

**EXPONENTIAL ATTRACTORS AND
FINITE-DIMENSIONAL REDUCTION FOR
NONAUTONOMOUS DYNAMICAL SYSTEMS.**

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ABSTRACT. We suggest in this article a new explicit algorithm allowing to construct exponential attractors which are uniformly Hölder continuous with respect to the variation of the dynamical system in some natural large class. Moreover, we extend this construction to nonautonomous dynamical systems (dynamical processes) treating in that case the exponential attractor as a uniformly exponentially attracting finite-dimensional time-dependent set in the phase space. In particular, this result shows that, for a wide class of nonautonomous equations of mathematical physics, the limit dynamics remains finite-dimensional no matter how complicated the dependence of the external forces on time is. We illustrate the main results of this article on the model example of a nonautonomous reaction-diffusion system in a bounded domain.

INTRODUCTION.

It is well known that the long-time behavior of dissipative dynamical systems generated by evolution equations of mathematical physics can be described in terms of the so-called global attractor \mathcal{A} which is, by definition, a compact invariant subset of the phase space Φ which attracts the images of all the bounded subsets as time goes to infinity, i.e., for every bounded subset $B \subset \Phi$,

$$(0.1) \quad \lim_{t \rightarrow \infty} \text{dist}_{\Phi}(S(t)B, \mathcal{A}) = 0,$$

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where $S(t)$ is the semigroup associated with the problem considered and dist is the nonsymmetric Hausdorff distance between sets. Thus, on the one hand, the attractor contains (in some sense) all the essential dynamics and, on the other hand, it is usually essentially smaller than the initial phase space. In particular, very often, its fractal (or Hausdorff) dimension is finite:

$$(0.2) \quad \dim_F(\mathcal{A}, \Phi) \leq C < \infty$$

and, consequently, in spite of the initial infinite-dimensionality of the phase space Φ , the reduced limit dynamics is finite-dimensional and can be effectively studied, using the concepts and methods of the classical theory of dynamical systems, see [1], [6], [15], [18], [23] and the references therein.

However, we recall that the approach based on the concept of *global* attractors has two rather essential drawbacks: on the one hand, the rate of convergence in (0.1) can be arbitrarily slow and it is usually very difficult (if not impossible) to estimate this rate in terms of the physical parameters of the problem and, on the other hand, the global attractor is, in general, only upper semicontinuous with respect to perturbations, so that the global attractor can change drastically under very small perturbations of the initial dynamical system. These drawbacks obviously lead to very essential difficulties in numerical simulations of global attractors and even make the global attractor, in some sense, unobservable.

In order to overcome these drawbacks, the concept of an *exponential* attractor has been suggested in [8]. By definition, an exponential attractor \mathcal{M} is a compact *semiinvariant* set of the phase space which is finite-dimensional (in the sense of (0.2)) and attracts exponentially the images of the bounded subsets of Φ , i.e., there exist a positive constant α and a monotonic function Q such that

$$(0.3) \quad \text{dist}_\Phi(S(t)B, \mathcal{M}) \leq Q(\|B\|_\Phi)e^{-\alpha t}, \quad t \geq 0,$$

for all the bounded subsets B of the phase space Φ . Moreover, in contrast to global attractors, the constant α and the function Q can be expressed explicitly in terms of the physical parameters of the problem. Thus, being still finite-dimensional, an exponential attractor allows to control the rate of convergence of $S(t)B$ as $t \rightarrow \infty$ and, as a consequence, an exponential attractor is more robust than the global attractor.

We note however that, in contrast to the global attractor, an exponential attractor is *not unique* and, consequently, the problem of "the best choice" of an exponential attractor is very important. In order to overcome the uniqueness problem, it would be very good to have a relatively simple (with respect to its possible numerical realization) algorithm which allows to construct, for every dynamical system S (belonging to some natural class), an exponential attractor $\mathcal{M} = \mathcal{M}_S$ such that the map $S \mapsto \mathcal{M}_S$ is, in some sense, regular (e.g., upper and lower semicontinuous at every point).

