + \beta|u|^2, \quad u \in \mathbb{R}^n, \quad C,\ \beta > 0.$$

Let us assume for a while that f is Lipschitz continuous. Then we have the uniqueness of solutions and, for every two solutions $u_1(t)$ and $u_2(t)$, the following estimate holds:

$$\|u_1(t) - u_2(t)\|_E \le Ce^{Kt}\|u_1(0) - u_2(0)\|_E, \quad t \ge 0, \tag{6.4}$$

where the constants C and K depend on $\|u_i(0)\|_E$, $i = 1,\ 2$.

In this classical case, the dissipative estimate (6.3) guarantees the existence of a compact absorbing set for the semigroup $S(t)$ associated with problem (6.2) via (6.1) (we recall that $\dim E < +\infty$). Thus, this semigroup possesses the global attractor $\mathcal{A}_{\mathrm{gl}}$ on E which has the usual structure,

$$\mathcal{A}_{\mathrm{gl}} = \mathcal{K}\big|_{t=0}, \tag{6.5}$$

where $\mathcal{K} \subset \mathcal{C}_b(\mathbb{R}, E)$ is the kernel (i.e., the set of all bounded complete trajectories of problem (6.2), see Subsection 2.1).

We now define a *trajectory* phase space for problem (6.2) as follows:

$$K_{\mathrm{tr}} := \{u \in \mathcal{C}_b(\mathbb{R}_+, E),\ u(t) = S(t)u_0,\ u_0 \in E,\ t \ge 0\}. \tag{6.6}$$

In other words, K_{tr} consists of all positive trajectories of (6.2) starting from all points $u_0 \in E$. Then, owing to the uniqueness, K_{tr} is isomorphic to E by the solution operator $\mathcal{S}u_0 := u(\cdot) = S(\cdot)u_0$,

$$\mathcal{S}\ :\ E \to \Phi_b := \mathcal{C}_b(\mathbb{R}_+, E),\ \mathcal{S}(E) = K_{\mathrm{tr}},\ \mathcal{S}^{-1}u = u(0),\ u \in K_{\mathrm{tr}},\ \mathcal{S}^{-1}K_{\mathrm{tr}} = E. \tag{6.7}$$

Furthermore, as it is not difficult to see, the semigroup $S(t)$ is conjugated to the time translations on K_{tr} under this isomorphism,

$$T_t\ :\ K_{\mathrm{tr}} \to K_{\mathrm{tr}},\ T_t = \mathcal{S} \circ S(t) \circ \mathcal{S}^{-1},\ T_t u(s) := u(t+s),\ t \ge 0,\ s \in \mathbb{R}. \tag{6.8}$$

We call the shifts semigroup $\{T_t,\ t \ge 0\}$ acting on the trajectory phase space the *trajectory* dynamical system associated with problem (6.2).

We now fix a class of bounded sets and a topology on K_{tr} via this isomorphism. Indeed, obviously, the set $B \subset E$ is bounded if and only if $\mathcal{S}(B)$ is bounded in Φ_b (see the dissipative estimate (6.3)) and $\mathcal{S}$ is an homeomorphism if we endow the phase space K_{tr} with the topology of $\Phi_{\mathrm{loc}} := \mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+, E)$ (due to the Lipschitz continuity (6.4)).

Thus, owing to the homeomorphism $\mathcal{S}$, the existence of the global attractor $\mathcal{A}_{\mathrm{gl}}$ for the semigroup $S(t)$ on E is *equivalent* to that of the $(\Phi_b, \Phi_{\mathrm{loc}})$-global attractor $\mathcal{A}_{\mathrm{tr}}$ of the trajectory dynamical system (T_t, K_{tr}) (we recall that a $(\Phi_b, \Phi_{\mathrm{loc}})$-attractor attracts the bounded subsets of Φ_b in the topology of Φ_{loc}, see [23]). We refer the attractor $\mathcal{A}_{\mathrm{tr}}$ as the *trajectory* attractor associated with problem (6.2). As usual, this trajectory attractor is also generated by the set $\mathcal{K}$ of all bounded complete trajectories of the problem,

$$\mathcal{A}_{\mathrm{tr}} = \mathcal{K}\big|_{t\ge 0},\quad \mathcal{A}_{\mathrm{gl}} = \mathcal{A}_{\mathrm{tr}}\big|_{t=0}. \tag{6.9}$$

A key observation here is that, although *crucial* for the usual global attractor $\mathcal{A}_{\mathrm{gl}}$, the uniqueness and continuity (6.4) are *not necessary* for the existence of the global attractor

$\mathcal{A}_{\rm tr}$ for the *trajectory* dynamical system $(T_t, K_{\rm tr})$ and can be relaxed. Indeed, the phase space $K_{\rm tr}$ is well-defined and the shifts semigroup T_t acts continuously on it, no matter whether or not the uniqueness holds (only the *dissipative* estimate (6.3) is necessary to ensure that $K_{\rm tr} \subset \Phi_b$; of course, we also need the continuity of f to ensure that the solutions exist and $K_{\rm tr}$ is not empty). Furthermore, the dissipative estimate (6.3) also guarantees that the set

$$B_{\rm tr} := \{u \in K_{\rm tr},\ \|u\|_{\Phi_b} \le R\}$$

is a Φ_b-absorbing set for T_t. Finally, this absorbing set is *compact* in the $\Phi_{\rm loc}$-topology (since E is finite dimensional and we have a uniform control on the norm of $\frac{du}{dt}$ for every $u \in B_{\rm tr}$ from equation (6.2)). Thus, the existence of the trajectory attractor $\mathcal{A}_{\rm tr}$ is verified when f is only continuous and we have the following theorem.

Theorem 6.1. *Let the nonlinearity f in* (6.2) *be continuous and let the dissipative estimate* (6.3) *be satisfied for all solutions. Then the trajectory dynamical system* $(T_t, K_{\rm tr})$ *possesses the* $(\Phi_b, \Phi_{\rm loc})$*-global attractor* $\mathcal{A}_{\rm tr}$ *(which is the trajectory attractor associated with problem* (6.2)*) which is generated by all bounded complete trajectories of the system,*

$$\mathcal{A}_{\rm tr} = \mathcal{K}\big|_{t\ge 0}. \tag{6.10}$$

It is also worth noting that, projecting this trajectory attractor $\mathcal{A}_{\rm tr}$, we obtain the global attractor $\mathcal{A}_{\rm gl}^{\rm m-v}$ for the multi-valued semigroup $S(t)$ associated with problem (6.2) in a standard way,

$$\mathcal{A}_{\rm gl}^{\rm m-v} = \mathcal{A}_{\rm tr}\big|_{t=0}, \tag{6.11}$$

see [51] for details.

Thus, we see that, although the trajectory approach usually essentially gives the same object as the multi-valued semigroup (see (6.11)), it allows, on the one hand, to avoid the use of "unfriendly" multi-valued maps and, on the other hand, to study the long time behavior for ill-posed problems by using the classical theory of attractors for single-valued semigroups. We also note that the trick consisting in passing from the usual to the trajectory dynamical system may be useful even when the uniqueness holds. In particular, the aforementioned l-trajectories method for estimating the dimension of global attractors and constructing exponential attractors is essentially based on this trick, see [156] and the references therein. Furthermore, this trick was also used in [170] to prove the persistence of hyperbolic trajectories when the perturbation is not small in the initial phase space, but only in some averaged time-integral norms.

However, it is worth mentioning that the above trajectory approach has not been applied to artificial problems like (6.2) (for which the nonuniqueness appears due to the lack of regularity on f), but to extremely complicated equations such as the three-dimensional Navier-Stokes equations and even to equations in compressible fluid mechanics for which only minimal information on the associated weak solutions is available. This leads to several unusual "common" delicate points in the theory which we would like to outline before passing to more relevant examples.

Remark 6.2. a) Very often, the dissipative estimate (6.3) can be verified not for every solution belonging to some function space, but *only* for some special weak solutions (e.g., obtained by Galerkin approximations, as for the three-dimensional Navier-Stokes equations). So, one should somehow exclude the "pathological", possibly non-dissipative, trajectories from the trajectory phase space $K_{\rm tr}$. By doing this, one should, however, take a special care to preserve the action of the shifts semigroup on K_{tr}. In particular, the direct way which consists in incorporating the dissipative estimate into the phase space K_{tr}, i.e., in defining K_{tr} as the set of all trajectories satisfying a dissipative estimate of the form (6.3), may fail for this very reason. Indeed, typically, for ill-posed problems, we can construct a solution which satisfies the energy inequality between $t = 0$ and any $t = T$ (which gives the dissipative estimate), but not between $t = \tau$ and $t = T$ for $\tau > 0$ (see the example of a damped wave equation below). So, in that case, we cannot verify a dissipative estimate of the form (6.3) starting from $t = \tau$ and, for this reason, we lose the invariance $T_t K_{\mathrm{tr}} \subset K_{\mathrm{tr}}$ which is crucial in the theory! This problem can be overcome (following [49]) by using, instead of (6.3), a weaker dissipativity assumption of the form

$$\|u(t)\|_E \le C_u e^{-\alpha t} + C_F \text{ or } \|T_t u\|_{\Phi_b} \le C_u e^{-\alpha t} + C_F,\ t \ge 0, \tag{6.12}$$

where the positive constants α and C_F are the same as in (6.3), except that C_u is now some constant depending on u (without specifying any relation with $u(0)$). Such dissipative inequalities are, obviously, invariant with respect to time shifts and the action of T_t on K_{tr} is recovered.