We now recall that the original construction of exponential attractors from [8] was based on the so-called squeezing property and was highly nonconstructive (indeed, Zorn's lemma had to be used). The lower semicontinuity property was also obtained, but only up to "time-shifts", which is factually equivalent to the following:

$$(0.4) \quad \lim_{\varepsilon \rightarrow 0} \text{dist}_\Phi(\mathcal{A}_0, \mathcal{M}_\varepsilon) = 0,$$

where \mathcal{A}_0 and \mathcal{M}_ε are the global attractor of the nonperturbed system and an exponential attractor of the perturbed one given by the above construction respectively.

An alternative, more explicit, construction of an exponential attractor has been suggested in [9]. This construction involves two Banach spaces H and H_1 such that H_1 is compactly embedded into H and requires the map $S = S(1)$ to satisfy the following *smoothing* property for the difference of two solutions:

$$(0.5) \quad \|Sh_1 - Sh_2\|_{H_1} \leq K\|h_1 - h_2\|_H, \quad \forall h_1, h_2 \in B,$$

where $B \subset \Phi \equiv H$ is a bounded absorbing set of the semigroup considered. Moreover, it was shown in [11] that, under natural assumptions, it is possible to construct a one-parametrical family of exponential attractors \mathcal{M}_ε , $\varepsilon \in [0, 1]$, associated with the one-parametrical family of semigroups S_ε satisfying (0.5) uniformly with respect to ε , such that

$$(0.6) \quad \text{dist}_H^{\text{symm}}(\mathcal{M}_\varepsilon, \mathcal{M}_0) \leq C\varepsilon^\kappa,$$

where $C, \kappa > 0$ and $\text{dist}^{\text{symm}}$ denotes the symmetric Hausdorff distance; see also [13] and [21].

It is however worth emphasizing that the above construction gives the lower semicontinuity only at one point S_0 ($\varepsilon = 0$). Moreover, the construction of the perturbed attractor \mathcal{M}_ε essentially uses not only the perturbed semigroup S_ε , but also the exponential attractor \mathcal{M}_0 of the nonperturbed dynamical system, see [11].

In the present article, we improve the above scheme of construction of an exponential attractor and give an explicit algorithm which allows to construct, for every map S satisfying (0.5), an exponential attractor \mathcal{M}_S such that the map $S \mapsto \mathcal{M}_S$ is Hölder continuous at *every point* (and, in particular, \mathcal{M}_S now *only* depends on S). To be more precise, we prove the following result.

Theorem 0.1. *For every map $S \in \mathbb{S}_{\delta, K}(B)$ (roughly speaking, the class $\mathbb{S}_{\delta, K}(B)$ consists of all the maps S which satisfy (0.5) for a fixed K , see Definition 1.1), there exists an exponential attractor $\mathcal{M} = \mathcal{M}_S$ which satisfies (0.2) and (0.3) uniformly with respect to $S \in \mathbb{S}_{\delta, K}(B)$ and such that, for every $S_1, S_2 \in \mathbb{S}_{\delta, K}(B)$, we have*

$$(0.7) \quad \text{dist}_H^{\text{symm}}(\mathcal{M}_{S_1}, \mathcal{M}_{S_2}) \leq C\|S_1 - S_2\|_{\mathbb{S}}^\kappa,$$

where the positive constants C and κ are independent of the concrete choice of the maps S_1 and S_2 .

The proof of this theorem is given in Section 1. Moreover, its extension to asymptotically smoothing systems is also discussed there.

We now turn to nonautonomous dynamical systems. In that case, instead of a semigroup, we have a so-called (dynamical) process $U(t, \tau)$ depending on two parameters $t, \tau \in \mathbb{R}$ (or $t, \tau \in \mathbb{Z}$ for discrete times), $t \geq \tau$, which are naturally interpreted as evolution maps from time τ to time t and, consequently, should satisfy

$$(0.8) \quad U(t, \tau) \circ U(\tau, s) = U(t, s), \quad t \geq \tau \geq s.$$

The asymptotic behavior of nonautonomous dynamical systems is essentially less understood and, to the best of our knowledge, the finite-dimensionality of the limit

dynamics was established only for some special (e.g. quasiperiodic) dependences of the external forces on time.

Indeed, there exist, at the present time, two principally different approaches for extending the concept of a *global* attractor to the nonautonomous case. The first one is based on the embedding of the nonautonomous dynamical system (0.8) into a larger *autonomous* one by using the skew-product technique. This approach naturally leads to the so-called *uniform* attractor \mathcal{A}^{un} which remains time-independent in spite of the fact that the dynamical system now depends explicitly on the time, see [3], [4], [6], [16] and [20]. In contrast to this, the second approach allows the attractor of the nonautonomous problem to be a time-dependent set as well, $t \mapsto \mathcal{A}(t)$, $t \in \mathbb{R}$. This leads to the so-called *pullback* attractor which is, by definition, a strictly invariant (i.e., $U(t, \tau)\mathcal{A}(\tau) = \mathcal{A}(t)$) family of compact subsets of the phase space which possesses the following pullback attraction property: for every bounded subset B of the phase space Φ and every $t \in \mathbb{R}$, we have

$$(0.9) \quad \lim_{s \rightarrow \infty} \text{dist}_{\Phi}(U(t, t-s)B, \mathcal{A}(t)) = 0,$$

see [2], [7], [17] and [22] for details.

We note however that both approaches described above are far from being perfect and have very essential drawbacks. Indeed, concerning the *uniform* attractor, in order to realize the reduction to an autonomous system via the skew-product technique, one usually needs to consider an auxiliary dynamical system generated by the temporal shifts acting on the so-called hull of all time-dependent external forces (see [3]). Unfortunately, in order to construct it, the external forces should be known for *all* times $t \in \mathbb{R}$, which, in some sense, violates the causal principle. Moreover, for more or less general external forces, this auxiliary artificial dynamical system is much more complicated than the initial nonautonomous dynamical system (in particular, it has infinite dimension and infinite topological entropy). This naturally leads to the artificial infinite-dimensionality of the uniform attractor and to the artificial complexity of the associated autonomous dynamics. In particular, the above "pathological" infinite-dimensionality appears even in the simple case of the following exponentially stable linear equation:

$$(0.10) \quad \partial_t u - \Delta_x u = h(t), \quad u|_{\partial\Omega} = 0,$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, whose dynamics is completely clear (one exponentially attracting trajectory), see [6].

Now, concerning the concept of a *pullback* attractor, although it works perfectly well for equation (0.10) (and even for much more general nonautonomous dynamical systems, see [2], [11], [24] and [25]), it has also an essential drawback related with the fact that the rate of convergence in (0.9) is not *uniform* with respect to t and, consequently, the "forward" convergence (as $t \rightarrow +\infty$) to the pullback attractor does not hold in general. Indeed, let us consider the following nonautonomous ODE:

$$(0.11) \quad y' = f(t, y), \quad \text{where } f(t, y) := \begin{cases} -y, & t \leq 0, \\ (-1 + 2t)y - ty^2, & t \in [0, 1], \\ y - y^2, & t \geq 1. \end{cases}$$

Then, on the one hand, the pullback attractor is reduced to zero, i.e., $\mathcal{A}(t) \equiv \{0\}$, $\forall t \in \mathbb{R}$, and, on the other hand, for $t \geq 1$, every nonzero trajectory starting from a sufficiently small neighborhood of zero leaves this neighborhood and never enters it again, which clearly contradicts our intuitive understanding of an "attractor".

Thus, although both approaches described above work perfectly well for some particular cases of nonautonomous dynamical systems, they do not give a reasonable description of the long-time behavior for general nonautonomous external forces. In order to overcome these difficulties, it seems natural to generalize the concept of an exponential attractor to the nonautonomous case, see [10], [14] and [19]. We note however that, in all these articles, the *uniform* attractor's approach was used in order to construct an exponential attractor for the nonautonomous system considered and, consequently, an (uniform) exponential attractor remained time-independent. Since, under this approach, an exponential attractor should contain the uniform attractor, all the drawbacks of uniform attractors (artificial infinite-dimensionality and high dynamical complexity) described above are preserved for exponential attractors.