b) In order to prove the existence of the global attractor, one usually uses a *compact* absorbing/attracting set $B_{\mathrm{tr}} \subset K_{\mathrm{tr}}$. The semi-compactness is usually not a problem, since the weak and weak-$*$ topologies are used, and immediately follows from energy estimates. The fact that the limit points of B_{tr} solve the equations is also not an essential problem, since, with a proper choice of the topology of Φ_{loc}, it can usually be done as in the proof of existence of a weak solution (which should be done before proving the existence of attractors!). However, since K_{tr} does not contain all the solutions of the problem, these limit points may not belong to K_{tr} and the existence of a compact absorbing set may be lost in such a procedure. For instance, without a special care, the limits of solutions which are all obtained by Galerkin approximations may not satisfy this property. Analogously, concerning the dissipative inequalities (6.12), if one defines an absorbing set in the natural way, namely, $B_{\mathrm{tr}} := \{u \in K_{\mathrm{tr}},\ \|u\|_{\Phi_b} \le R\}$, then it may very well be *not closed*, since an estimate of the form (6.12) may be lost under the limit procedure. Thus, the closure of the absorbing/attracting set indeed requires an additional attention. These considerations show that the use of the space Φ_b to define the class of bounded sets is *not sufficient* and more general abstract definitions of "bounded" sets should be used instead, see Definition 2.15. In particular, for dissipative inequalities of the form (6.12), it is sufficient to define the class of bounded sets in the following natural way:

$$B \subset K_{\mathrm{tr}} \text{ is "bounded" if and only if } C_u \le C_B < +\infty,\ u \in B. \tag{6.13}$$

In other words, B is "bounded" if there exists a uniform constant C_B such that (6.12) holds with C_u replaced by C_B, for every $u \in B$. Then the existence of a "bounded" absorbing set is an immediate consequence of (6.12) and such estimates are preserved under the limit procedure, see [51] for details. This problem, when K_{tr} only consists of solutions obtained by some (e.g., Galerkin) approximation scheme can also be solved in a similar way, see [243] and the examples below.

Example 6.3. Here, we briefly consider the application of the trajectory approach to the three-dimensional Navier-Stokes equations in a bounded domain Ω (see [49], [51], and

[214] for more detailed expositions),

$$\begin{cases} \partial_t u + (u, \nabla_x)u = \nu\Delta_x u - \nabla_x p + g, \\ \operatorname{div} u = 0, \ u\big|_{\partial\Omega} = 0, \ u\big|_{t=0} = u_0. \end{cases} \tag{6.14}$$

Let, as usual, H and H_1 be the closures of the smooth divergent free vector fields in Ω which vanish on the boundary in the metrics of $L^2(\Omega)^3$ and $W^{1,2}(\Omega)^3$, respectively. Then, as is well-known (see, e.g., [51], [146], and [220]), for every $u_0 \in H$, the Navier-Stokes problem possesses at least one global weak energy solution

$$u \in \Phi_b := L^\infty(\mathbb{R}_+, H) \cap L^2_b(\mathbb{R}_+, H_1)$$

which satisfies, in addition, an *energy inequality* in the following differential form:

$$1/2\frac{d}{dt}\|u(t)\|^2_H + \nu\|\nabla_x u(t)\|^2_H \le (u, g)_H. \tag{6.15}$$

To be more precise, this inequality should be understood in the sense of distributions, i.e.,

$$-1/2\int_0^{+\infty} \|u(t)\|^2_H \cdot \phi'(t)\,dt + \nu\int_0^{+\infty} \|\nabla_x u(t)\|^2 \cdot \phi(t)\,dt \le \int_0^{+\infty} \phi(t)\cdot(g, u(t))_H\,dt \tag{6.16}$$

holds for every $\phi \in \mathcal{C}_0^\infty(\mathbb{R}_+)$ such that $\phi(t) \ge 0$. In particular, this energy inequality implies that, for almost every $t, \ \tau \in \mathbb{R}_+$, $t \ge \tau$, the following dissipative estimate holds:

$$\|u(t)\|^2_H + \nu\int_\tau^t e^{-\alpha(s-\tau)}\|\nabla_x u(s)\|^2_H\,ds \le \|u(\tau)\|^2_H e^{-\alpha(t-\tau)} + C\|g\|^2_{L^2}, \tag{6.17}$$

for some positive constants C and α which only depend on ν and Ω. Since the existence or the nonexistence of other weak solutions $u \in \Phi_b$ (which do not satisfy the energy inequality and, thus, are non-dissipative) is not known yet, it is natural to define the trajectory phase space $K_{\rm tr}$ as the set of all weak solutions satisfying this energy inequality,

$$K_{\rm tr} := \{u \in \Phi_b, \ u \text{ solves (6.14) and satisfies (6.15)}\}. \tag{6.18}$$

Indeed, since the energy inequality is shift-invariant, the phase space $K_{\rm tr}$ thus defined is also invariant with respect to the shifts semigroup T_t, $T_t : K_{\rm tr} \to K_{\rm tr}$, and, therefore, the trajectory dynamical system $(T_t, K_{\rm tr})$ is well-defined. Furthermore, the dissipative estimate (6.17) implies that

$$\|T_t u\|^2_{\Phi_b} \le C\|u\|^2_{L^\infty(\mathbb{R}_+,H)}e^{-\alpha t} + C\|g\|^2_{L^2}, \ t \ge 0, \tag{6.19}$$

for every $u \in K_{\rm tr}$ and for positive constants C and α which are independent of u and t. Therefore, the R-ball in Φ_b, intersected with $K_{\rm tr}$,

$$B_R := \{u \in K_{\rm tr}, \ \|u\|_{\Phi_b} \le R\},$$

is a Φ_b-absorbing set for the trajectory semigroup T_t on $K_{\rm tr}$ if R is large enough. Thus, there only remains to fix the topology of $\Phi_{\rm loc}$ on $K_{\rm tr}$ in such a way that this ball is compact. To be more precise, we set

$$\Phi_{\rm loc} := L^{\infty,w^*}_{\rm loc}(\mathbb{R}_+, H) \cap L^{2,w}_{\rm loc}(\mathbb{R}_+, H_1),$$

where w and w^* denote the weak and weak-$*$ topologies, respectively. We recall that a sequence u_n converges to u in the space $\Phi_{\rm loc}$ if and only if, for every $T > 0$, the sequence $u_n\big|_{[0,T]}$ converges to $u\big|_{[0,T]}$ weakly in $L^2([0,T], H_1)$ and weakly-$*$ in $L^\infty([0,T], H)$, see [51] for details. Then every bounded subset of Φ_b is precompact and metrizable in $\Phi_{\rm loc}$, see, e.g., [208], and we only need to verify that B is closed in $K_{\rm tr}$ in the $\Phi_{\rm loc}$-topology. As already mentioned in Remark 6.2, this can be done as in the justification of the passage to the limit $N \to +\infty$ in Galerkin approximations for weak energy solutions u, see [51] for details. Thus, the assumptions of the $(\Phi_b, \Phi_{\mathrm{loc}})$-attractor's existence theorem (see Theorem 2.20 and [23]) are verified and, consequently, the trajectory dynamical system (T_t, K_{tr}) possesses the global attractor $\mathcal{A}_{\mathrm{tr}}$ which attracts the bounded subsets of Φ_b in the topology of Φ_{loc}. As usual, the trajectory attractor $\mathcal{A}_{\mathrm{tr}}$ is generated by all bounded complete solutions of the Navier-Stokes system (of course, satisfying the energy inequality) via (6.10) and its restriction at $t = 0$ gives the global attractor of the associated semigroup of multi-valued maps (i.e., (6.11) holds), see [49] and [51] for details. To conclude with the Navier-Stokes equations, we mention that, although the above trajectory attractor attracts the bounded subsets of Φ_b in the *weak* topology of Φ_{loc} only, this weak convergence implies the *strong* convergence in slightly larger spaces (due to compactness arguments). In particular, for every bounded subset $B \subset K_{\mathrm{tr}}$, every $T \in \mathbb{R}_+$, and every $\delta > 0$, we have

$$\lim_{t\to+\infty} \mathrm{dist}_{\mathcal{C}([0,T],H_{-\delta})\cap L^2([0,T],H_{1-\delta})}(T_tB\big|_{[0,T]}, \mathcal{A}_{\mathrm{tr}}\big|_{[0,T]}) = 0, \tag{6.20}$$

where H_s is a scale of Hilbert spaces associated with the Stokes operator in Ω, see [51]. We also recall that, although nothing is known, in general, concerning the additional regularity and/or the structure of the attractor $\mathcal{A}_{\mathrm{tr}}$ of the three-dimensional Navier-Stokes problem, there are several special cases for which such results can be proven. In particular, for thin domains $\Omega = [-h, h] \times \Omega_0$, where Ω_0 is a bounded two-dimensional domain and h is a small parameter (depending on ν), endowed with Dirichlet boundary conditions on $\partial\Omega_0$ and Neumann boundary conditions on $\{-h, h\} \times \Omega$, the smoothness $\mathcal{A}_{\mathrm{tr}} \subset \mathcal{C}_b(\mathbb{R}_+, H_1)$, which is enough for the uniqueness on the attractor, follows from [199]. Another nontrivial example is a three-dimensional Navier-Stokes system with an additional rotation term,

$$\partial_t u + (u, \nabla_x)u + \omega \times u = \nu\Delta_x u - \nabla_x p + g$$

in the domain $\Omega = [0, T_1] \times [0, T_2] \times [0, T_3]$ with periodic boundary conditions. As proven in [18], if ω is large enough and the periods T_i, $i = 1,\ 2,\ 3$, satisfy some non-resonance conditions, the attractor $\mathcal{A}_{\mathrm{tr}}$ is also smooth and the uniqueness holds on the attractor.