In the present article, we give a systematic study of exponential attractors of nonautonomous systems based on the concept of a *nonautonomous* (pullback) attractor. Thus, in our approach, an exponential attractor of a nonautonomous dynamical system is also time-dependent. To be more precise, a family $t \mapsto \mathcal{M}(t)$ of compact semiinvariant (i.e., $U(t, \tau)\mathcal{M}(\tau) \subset \mathcal{M}(t)$) sets of the dynamical process (0.8) is an (nonautonomous) exponential attractor if

1) The fractal dimension of all the sets $\mathcal{M}(t)$ is finite and uniformly bounded with respect to t :

$$(0.12) \quad \dim_F(\mathcal{M}(t), \Phi) \leq C < \infty.$$

2) There exist a positive constant α and a monotonic function Q such that, for every $t \in \mathbb{R}$, $s \geq 0$ and every bounded subset B of Φ ,

$$(0.13) \quad \text{dist}_\Phi(U(t+s, t)B, \mathcal{M}(t+s)) \leq Q(\|B\|_\Phi)e^{-\alpha s}.$$

We emphasize that the convergence in (0.13) is *uniform* with respect to $t \in \mathbb{R}$ and, consequently, under this approach, we indeed overcome the main drawback of *pullback* global attractors. Thus, according to (0.12) and (0.13), a nonautonomous exponential attractor (if it exists) gives indeed a reasonable extension of a finite-dimensional reduction principle to general nonautonomous systems.

We also study perturbations of nonautonomous exponential attractors. To be more precise, we prove the following nonautonomous analogue of Theorem 0.1 which can be considered as the main result of the article.

Theorem 0.2. *For every discrete dynamical process $U = U(n, m)$, $n, m \in \mathbb{Z}$, $n \geq m$, which satisfies the following property: $U(n) := U(n+1, n) \in \mathbb{S}_{\delta, K}(B)$ for every $n \in \mathbb{Z}$ (the class $\mathbb{S}_{\delta, K}(B)$ is the same as in Theorem 0.1, see Definition 1.1), there exists an exponential attractor $n \mapsto \mathcal{M}_U(n)$ which satisfies estimates (0.12) and (0.13) uniformly with respect to all the processes belonging to this class. Moreover, for every processes U_1 and U_2 satisfying the above property, we have*

$$(0.14) \quad \text{dist}_H^{\text{symm}}(\mathcal{M}_{U_1}(n), \mathcal{M}_{U_2}(n)) \leq C \sup_{s \in \mathbb{N}} \left\{ e^{-\beta s} \|U_1(n-s) - U_2(n-s)\|_{\mathbb{S}}^\kappa \right\},$$

where the positive constants C , β and κ are independent of the concrete choice of U_1 and U_2 .

We illustrate this result on the model example of a reaction-diffusion system in a bounded domain Ω :

$$(0.15) \quad \partial_t u = a \Delta_x u - f(u) + g(t), \quad u|_{t=\tau} = u_\tau, \quad u|_{\partial\Omega} = 0.$$

Here, $u = (u^1, \dots, u^k)$ is an unknown vector-valued function, Δ_x is the Laplacian with respect to the variable x , a is a diffusion matrix, $f(u)$ is a given nonlinear interaction function and $g \in L^\infty(\mathbb{R}, L^2(\Omega))$ are given time-dependent external forces.

Applying the above abstract result to the dynamical processes $U_g(t, \tau)$ associated with problem (0.1), we prove (in Section 3) that, under natural assumptions on the nonlinearity f and the diffusion matrix a , for every g satisfying

$$(0.16) \quad \|g\|_{L^\infty(\mathbb{R}, L^2(\Omega))} \leq K'$$

(where K' is some fixed number), there exists an exponential attractor $t \mapsto \mathcal{M}_g(t)$ which satisfies (0.12) and (0.13) uniformly with respect to g enjoying (0.16). Moreover, for every external forces g_1 and g_2 satisfying (0.16), we have the following estimate:

$$(0.17) \quad \text{dist}_H^{\text{symm}}(\mathcal{M}_{g_1}(t), \mathcal{M}_{g_2}(t)) \leq C \left(\int_{-\infty}^t e^{-\beta(t-s)} \|g_1(s) - g_2(s)\|_{L^2(\Omega)}^2 ds \right)^\kappa,$$

where $H := H_0^1(\Omega)$ and the positive constants C , β and κ only depend on K' and are independent of the concrete choice of g_1 and g_2 .