Example 6.4. As a next example, we consider "the second" (after the three-dimensional Navier-Stokes equations) classical ill-posed problem, namely, a damped wave equation with a supercritical nonlinearity,

$$\begin{cases} \varepsilon\partial_t^2 u + \gamma\partial_t u - \Delta_x u + f(u) = g, \\ u\big|_{\partial\Omega} = 0,\ u\big|_{t=0} = u_0,\ \partial_t u\big|_{t=0} = u_0', \end{cases} \tag{6.21}$$

in a bounded smooth domain Ω of $\mathbb{R}^3$. Here, ε and γ are positive parameters, $g \in L^2(\Omega)$ corresponds to given external forces and the nonlinearity $f \in \mathcal{C}^2(\mathbb{R})$ is assumed to satisfy the following dissipative and growth assumptions:

$$1.\ f'(u) \geq -C + C_1|u|^{p-1}, \quad 2.\ |f''(u)| \leq C(1 + |u|^{p-2}), \tag{6.22}$$

$C,\ C_1 > 0$, $u \in \mathbb{R}$, $p \geq 0$. It is well-known, see, e.g., [23], that, in the subcritical $p < 3$ and critical $p = 3$ cases, problem (6.21) is well-posed in the energy phase space $W_0^{1,2}(\Omega) \times L^2(\Omega)$ and possesses the global attractor $\mathcal{A}$, see also [122], [139], [220], and the references therein. In contrast to this, in the supercritical case $p > 3$, the well-posedness of (6.21) in a proper phase space is still an open problem (the limit exponent $p = 3$ can be shifted till $p = 5$ when $\Omega = \mathbb{R}^3$, see [89], but, to the best of our knowledge, this result is not known for bounded domains). On the other hand, it is well-known (see, e.g., [51] and [146]) that, for every $(u_0, u_0') \in E := W_0^{1,2}(\Omega) \cap (L^{p+1}(\Omega) \times L^2(\Omega))$, equation (6.21) possesses at least one global weak energy solution

$$u \in \Phi_b := L^\infty(\mathbb{R}_+, W_0^{1,2}(\Omega) \cap L^{p+1}(\Omega)) \times W^{1,\infty}(\mathbb{R}_+, L^2(\Omega))$$

which satisfies the dissipative estimate

$$\|(u(t), \partial_t u(t))\|_E \le Q(\|(u(0), \partial_t u(0))\|_E)e^{-\alpha t} + Q(\|g\|_{L^2}),\ t \ge 0, \tag{6.23}$$

where the positive constant α and monotonic function Q are independent of u. As shown in [49] and [51], this is sufficient to verify the existence of the trajectory attractor. However, we are now exactly in the situation mentioned in Remark 6.2. Indeed, in contrast to the three-dimensional Navier-Stokes equations, here, the dissipativity estimate cannot be formulated as a differential inequality similar to (6.15) and the dissipative estimate (6.23) is the best known one. As already mentioned, this estimate is *not* shift-invariant and, therefore, *cannot* be directly used to define the space $K_{\rm tr}$ (this is related to the fact that, when constructing the solution $u(t)$, e.g., by Galerkin approximations, we can easily guarantee that $u_n(0)$ converges *strongly* to $u(0)$ in E, but, for $u_n(\tau)$ with $\tau > 0$, we only have a weak convergence, which does not yield the convergence of the norms and, consequently, $t = 0$ cannot be replaced by $t = \tau$ in (6.23)). Thus, following the general scheme described in Remark 6.2, we consider a dissipative estimate in a weaker, but shift-invariant form,

$$\|T_t u\|_{\Phi_b} \le C_u e^{-\alpha t} + C_*,\ C_* = Q(\|g\|_{L^2}),\ \alpha > 0,\ t \ge 0, \tag{6.24}$$

where C_u is a constant which depends on u, and define the trajectory phase space $K_{\rm tr}$ by using this dissipative estimate,

$$K_{\rm tr} := \{u \in \Phi_b,\ u \text{ solves (6.21) and satisfies (6.24)}\}.$$

Thus, the trajectory dynamical system $(T_t, K_{\rm tr})$ is well-defined. Furthermore, following the general scheme, we also define a class of "bounded" sets via (6.13). Then the existence of a "bounded" absorbing set, e.g., of the form $B := \{u \in K_{\rm tr},\ C_u \le 1\}$, immediately follows from the dissipative estimate (6.24) and the definition of "bounded" sets. So, there only remains to fix a topology on $K_{\rm tr}$ in such a way that the absorbing set B is compact. This can be done by using the local weak-$*$ topology on Φ_b, exactly as in the case of the three-dimensional Navier-Stokes equations, namely,

$$\Phi_{\rm loc} := L^{\infty,w^*}_{\rm loc}(\mathbb{R}_+, W_0^{1,2}(\Omega) \cap L^{p+1}(\Omega)) \times L^{\infty,w^*}_{\rm loc}(\mathbb{R}_+, L^2(\Omega)). \tag{6.25}$$

Then B is precompact and metrizable in $\Phi_{\rm loc}$, since it is bounded in Φ_b, see [208], and the fact that it is closed in $K_{\rm tr}$ can be verified in a standard way, see [49] and [51] for details. Thus, the assumptions of the attractor's existence theorem (see Theorem 2.20) are verified and the semigroup $(T_t, K_{\rm tr})$ possesses the global attractor $\mathcal{A}_{\rm tr}$ (i.e., the trajectory attractor) which attracts the "bounded" (in the sense of (6.13)) subsets of $K_{\rm tr}$ in the topology of $\Phi_{\rm loc}$. Again, the trajectory attractor $\mathcal{A}_{\rm tr}$ is generated by all bounded complete solutions (satisfying $\|u\|_{\Phi_b} \le C_*$) via (6.10) and its restriction at $t = 0$ gives the global attractor of the associated semigroup of multi-valued maps constructed in [20] (i.e., (6.11) holds), see [49] and [51] for details.

Example 6.5. In this example, we consider, following [243], an alternative way to construct a trajectory attractor for the damped wave equation (6.21) which a priori contains a "smaller number" of possible pathological solutions and, as a consequence, some reasonable results concerning its structure are available. To this end, we first recall a construction of Galerkin approximations for (6.21). Let $\{e_k\}_{k=1}^{+\infty}$ be an orthonormal basis in $L^2(\Omega)$ (say, generated by the eigenvectors of the Laplacian with Dirichlet boundary conditions) and denote by P_N the orthoprojector onto the first N vectors of this basis. Then the N-th Galerkin approximation for (6.21) reads

$$\epsilon\partial_t^2 u_N + \gamma\partial_t u_N - \Delta_x u_N + P_N f(u_N) = P_N g,\ \ u_N \in P_N L^2(\Omega). \tag{6.26}$$

Actually, the weak energy solutions mentioned in the previous example are usually constructed by solving the Galerkin ODEs (6.26) and by then passing to the limit $N \to +\infty$ in a proper sense, namely,

$$u := \Phi_{\mathrm{loc}} - \lim_{k\to+\infty} u_{N_k}, \tag{6.27}$$

where the space Φ_{loc} is the same as in the previous example, see (6.25). The main idea is now to restrict ourselves to the solutions which can be obtained via (6.27) only and to define the trajectory phase space K_{tr} as follows:

$$K_{\mathrm{tr}} := \{u \in \Phi_b,\ u \text{ solves (6.21) and is obtained via (6.27)}\}. \tag{6.28}$$

The main problem here is that the weak limit of solutions which can all be obtained by the above Galerkin approximations may a priori not satisfy this property, so that the usual bounded subsets of Φ_b may not be closed in K_{tr}. In order to overcome this difficulty and to define the proper class of "bounded" sets, we need to introduce the following functional on K_{tr}:

$$M(u) := \inf\ \{\liminf_{k\to+\infty} \|(u_{N_k}(0), \partial_t u_{N_k}(0))\|_E,\ u = \Phi_{\mathrm{loc}} - \lim_{k\to+\infty} u_{N_k}\}, \tag{6.29}$$

where the infimum is taken over all sequences u_{N_k} of Galerkin solutions which converge weakly to a given solution u. We now define the class of M-bounded sets of K_{tr} as the sets on which the functional M is uniformly bounded. Then, as shown in [243], the trajectory dynamical system (T_t, K_{tr}) possesses an M-bounded absorbing set and the weak limit of a sequence u_n belonging to any M-bounded set belongs to K_{tr} (i.e., it can be obtained by the above Galerkin approximations). Thus, according to the abstract attractor's existence theorem, the trajectory dynamical system (T_t, K_{tr}) possesses the global attractor $\mathcal{A}_{\mathrm{tr}}^{\mathrm{Gal}}$ which attracts all M-bounded sets in the topology of Φ_{loc}. It is worth noting once more that, in contrast to the trajectory attractor $\mathcal{A}_{\mathrm{tr}}$ constructed above, this new attractor $\mathcal{A}_{\mathrm{tr}}^{\mathrm{Gal}} \subset \mathcal{A}_{\mathrm{tr}}$ possesses several good properties which are not available for $\mathcal{A}_{\mathrm{tr}}$ and, in particular,

1) it is connected in Φ_{loc} (since the simplest M-bounded sets $B_R := \{u \in K_{\mathrm{tr}},\ M(u) \le R\}$ are connected ; this follows from the fact that they can be approximated, in Φ_{loc}, by analogous sets for the Galerkin approximations which are clearly connected) ;

2) every complete trajectory belonging to this attractor tends in a proper sense to the set of equilibria as time goes to plus or minus infinity ;

3) every complete trajectory $u(t)$ on $\mathcal{A}_{\mathrm{tr}}^{\mathrm{Gal}}$ is *smooth* for sufficiently small t, i.e., there exists $T = T_u$ such that $u(t) \in W^{2,2}(\Omega) \subset \mathcal{C}(\Omega)$ for $t \le T_u$, and every solution is unique (in the above class) as long as it is smooth. So, the only way for a singular solution to appear on the attractor $\mathcal{A}_{\mathrm{tr}}^{\mathrm{Gal}}$ is by a blow up of a strong solution, see [243] for details ;

4) as proven in [243], the whole attractor $\mathcal{A}_{\mathrm{tr}}^{\mathrm{Gal}}$ is *smooth* if the coefficient $\varepsilon > 0$ is small enough, $\mathcal{A}_{\mathrm{tr}}^{\mathrm{Gal}} \subset \mathcal{C}_b(\mathbb{R}_+, W^{2,2}(\Omega))$.

A drawback of such a construction is that $\mathcal{A}_{\mathrm{tr}}^{\mathrm{Gal}}$ depends on the concrete approximation scheme (e.g., different Galerkin bases may lead to different attractors). However, the attractor $\mathcal{A}_{\rm tr}$ constructed in the previous example also depends a priori on the artificial constant C_* in the dissipative estimate (6.24).

Remark 6.6. To conclude, we note that the above trajectory approach has been successfully applied not only to ill-posed *evolutionary* problems, but also to elliptic boundary value problems in unbounded domains (for which the nonuniqueness does not appear as a consequence of poorly understood analytical properties of the equations under study, but is related to the classical ill-posedness of the Cauchy problem for elliptic equations), see [169], [225], and [226] for trajectory attractors for elliptic problems in cylindrical domains and [27] and [235] for more general classes of unbounded domains. Finally, we also note that most of the results considered in this subsection can naturally be extended to nonautonomous ill-posed problems as well, see [51] for details.

References

[1] F. Abergel, *Existence and finite dimensionality of the global attractor for evolution equations on unbounded domains*, J. Diff. Eqns. **83** (1990), 85–108.

[2] V. Afraimovich, A.V. Babin, and S.N. Chow, *Spatial chaotic structure of attractors of reaction-diffusion systems*, Trans. Amer. Math. Soc. **348** (1996), 5031–5063.

[3] A. Afendikov and A. Mielke, *Multi-pulse solutions to the Navier-Stokes problem between parallel plates*, Z. Angew. Math. Phys. **52** (2001), 79–100.

[4] A. Afendikov and A. Mielke, *Bifurcation of homoclinic orbits to a saddle-focus in reversible systems with* SO(2)*-symmetry*, J. Diff. Eqns. **159** (1999), 370–402.

[5] V. Afraimovich and B. Fernandez, *Topological properties of linearly coupled map lattices*, Nonlinearity **13** (2000), 973–993.

[6] M. Aida, M. Efendiev, and A. Yagi, *Quasilinear abstract parabolic evolution equations and exponential attractors*, Osaka Math. J. **42** (2005), 101–132.

[7] M. Aida and A. Yagi, *Global stability of approximation for exponential attractors*, Funkcial. Ekvac. **47** (2004), 251–276.

[8] S. Angenent, *The shadowing lemma for elliptic PDE, in Dynamics of infinite-dimensional systems (Lisbon, 1986), NATO Adv. Sci. Inst. F. Comput. Systems Sci., Vol. 37, Springer-Verlag, Berlin, 1987.*

[9] J. Arrieta, J. Cholewa, T. Dlotko, and A. Rodríguez-Bernal*, Asymptotic behavior and attractors for reaction-diffusion equations in unbounded domains,* Nonlinear Anal. **56** *(2004), 515–554.*

[10] A.V. Babin*, Asymptotic expansions at infinity of a strongly perturbed Poiseuille flow,* Adv. Soviet Math. **10** *(1992), 1–83.*

[11] A.V. Babin*, The attractor of a Navier-Stokes system in an unbounded channel-like domain,* J. Dyn. Diff. Eqns. **4** *(1992), 555–584.*

[12] A.V. Babin*, The attractor of a generalized semigroup generated by an elliptic equation in a tube domain,* Russian Acad. Sci. Izv. Math. **44** *(1995), 207–223.*

[13] A. Babin*, Inertial manifolds for travelling-wave solutions of reaction-diffusion systems,* Comm. Pure Appl. Math. **48** *(1995), 167–198.*

[14] A.V. Babin*, On space-chaotic solutions to scalar parabolic equations with modulated nonlinearities,* Russian J. Math. Phys. **3** *(1995), 389–392.*

[15] A.V. Babin*, Topological invariants and solutions with a high complexity for scalar semilinear PDE,* J. Dyn. Diff. Eqns. **12** *(2000), 599–646.*

[16] A.V. Babin*, Global attractors in PDE, in Handbook of dynamical systems, Vol. 1B, Elsevier, Amsterdam, 983–1085, 2006.*

[17] A. Babin and L. Bunimovich*, Stable chaotic waves generated by hyperbolic PDEs,* Nonlinearity **9** *(1996), 853–875.*

[18] A. Babin, A. Mahalov, and B. Nicolaenko*, Regularity and integrability of* 3*D Euler and Navier-Stokes equations for rotating fluids,* Asymp. Anal. **15** *(1997), 103–150.*

[19] A. Babin and B. Nicolaenko, *Exponential attractors of reaction-diffusion systems in an unbounded domain,* J. Dyn. Diff. Eqns. **7** *(1995), 567–590.*

[20] A.V. Babin and M.I. Vishik, *Maximal attractors of semigroups corresponding to evolutionary differential equations (Russian),* Mat. Sb. (N.S.) **126** *(1985), 397–419.*

[21] A.V. Babin and M.I. Vishik, *Unstable invariant sets of semigroups of nonlinear operators and their perturbations (Russian),* Uspekhi Mat. Nauk. **41** *(1986), 3–34.*

[22] A.V. Babin and M.I. Vishik, *Attractors of partial differential evolution equations in an unbounded domain,* Proc. Roy. Soc. Edinburgh Sect. A **116** *(1990), 221–243.*

[23] A.V. Babin and M.I. Vishik, *Attractors of evolution equations, North-Holland, Amsterdam, 1992.*

[24] J.M. Ball, *Continuity properties and global attractors of generalized semiflows and the Navier-Stokes equations,* J. Nonlinear Sci. **7** *(1997), 475–502.*

[25] J.M. Ball, *Global attractors for damped semilinear wave equations,* Discrete Cont. Dyn. Systems **10** *(2004), 31–52.*

[26] J. Barrow-Green, *Poincaré and the three-body problem, History of Mathematics, Vol. 11, Amer. Math. Soc., Providence, RI, 1999.*

[27] H. Berestycki, M. Efendiev, and S. Zelik, *Dynamical approach for positive solutions of semilinear elliptic problems in unbounded domains,* Submitted.

[28] P. Bergé, Y. Pomeau, and C. Vidal, *L'ordre dans le chaos, Hermann, Paris, 1984.*

[29] A. Biryuk, *Spectral properties of solutions of Burgers equation with small dissipation,* Funct. Anal. Appl. **35** *(2001), 1–12.*

[30] R. Boas, *Entire functions, Academic Press, New York, 1954.*

[31] L. Brevdo and K. Kirchgässner, *Structure formation in a zonal barotropic current: a treatment via the centre manifold reduction,* R. Soc. London Proc. Ser. A Math. Phys. Eng. Sci. **455** *(1999), 2021–2054.*

[32] P. Brunovský and B. Fiedler, *Connecting orbits in scalar reaction diffusion equations. II. The complete solution,* J. Diff. Eqns. **81** *(1989), 106–135.*

[33] M. Bulíček, J. Málek, and D. Pražák, *On the dimension of the attractor for a class of fluids with pressure dependent viscosities,* Comm. Pure Appl. Anal. **4** *(2005), 805–822.*

[34] L.A. Bunimovich, *Space-time chaos in spatially continuous systems,* Phys. D **131** *(1999), 31–37.*

[35] L.A. Bunimovich and Ya.G. Sinaĭ, *Spacetime chaos in coupled map lattices,* Nonlinearity **1** *(1988), 491–516.*

[36] M. Cabral, R. Rosa, and R. Temam, *Existence and dimension of the attractor of the Bénard problem on channel-like domains,* Discrete Cont. Dyn. Systems **10** *(2004), 89–116.*

[37] T. Caraballo, P. Kloeden, and J. Real, *Pullback and forward attractors for a damped wave equation with delays,* Stochastics and Dynamics **4** *(2004), 405–423.*

[38] T. Caraballo and J.A. Langa, *On the upper semicontinuity of cocycle attractors for non-autonomous and random dynamical systems,* Dyn. Cont. Discrete Impulsive Systems A **10** *(2003), 491–513.*

[39] T. Caraballo, J.A. Langa, V.S. Melnik, and J. Valero, *Pullback attractors of nonautonomous and stochastic multivalued dynamical systems,* Set-Valued Anal. **11** *(2003), 153–201.*

[40] T. Caraballo, J.A. Langa, and J. Valero, *The dimension of attractors of nonautonomous partial differential equations,* ANZIAM J. **45** *(2003), 207–222.*

[41] T. Caraballo, G. Lukaszewicz, and J. Real, *Pullback attractors for nonautonomous 2D-Navier-Stokes equations in some unbounded domains,* C. R. Acad. Sci. Paris Série I Math. **342** *(2006), 263–268.*

[42] D. Cheban, P. Kloeden, and B. Schmalfuss, *The relationship between pullback, forwards and global attractors of nonautonomous dynamical systems,* Nonlinear Dyn. Syst. Theory **2** *(2002), 9–28.*

[43] V.V. Chepyzhov and A.A. Ilyin, *On the fractal dimension of invariant sets: applications to Navier-Stokes equations,* Discrete Cont. Dyn. Systems **10** *(2004), 117–135.*

[44] V.V. Chepyzhov and A.Yu. Goritsky, *Explicit construction of integral manifolds with exponential tracking,* Appl. Anal. **71** *(1999), 237–252.*

[45] V.V. Chepyzhov, A.Yu. Goritsky, and M.I. Vishik, *Integral manifolds and attractors with exponential rate for nonautonomous hyperbolic equations with dissipation,* Russian J. Math. Phys. **12** *(2005), 17–39.*

[46] V.V. Chepyzhov and M.I. Vishik, *A Hausdorff dimension estimate for kernel sections of nonautonomous evolution equations,* Indiana Univ. Math. J. **42** *(1993), 1057–1076.*

[47] V.V. Chepyzhov and M.I. Vishik, *Attractors of nonautonomous dynamical systems and their dimension,* J. Math. Pures Appl. **73** *(1994), 279–333.*

[48] V.V. Chepyzhov and M.I. Vishik, *Trajectory attractors for the* 2*D Navier-Stokes system and some generalizations,* Topol. Methods Nonlinear Anal. **8** *(1996), 217–243.*

[49] V.V. Chepyzhov and M.I. Vishik, *Evolution equations and their trajectory attractors,* J. Math. Pures Appl. **76** *(1997), 913–964.*

[50] V.V. Chepyzhov and M.I. Vishik, *Perturbation of trajectory attractors for dissipative hyperbolic equations, in The Mazya anniversary collection, Vol. 2 (Rostock, 1998), 33–54, Oper. Theory Adv. Appl., 110, Birkhaüser, Basel, 1999.*

[51] V.V. Chepyzhov and M.I. Vishik, *Attractors of equations of mathematical physics, Amer. Math. Soc., Providence, RI, 2002.*

[52] A. Cheskidov and C. Foias, *On global attractors of the 3D Navier-Stokes equations,* J. Diff. Eqns. **231** *(2006), 714–754.*

[53] J.W. Cholewa and T. Dlotko, *Global attractors in abstract parabolic problems, London Math. Soc. Lecture Notes Series, Vol. 278, Cambridge University Press, Cambridge, 2000.*

[54] J.W. Cholewa and T. Dlotko, *Strongly damped wave equation in uniform spaces,* Nonlinear Anal. **64** *(2006), 174–187.*

[55] I. Chueshov, M. Eller, and I. Lasiecka, *Finite dimensionality of the attractor for a semilinear wave equation with nonlinear boundary dissipation,* Comm. Partial Diff. Eqns. **29** *(2004), 1847–1876.*

[56] I. Chueshov and I. Lasiecka, *Attractors for second-order evolution equations with a nonlinear damping,* J. Dyn. Diff. Eqns. **16** *(2004), 469–512.*

[57] P. Collet and J.-P. Eckmann, *The definition and measurement of the topological entropy per unit volume in parabolic PDEs,* Nonlinearity **12** *(1999), 451–473.*

[58] P. Collet and J.-P. Eckmann, *Extensive properties of the complex Ginzburg-Landau equation,* Comm. Math. Phys. **200** *(1999), 699–722.*

[59] P. Collet and J.-P. Eckmann, *Topological entropy and ε-entropy for damped hyperbolic equations,* Ann. Henri Poincaré **1** *(2000), 715–752.*

[60] P. Constantin, C. Foias, B. Nicolaenko, and R. Temam, *Integral and inertial manifolds for dissipative partial differential equations, Applied Mathematical Sciences, Vol. 70, Springer-Verlag, New York, 1988.*

[61] M. Conti and V. Pata, *Weakly dissipative semilinear equations of viscoelasticity,* Comm. Pure Appl. Anal. **4** *(2005), 705–720.*

[62] M. Conti, V. Pata, and M. Squassina, *Singular limit of differential systems with memory,* Indiana Univ. Math. J. **55** *(2006), 169–215.*

[63] M. Conti, V. Pata, and M. Squassina, *Singular limit of dissipative hyperbolic equations with memory,* Discrete Cont. Dyn. Systems, **Suppl.** *(2005), 200–208.*

[64] H. Crauel and F. Flandoli, *Attractors for random dynamical systems,* Probab. Theory Related Fields **100** *(1994), 365–393.*

[65] Z. Dai and B.L. Guo, *Global attractor of nonlinear strain waves in elastic waveguides,* Acta Math. Sci. Series B Engl. Ed. **20** *(2000), 322–334.*

[66] R.L. Devaney, *An introduction to chaotic dynamical systems, Reprint of the second (1989) edition, Studies in Nonlinearity, Westview Press, Boulder, CO, 2003.*

[67] J.-P. Eckmann and J. Rougemont, *Coarsening by Ginzburg-Landau dynamics,* Comm. Math. Phys. **199** *(1998), 441–470.*

[68] A. Eden, C. Foias, B. Nicolaenko, and R. Temam, *Exponential attractors for dissipative evolution equations, Research in Applied Mathematics, Vol. 37, John-Wiley, New York, 1994.*

[69] M. Efendiev and A. Miranville, *Finite-dimensional attractors for reaction-diffusion equations in R^n with a strong nonlinearity,* Discrete Cont. Dyn. Systems **5** *(1999), 399–424.*

[70] M. Efendiev and A. Miranville, *The dimension of the global attractor for dissipative reaction-diffusion systems,* Appl. Math. Letters **16** *(2003), 351–355.*

[71] M. Efendiev, A. Miranville, and S. Zelik, *Exponential attractors for a nonlinear reaction-diffusion system in R^3,* C. R. Acad. Sci. Paris Série I Math. **330** *(2000), 713–718.*

[72] M. Efendiev, A. Miranville, and S. Zelik, *Infinite dimensional attractors for a non-autonomous reaction-diffusion system,* Math. Nachr. **248-249** *(2003), 72–96.*

[73] M. Efendiev, A. Miranville, and S. Zelik, *Exponential attractors for a singularly perturbed Cahn-Hilliard system,* Math. Nach. **272** *(2004), 11–31.*

[74] M. Efendiev, A. Miranville, and S. Zelik, *Global and exponential attractors for nonlinear reaction-diffusion systems in unbounded domains,* Proc. Roy. Soc. Edinburgh Sect. A **134** *(2004), 271–315.*

[75] M. Efendiev, A. Miranville, and S. Zelik, *Infinite-dimensional exponential attractors for nonlinear reaction-diffusion systems in unbounded domains and their approximation,* R. Soc. London Proc. Ser. A Math. Phys. Eng. Sci. **460** *(2004), 1107–1129.*

[76] M. Efendiev, A. Miranville, and S. Zelik, *Exponential attractors and finite-dimensional reduction for non-autonomous dynamical systems,* Proc. Roy. Soc. Edinburgh Sect. A **135** *(2005), 703–730.*

[77] M. Efendiev and A. Yagi, *Continuous dependence on a parameter of exponential attractors for chemotaxis-growth system,* J. Math. Soc. Japan **57** *(2005), 167–181.*

[78] M. Efendiev and S. Zelik, *The attractor for a nonlinear reaction-diffusion system in an unbounded domain,* Comm. Pure Appl. Math. **54** *(2001), 625–688.*

[79] M. Efendiev and S.V. Zelik, *Upper and lower bounds for the Kolmogorov entropy of the attractor for the RDE in an unbounded domain,* J. Dyn. Diff. Eqns. **14** *(2002), 369–403.*

[80] M. Efendiev and S. Zelik, *Attractors of the reaction-diffusion systems with rapidly oscillating coefficients and their homogenization,* Ann. Henri Poincaré **6** *(2002), 961–989.*

[81] M. Efendiev and S. Zelik, *Finite-dimensional attractors and exponential attractors for degenerate doubly nonlinear equations,* Preprint.

[82] S.-I. Ei, *The motion of weakly interacting pulses in reaction-diffusion systems,* J. Dyn. Diff. Eqns. **14** *(2002), 85–137.*

[83] P. Fabrie and C. Galusinski, *Exponential attractors for the slightly compressible* 2*D-Navier-Stokes,* Discrete Cont. Dyn. Systems **2** *(1996), 315–348.*

[84] P. Fabrie and C. Galusinski, *Exponential attractors for a partially dissipative reaction system,* Asymp. Anal. **12** *(1996), 329–354.*

[85] P. Fabrie, C. Galusinski, and A. Miranville, *Uniform inertial sets for damped wave equations,* Discrete Cont. Dyn. Systems **6** *(2000), 393–418.*

[86] P. Fabrie, C. Galusinski, A. Miranville, and S. Zelik, *Uniform exponential attractors for a singularly perturbed damped wave equation,* Discrete Cont. Dyn. Systems **10** *(2004), 211–238.*

[87] K. Falconer, *Fractal geometry. Mathematical foundations and applications, Second edition, John Wiley & Sons, Hoboken, NJ, 2003.*

[88] A. Favini, A. Lorenzi, and A. Yagi, *Exponential attractors for semiconductor equation, in Differential equations: inverse and direct problems, A. Favini and A. Lorenzi eds., Lecture Notes in Pure and Applied Mathematics, Vol. 251, Chapman & Hall, Boca Raton, FL, 111–130, 2006.*

[89] E. Feireisl, *Asymptotic behaviour and attractors for a semilinear damped wave equation with supercritical exponent,* Proc. Roy. Soc. Edinburgh Sect. A **125** *(1995), 1051–1062.*

[90] E. Feireisl, *Bounded locally compact global attractors for semilinear damped wave equations on* $\mathbb{R}^n$, Diff. Int. Eqns. **9** *(1996), 1147–1156.*

[91] E. Feireisl, *Asymptotic behaviour and attractors for degenerate parabolic equations on unbounded domains,* Nonlinear Anal. **26** *(1996), 1535–1546.*

[92] E. Feireisl, *Propagation of oscillations, complete trajectories and attractors for compressible flows,* Nonlinear Diff. Eqns. Appl. **10** *(2003), 33–55.*

[93] E. Feireisl, Ph. Laurençot, F. Simondon, and H. Touré, *Compact attractors for reaction-diffusion equations in* R^n, C. R. Acad. Sci. Paris Série I **319** *(1994), 147–151.*

[94] B. Fiedler and C. Rocha, *Orbit equivalence of global attractors of semilinear parabolic differential equations,* Trans. Amer. Math. Soc. **352** *(2000), 257–284.*

[95] B. Fiedler and C. Rocha, *Realization of meander permutations by boundary value problems,* J. Diff. Eqns. **156** *(1999), 282–308.*

[96] C. Foias, O. Manley, and R. Temam, *Modelling of the interaction of small and large eddies in two-dimensional turbulent flows,* RAIRO Modél. Math. Anal. Numér. **22** *(1988), 93–118.*

[97] C. Foias and E. Olson, *Finite fractal dimension and Hölder-Lipschitz parametrization,* Indiana Univ. Math. J. **45** *(1996), 603–616.*

[98] C. Foias, G.R. Sell, and R. Temam, *Inertial manifolds for nonlinear evolution equations,* J. Diff. Eqns. **73** *(1988), 309–353.*

[99] C. Foias and R. Temam, *Approximation of attractors by algebraic or analytic sets,* SIAM J. Math. Anal. **25** *(1994), 1269–1302.*

[100] Th. Gallay and S. Slijepčević, *Energy flow in formally gradient partial differential equations on unbounded domains,* J. Dyn. Diff. Eqns. **13** *(2001), 757–789.*

[101] C. Galusinski, *Existence and continuity of uniform exponential attractors of the singularly perturbed Hodgkin-Huxley system,* J. Diff. Eqns. **144** *(1998), 99–169.*

[102] B. García-Archilla, J. Nuvo, and E.S. Titi, *An approximate inertial manifolds approach to postprocessing the Galerkin method for the Navier-Stokes system,* Math. Comp. **68** *(1999), 893–911.*

[103] S. Gatti, C. Giorgi, and V. Pata, *Navier-Stokes limit of Jeffreys type flows,* Phys. D **203** *(2005), 55–79.*

[104] S. Gatti, M. Grasselli, A. Miranville, and V. Pata, *Hyperbolic relaxation of the viscous Cahn-Hilliard equation in 3-D,* Math. Models Methods Appl. Sci. **15** *(2005), 165–198.*

[105] S. Gatti, M. Grasselli, A. Miranville, and V. Pata, *On the hyperbolic relaxation of the one-dimensional Cahn-Hilliard equation,* J. Math. Anal. Appl. **312** *(2005), 230–247.*

[106] S. Gatti, M. Grasselli, A. Miranville, and V. Pata, *Memory relaxation of first order evolution equations,* Nonlinearity **18** *(2005), 1859–1883.*

[107] S. Gatti, M. Grasselli, A. Miranville, and V. Pata, *A construction of a robust family of exponential attractors,* Proc. Amer. Math. Soc. **134** *(2006), 117–127.*

[108] S. Gatti, M. Grasselli, A. Miranville, and V. Pata, *Memory relaxation of the one-dimensional Cahn-Hilliard equation, in Dissipative phase transitions, P. Colli, N. Kenmochi, and J. Sprekels eds., Advances in Mathematics for Applied Sciences, Vol. 71, World Sci. Publishing, River Edge, NJ, 2006.*

[109] S. Gatti, M. Grasselli, and V. Pata, *Exponential attractors for a phase-field model with memory and quadratic nonlinearity,* Indiana Univ. Math. J. **53** *(2004), 719–754.*

[110] S. Gatti, M. Grasselli, and V. Pata, *Exponential attractors for a conserved phase-field system with memory,* Phys. D **189** *(2004), 31–48.*

[111] S. Gatti, M. Grasselli, V. Pata, and M. Squassina, *Robust exponential attractors for a family of nonconserved phase-field systems with memory,* Discrete Cont. Dyn. Systems **12** *(2005), 1019–1029.*

[112] S. Gatti and A. Miranville, *Asymptotic behavior of a phase-field system with dynamic boundary conditions, in Differential equations: inverse and direct problems, A. Favini and A. Lorenzi eds., Lecture Notes in Pure and Applied Mathematics, Vol. 251, Chapman & Hall, Boca Raton, FL, 149–170, 2006.*

[113] M. Grasselli, A. Miranville, V. Pata, and S. Zelik, *Well-posedness and long time behavior of a parabolic-hyperbolic phase-field system with singular potentials,* Math. Nach., To appear.

[114] M. Grasselli and V. Pata, *Robust exponential attractors for a phase-field system with memory,* J. Evol. Eqns. **5** *(2005), 465–483.*

[115] J.-M. Ghidaglia, *A note on the strong convergence towards attractors of damped KdV equations,* J. Diff. Eqns. **110** *(1994), 356–359.*

[116] G. Goren, J.-P. Eckmann, and I. Procaccia, *Scenario for the onset of space-time chaos,* Phys. Rev. E **57** *(1998), 4106–4134.*

[117] A.Yu. Goritsky and M.I. Vishik, *Integral manifolds for nonautonomous equations,* Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. **21** *(1997), 109–146.*

[118] O. Goubet, *Asymptotic smoothing effect for weakly damped forced Korteweg-de Vries equations,* Discrete Cont. Dyn. Systems **6** *(2000), 625–644.*

[119] O. Goubet and I. Moise, *Attractor for dissipative Zakharov system,* Nonlinear Anal. **31** *(1998), 823–847.*

[120] O. Goubet and R. Rosa, *Asymptotic smoothing and global attractor of a weakly damped, forced KdV equation on the real line,* J. Diff. Eqns. **185** *(2002), 25–53.*

[121] J.K. Hale, *Asymptotic bahavior and dynamics in infinite dimensions, in Nonlinear differential equations, Papers from the seventh congress on Differential equations and applications held at the University of Granada, Granada, September 24–28, 1984, J.K. Hale and P. Martínez-Amores eds., Research Notes in Mathematics, Vol. 132, Pitman (Advanced Publishing Program), Boston, MA, 1–42, 1985.*

[122] J.K. Hale, *Asymptotic behavior of dissipative systems, Math. Surveys Monographs, Vol. 25, Amer. Math. Soc., Providence, RI, 1988.*

[123] J.K. Hale and G. Raugel, *Limits of semigroups depending on parameters,* Resenhas **1** *(1993), 1–45.*

[124] A. Haraux, *Systèmes dynamiques dissipatifs et applications, Masson, Paris, 1991.*

[125] B.R. Hunt and V.Y. Kaloshin, *Regularity of embeddings of infinite-dimensional fractal sets into finite-dimensional spaces,* Nonlinearity **12** *(1999), 1263–1275.*

[126] B.R. Hunt, T. Sauer, and J.A. Yorke, *Prevalence: a translation-invariant "almost every" for infinite-dimensional spaces,* Bull. Amer. Math. Soc. **27** *(1992), 217–238.*

[127] P. Kaplický and D. Pražák, *Differentiability of the solution operator and the dimension of the global attractor for certain power-law fluids,* J. Math. Anal. Appl. **326** *(2006), 75–87.*

[128] N.I. Karachalios and N.M. Stavrakakis, *Global attractor for the weakly damped driven Schrödinger equation in $H^2(\mathbb{R})$,* Nonlinear Diff. Eqns. Appl. **9** *(2002), 347–360.*

[129] N.I. Karachalios and N.B. Zographopoulos, *Convergence towards attractors for a degenerate Ginzburg-Laudau equation,* Z. Angew. Math. Phys. **56** *(2005), 11–30.*

[130] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems, Cambridge University Press, Cambridge, 1995.*

[131] K. Kirchgässner, *Wave-solutions of reversible systems and applications,* J. Diff. Eqns. **45** *(1982), 113–127.*

[132] P. Kloeden, *Pullback attractors of nonautonomous semidynamical systems,* Stochastics and Dynamics **3** *(2003), 101–112.*

[133] N. Koksch and S. Siegmund, *Pullback attracting inertial manifolds for nonautonomous dynamical systems,* J. Dyn. Diff. Eqns. **14** *(2002), 889–941.*

[134] N. Koksch and S. Siegmund, *Cone invariance and squeezing properties for inertial manifolds of nonautonomous evolution equations,* Banach Center Publ. **60** *(2003), 27–48.*

[135] A.N. Kolmogorov and V. Tikhomirov, *ϵ-entropy and ϵ-capacity of sets in functional spaces, in Selected works of A.N. Kolmogorov, Vol. III, Kluwer Academic, Dordrecht, 1993.*

[136] I.N. Kostin, *Rate of attraction to a non-hyperbolic attractor,* Asymp. Anal. **16** *(1998), 203–222.*

[137] O.A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow, Second edition, Gordon and Breach, New York, 1969.*

[138] O.A. Ladyzhenskaya, *Finding minimal global attractors for the Navier-Stokes equations and other partial differential equations,* Russian Math. Surveys **42** *(1987), 27–73.*

[139] O.A. Ladyzhenskaya, *Attractors of nonlinear evolution problems with dissipation,* J. Soviet Math. **40** *(1988), 632–640.*

[140] O.A. Ladyzhenskaya, *Attractors for semigroups and evolution equations, Lezioni Lincee, Cambridge University Press, Cambridge, 1991.*

[141] J.A. Langa, J.C. Robinson, and A. Suarez, *Bifurcation from zero of a complete trajectory for nonautonomous logistic PDEs,* Internat. J. Bifur. Chaos Appl. Sci. Engrg. **15** *(2005), 2663–2669.*

[142] J.A. Langa and B. Schmalfuss, *Finite dimensionality of attractors for non-autonomous dynamical systems given by partial differential equations,* Stochastics and Dynamics **4** *(2004), 385–404.*

[143] Le Dung and B. Nicolaenko, *Exponential attractors in Banach spaces,* J. Dyn. Diff. Eqns. **13** *(2001), 791–806.*

[144] D.S. Li and P.E. Kloeden, *On the dynamics of nonautonomous periodic general dynamical systems and differential inclusions,* J. Diff. Eqns. **224** *(2006), 1–38.*

[145] E. Lindenstrauss and B. Weiss, *Mean topological dimension,* Israel J. Math. **115** *(2000), 1–24.*

[146] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Paris, 1969.*

[147] V.X. Liu, *A sharp lower bound for the Hausdorff dimension of the global attractors of the 2D Navier-Stokes equations,* Comm. Math. Phys. **158** *(1993), 327–339.*

[148] V.X. Liu, *Remarks on the Navier-Stokes equations on the two- and three-dimensional torus,* Comm. Partial Diff. Eqns. **19** *(1994), 873–900.*

[149] E.N. Lorenz, *Deterministic nonperiodic flow,* J. Atmospheric Sci. **20** *(1963), 130–141.*

[150] S. Lu, *Attractors for nonautonomous 2D Navier-Stokes equations with less regular normal forces,* J. Diff. Eqns. **230** *(2006), 196–212.*

[151] K. Lu and B. Wang, *Global attractors for the Klein-Gordon-Schrödinger equation in unbounded domains,* J. Diff. Eqns. **170** *(2001), 281–316.*

[152] S. Lu, H. Wu, and C. Zhong, *Attractors for nonautonomous 2D Navier-Stokes equations with normal external forces,* Discrete Cont. Dyn. Systems **13** *(2005), 701–719.*

[153] Q. Ma, S. Wang, and C. Zhong, *Necessary and sufficient conditions for the existence of global attractors for semigroups and applications,* Indiana Univ. Math. J. **51** *(2002), 1541–1559.*

[154] S. Ma and C. Zhong, *The attractors for weakly damped non-autonomous hyperbolic equations with a new class of external forces,* Discrete Cont. Dyn. Systems **18** *(2007), 53–70.*

[155] J. Málek and J. Nečas, *A finite-dimensional attractor for three-dimensional flow of incompressible fluids,* J. Diff. Eqns. **127** *(1996), 498–518.*

[156] J. Málek and D. Pražák, *Large time behavior via the method of l-trajectories,* J. Diff. Eqns. **18** *(2002), 243–279.*

[157] J. Málek and D. Pražák, *On the dimension of the global attractor for the modified Navier-Stokes equations, in Nonlinear Problems of the Mathematical Physics and Related Topics, M.S. Birman et al. eds., Kluwer Academic, Dordrecht, 2002.*

[158] P. Manneville, *Dissipative structures and weak turbulence, Perspectives in Physics, Academic Press, Boston, MA, 1990.*

[159] P. Manneville, *Dynamical systems, temporal vs. spatio-temporal chaos, and climate, in Nonlinear dynamics and pattern formation in the natural environment (Noordwijkerhout, 1994), Pitman Res. Notes Math. Ser., Vol. 335, 168–187, Longman, Harlow, 1995.*

[160] K. Matsuura, *Exponential attractors for 2D magneto-micropolar fluid flow in a bounded domain,* Discrete Cont. Dyn. Systems, **Suppl.** *(2005), 634–641.*

[161] V. Melnik, *Multivalued semiflows and their attractors (Russian),* Dokl. Akad. Nauk. **343** *(1995), 302–305.*

[162] V. Melnik and J. Valero, *On attractors of multi-valued semi-flows and differential inclusions,* Set-Valued Anal. **6** *(1998), 83–111.*

[163] S. Merino, *On the existence of the global attractor for semilinear RDE on* $\mathbb{R}^n$*, J. Diff. Eqns.* **132** *(1996), 87–106.*

[164] A. Mielke, *The complex Ginzburg-Landau equation on large and unbounded domains: sharper bounds and attractors,* Nonlinearity **10** *(1997), 199–222.*

[165] A. Mielke, *Essential manifolds for an elliptic problem in an infinite strip,* J. Diff. Eqns. **110** *(1994), 322–355.*

[166] A. Mielke, *The Ginzburg-Landau equation in its role as a modulation equation, in Handbook of Dynamical Systems, Vol. 2, B. Fiedler ed., 759–834, Elsevier, Amsterdam, 2002.*

[167] A. Mielke and P. Holmes, *Spatially complex equilibria of buckled rods,* Arch. Rational Mech. Anal. **101** *(1988), 319–348.*

[168] A. Mielke and G. Schneider, *Attractors for modulation equations on unbounded domains-existence and comparison,* Nonlinearity **8** *(1995), 743–768.*

[169] A. Mielke and S. Zelik, *Infinite-dimensional trajectory attractors for elliptic boundary-value problems on cylindrical domains,* Russian Math. Surveys **57** *(2002), 753-784.*

[170] A. Mielke and S. Zelik, *Infinite-dimensional hyperbolic sets and spatio-temporal chaos in reaction-diffusion systems in* $\mathbb{R}^n$, J. Dyn. Diff. Eqns., To appear.

[171] A. Mielke and S. Zelik, *Weak interaction in multi-pulse structures and space-time chaos in reaction-diffusion equations,* Mem. Amer. Math. Soc., To appear.

[172] J. Milnor, *On the entropy geometry of cellular automata,* Complex Systems **2** *(1988), 357–385.*

[173] Gh. Minea, *Remarques sur l'unicité de la solution stationnaire d'une équation de type Navier-Stokes,* Rev. Roumaine Math. Pures Appl. **21** *(1976), 1071–1075.*

[174] A. Miranville, *Lower bound on the dimension of the attractor for the Bénard problem with free surfaces,* Nonlinear Anal. **25** *(1995), 1079–1094.*

[175] A. Miranville, *Exponential attractors for a class of evolution equations by a decomposition method,* C. R. Acad. Sci. Paris Série I Math. **328** *(1999), 145–150.*

[176] A. Miranville, *Finite dimensional global attractors for a class of doubly nonlinear parabolic equations,* Cent. Eur. J. Math. **4** *(2006), 163–182.*

[177] A. Miranville and V. Pata, *On the semilinear wave equation with locally distributed damping, in The proceedings of third Polish-Japanese days on Mathematical approach to nonlinear phenomena: modelling, analysis and simulations, T. Aiki, N. Kenmochi, M. Niezgódka, and M. Otani eds., Gakuto International Series on Mathematical Sciences and Applications, Vol. 23, Gakkotosho, Tokyo, Japan, 188-197, 2005.*

[178] A. Miranville and S. Zelik, *Robust exponential attractors for singularly perturbed phase-field type equations,* Electron. J. Diff. Eqns. **2002** *(2002), 1–28.*

[179] A. Miranville and S. Zelik, *Robust exponential attractors for Cahn-Hilliard type equations with singular potentials,* Math. Methods Appl. Sci. **27** *(2004), 545–582.*

[180] A. Miranville and S. Zelik, *Exponential attractors for the Cahn-Hilliard equation with dynamic boundary conditions,* Math. Methods Appl. Sci. **28** *(2005), 709–735.*

[181] A. Miranville and S. Zelik, *Finite-dimensionality of attractors for degenerate equations of elliptic-parabolic type,* Preprint.

[182] A. Miranville and M. Ziane, *On the dimension of the attractor for the Bénard problem with free surfaces,* Russian J. Math. Phys. **5** *(1997), 489–502.*

[183] I. Moise and R. Rosa, *On the regularity of the global attractor of a weakly damped, forced Korteweg-de Vries equation,* Adv. Diff. Eqns. **2** *(1997), 257–296.*

[184] I. Moise, R. Rosa, and X. Wang, *Attractors for non-compact semigroups via energy equations,* Nonlinearity **11** *(1998), 1369–1393.*

[185] I. Moise, R. Rosa, and X. Wang, *Existence of uniform attractors for noncompact nonautonomous dissipative systems via energy equations,* Discrete Cont. Dyn. Systems **10** *(2004), 473–496.*

[186] X. Mora and J. Solà-Morales, *Existence and non-existence of finite-dimensional globally attracting invariant manifolds in semilinear damped wave equations, in Dynamics of infinite-dimensional systems (Lisbon, 1986), NATO Adv. Sci. Inst. F. Comput. Systems Sci., Vol. 37, Springer-Verlag, Berlin, 1987.*

[187] V. Pata and M. Squassina, *On the strongly damped wave equation,* Comm. Math. Phys. **253** *(2005), 511–533.*

[188] V. Pata and S. Zelik, *Global and exponential attractors for 3-D wave equations with displacement dependent damping,* Math. Methods Appl. Sci. **29** *(2006), 1291–1306.*

[189] V. Pata and S. Zelik, *A result on the existence of global attractors for semigroups of closed operators,* Comm. Pure Appl. Anal., To appear.

[190] L. Peletier and W. Troy, *Spatial patterns. Higher order models in physics and mechanics, Progress in nonlinear differential equations and their applications, Vol. 45, Birkhaüser Boston, Boston, MA, 2001.*

[191] Ya.B. Pesin and Ya.G. Sinaĭ, *Space-time chaos in the system of weakly interacting hyperbolic systems,* J. Geom. Phys. **5** *(1988), 483–492.*

[192] Ya.B. Pesin and Ya.G. Sinaĭ, *Space-time chaos in chains of weakly interacting hyperbolic mappings, Adv. Soviet Math., 3, Dynamical systems and statistical mechanics (Moscow, 1991), 165–198, Amer. Math. Soc., Providence, RI, 1991.*

[193] D. Pražák, *On finite fractal dimension of the global attractor for the wave equation with nonlinear damping,* J. Dyn. Diff. Eqns. **14** *(2002), 763–776.*

[194] D. Pražák, *A necessary and sufficient condition on the existence of an exponential attractor,* Cent. Eur. J. Math. **1** *(2003), 411–417.*

[195] D. Pražák, *On the dimension of the attractor for the wave equation with nonlinear damping,* Comm. Pure Appl. Anal. **4** *(2005), 165–174.*

[196] M.I. Rabinovich, A.B. Ezersky, and P.D. Weidman, *The dynamics of patterns, World Sci. Publishing, River Edge, NJ, 2000.*

[197] P.H. Rabinowitz, *Multibump solutions of a semilinear elliptic PDE on* $\mathbb{R}^n$*, Degenerate diffusions (Minneapolis, MN, 1991), 175–185, IMA Vol. Math. Appl., Vol. 47, Springer, New York, 1993.*

[198] G. Raugel, *Global attractors of partial differential equations, in Handbook of dynamical systems, Vol. 2, North-Holland, Amsterdam, 885–982, 2002.*

[199] G. Raugel and G.R. Sell, *Navier-Stokes equations in thin* 3*D domains. III. Existence of a global attractor, in Turbulence in fluid flows, 137–163, IMA Vol. Math. Appl., Vol. 55, Springer, New York, 1993.*

[200] J.C. Robinson, *Infinite-dimensional dynamical systems, Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2001.*
[201] R. Rosa, *The global attractor for the 2D Navier-Stokes flow on some unbounded domains,* Nonlinear Anal. **32** *(1998), 71–85.*
[202] R. Rosa, *The global attractor of a weakly damped, forced Korteweg-de Vries equation in* $H^1(\mathbb{R})$, Mat. Contemp. **19** *(2000), 129–152.*
[203] R. Rosa and R. Temam, *Inertial manifolds and normal hyperbolicity,* Acta Appl. Math. **45** *(1996), 1–50.*
[204] R. Rossi, A. Segatti, and U. Stefanelli, *Attractors for gradient flows of non convex functionals and applications,* Preprint.
[205] O. Rössler, *An equation for continuous chaos,* Phys. Letters A **57** *(1978), 397–398.*
[206] A. Rougirel, *Convergence to steady state and attractors for doubly nonlinear equations,* Preprint.
[207] B. Sandstede, *Stability of travelling waves, in Handbook of dynamical systems, Vol. 2, 983–1055, North-Holland, Amsterdam, 2002.*
[208] H. Schaefer and M. Wolff, *Topological vector spaces, Second edition, Springer-Verlag, New York, 1999.*
[209] G. Schimperna and A. Segatti, *Attractors for the semiflow associated with a class of doubly nonlinear parabolic equations,* Preprint.
[210] B. Schmalfuss, *Attractors for the non-autonomous dynamical systems, in International Conference on Differential Equations, Vol. 1, 2 (Berlin, 1999), World Sci. Publishing, River Edge, NJ, 684–689, 2000.*
[211] A. Segatti, *Global attractor for a class of doubly nonlinear abstract evolution equations,* Discrete Cont. Dyn. Systems **14** *(2006), 801–820.*
[212] A. Segatti, *On the hyperbolic relaxation of the Cahn-Hilliard equation in 3-D: approximation and long time behaviour,* Math. Models Methods Appl. Sci., To appear.
[213] G.R. Sell, *Global attractors for the three-dimensional Navier-Stokes equations,* J. Dyn. Diff. Eqns. **8** *(1996), 1–33.*
[214] G.R. Sell and Y. You, *Dynamics of evolutionary equations, Applied Mathematical Sciences, Vol. 143, Springer-Verlag, New York, 2002.*
[215] F. Shuhong, *Finite dimensional behavior of periodic and asymptotically periodic processes,* Nonlinear Anal. **28** *(1997), 1785–1797.*
[216] S. Slijepčević, *Extended gradient systems: dimension one,* Discrete Cont. Dyn. Systems **6** *(2000), 503–518.*
[217] C. Sun, D. Cao, and J. Duan, *Non-autonomous dynamics of wave equations with nonlinear damping and critical nonlinearity,* Nonlinearity **19** *(2006), 2645–2665.*
[218] Y. Takei, T. Tsujikawa, and A. Yagi, *Numerical computations and pattern formation for absorbate-induced phase transition models,* Sci. Math. Jpn. **61** *(2005), 525–534.*
[219] Y. Takei, M. Efendiev, T. Tsujikawa, and A. Yagi, *Exponential attractor for an adsorbate-induced phase transition model in non smooth domain,* Osaka Math. J. **43** *(2006), 215–237.*
[220] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics, Second edition, Applied Mathematical Sciences, Vol. 68, Springer-Verlag, New York, 1997.*
[221] R. Temam, *Navier-Stokes Equations, AMS-Chelsea Series, Amer. Math. Soc., Providence, RI, 2001.*
[222] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators, North-Holland, Amsterdam, 1978.*
[223] D. Turaev and S. Zelik, *Sinai-Bunimovich space-time chaos in 1D Ginzburg-Landau equations: the homogeneous case,* In preparation.
[224] M.I. Vishik and V.V. Chepyzhov, *Kolmogorov* ϵ*-entropy of attractors of reaction-diffusion systems,* Mat. Sb. **189** *(1998), 235–263.*
[225] M.I. Vishik, B.-W. Schulze, I. Witt, and S.V. Zelik, *The trajectory attractor for a nonlinear elliptic system in a cylindrical domain with a piecewise smooth boundary,* Rend. Accad. Naz. Sci XL Mem. Mat. Appl. **23** *(1999), 125–166.*
[226] M.I. Vishik and S.V. Zelik, *The trajectory attractor for a nonlinear elliptic system in an unbounded domain,* Mat. Sb. **187** *(1996), 21–56.*

[227] M.I. Vishik and S.V. Zelik, *The regular attractor for a nonlinear elliptic system in an unbounded domain,* Mat. Sb. **190** *(1999), 23–58.*

[228] M.I. Vishik and S.V. Zelik, *Nonautonomous perturbations of regular attractors,* In preparation.

[229] B. Wang, *Regularity of attractors for the Benjamin-Bona-Mahoney equation,* J. Phys. A **31** *(1998), 7635–7645.*

[230] B. Wang, *Attractors for reaction-diffusion equations in unbounded domains,* Phys. D **128** *(1999), 41–52.*

[231] B. Wang and H. Lange, *Attractors for the Klein-Gordon-Schrödinger equation,* J. Math. Phys. **40** *(1999), 2445–2457.*

[232] X. Wang, *An energy equation for the weakly damped driven nonlinear Schrödinger equations and its application to their attractors,* Phys. D **88** *(1995), 167–175.*

[233] Y. Wang, C. Zhong, and S. Zhou, *Pullback attractors of nonautonomous dynamical systems,* Discrete Cont. Dyn. Systems **16** *(2006), 587–614.*

[234] Y. Wang and S. Zhou, *Kernel sections and uniform attractors of multi-valued semiprocesses,* J. Diff. Eqns. **232** *(2007), 573–622.*

[235] S.V. Zelik, *The trajectory attractor of the nonlinear elliptic system in an unbounded domain,* Math. Notes **63** *(1998), 135–137.*

[236] S.V. Zelik, *An attractor of a nonlinear system of reaction-diffusion equations in R^n and estimates for its ϵ-entropy,* Math. Notes **65** *(1999), 790–792.*

[237] S. Zelik, *The attractor for a nonlinear reaction-diffusion system with a supercritical nonlinearity and it's dimension,* Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. **24** *(2000), 1–25.*

[238] S.V. Zelik, *The attractor for a nonlinear hyperbolic equation in the unbounded domain,* Discrete Cont. Dyn. Systems **7** *(2001), 593–641.*

[239] S.V. Zelik, *The attractor for a nonlinear reaction-diffusion system in the unbounded domain and Kolmogorov's ϵ-entropy,* Math. Nachr. **232** *(2001), 129–179.*

[240] S.V. Zelik, *The attractors of reaction-diffusion systems in unbounded domains and their spatial complexity,* Comm. Pure Appl. Math. **56** *(2003), 584–637.*

[241] S. Zelik, *Formally gradient reaction-diffusion systems in $\mathbb{R}^n$ have zero spatio-temporal topological entropy,* Discrete Cont. Dyn. Systems, **Suppl.** *(2003), 960–966.*

[242] S.V. Zelik, *Multiparametrical semigroups and attractors of reaction-diffusion systems in $\mathbb{R}^n$*, Proc. Moscow Math. Soc. **65** *(2004), 69–130.*

[243] S. Zelik, *Asymptotic regularity of solutions of singularly perturbed damped wave equations with supercritical nonlinearities,* Discrete Cont. Dyn. Systems **11** *(2004), 351–392.*

[244] S.V. Zelik, *Spatial and dynamical chaos generated by reaction-diffusion systems in unbounded domains,* J. Dyn. Diff. Eqns., To appear.

[245] S. Zelik, *Nondecaying solutions of 2D Navier-Stokes equations in a strip,* Submitted.

[246] S. Zelik, *Weak solutions for the 3D Navier-Stokes equations in cylindrical domains: the spatially non-decaying case,* In preparation.

[247] C.-K. Zhong, M.-H. Yang, and C.-Y. Sun, *The existence of the global attractors for the norm-to-weak continuous semigroup and application to the nonlinear reaction-diffusion equations,* J. Diff. Eqns. **223** *(2006), 367–399.*

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