In particular, applying (0.17) to the external forces $g(\cdot)$ and $g(\cdot + \tau)$ for some $\tau \in \mathbb{R}$, we deduce

$$(0.18) \quad \text{dist}_H^{\text{symm}}(\mathcal{M}_g(t + \tau), \mathcal{M}_g(t)) \leq C \left(\int_{-\infty}^t e^{-\beta(t-s)} \|g(s) - g(s + \tau)\|_{L^2(\Omega)}^2 ds \right)^\kappa.$$

Estimate (0.18) shows, in particular, that, if the external forces g are time-independent (time-periodic or almost periodic), the same will hold for the exponential attractor $t \mapsto \mathcal{M}_g(t)$. Thus, the dependence of the exponential attractors $\mathcal{M}_g(t)$ on the time reflects in a right way the dependence of the external forces on the time. Moreover, estimates (0.17) and (0.18) show that the exponential attractor $\tau \mapsto \mathcal{M}_g(\tau)$ is independent of the future (of the values of the external forces for $t \geq \tau$), so that the causal principle is satisfied, and the dependence of $\mathcal{M}_g(\tau)$ on the past (on the external forces $g(\tau - s)$, $s > 0$) decays exponentially with respect to the time passed (with respect to s) in a complete agreement with our physical intuition. We also establish that the function $t \mapsto \mathcal{M}_g(t)$ is uniformly Hölder continuous with respect to t in the Hausdorff metric.

It is finally worth noting that the natural formula

$$(0.19) \quad \mathcal{M}_g^{\text{un}} := \left[\cup_{t \in \mathbb{R}} \mathcal{M}_g(t) \right]_{H_0^1(\Omega)},$$

where $[\cdot]_V$ denotes the closure in the space V , gives the so-called (infinite-dimensional) *uniform* exponential attractor for problem (0.15) introduced in [10]. We also

note that, concerning *global* attractors, the analogues of formulae (0.12-0.13) and (0.17-0.18) are verified only when the nonautonomous process is a small perturbation of an autonomous semigroup possessing the so-called regular attractor, see [1], [12] and [24], and is false in general.

In the present article, we restrict ourselves to the simplest model example (0.15) of a nonautonomous reaction-diffusion system, although our scheme has a universal nature and seems to be applicable to all known (by the authors) classes of equations of mathematical physics for which the finite-dimensionality of global/pullback attractors can be established (e.g., the 2D Navier-Stokes system, damped hyperbolic equations, the Cahn-Hilliard equation, etc.). More complicated examples, including damped hyperbolic equations and some kind of singularly perturbed problems, e.g., with external forces rapidly oscillating in space or time, will be considered in forthcoming articles.

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§1 A CONSTRUCTION OF EXPONENTIAL ATTRACTORS: THE AUTONOMOUS CASE.

In this section, we give a construction of a family of exponential attractors for abstract semigroups which will be Hölder continuous with respect to perturbations (the nonautonomous case will be considered in the next section). To this end, we first define the admissible class of semigroups.

Definition 1.1. Let H and H_1 be two Banach spaces such that H_1 is compactly embedded into H and let B be a bounded subset of H_1 . For given positive constants δ and K , we define a class $\mathbb{S}_{\delta,K}(B)$ of nonlinear operators $S : H_1 \rightarrow H_1$ as follows. An operator $S \in \mathbb{S}_{\delta,K}(B)$ if

1) The operator S maps a δ -neighborhood $\mathcal{O}_\delta(B)$ of the set B into B :

$$(1.1) \quad S : \mathcal{O}_\delta(B) \rightarrow B,$$

where the neighborhood is taken in the topology of H_1 .

2) For every points $h_1, h_2 \in \mathcal{O}_\delta(B)$, we have

$$(1.2) \quad \|Sh_1 - Sh_2\|_{H_1} \leq K\|h_1 - h_2\|_H.$$

Here, the metric in the space $\mathbb{S}_{\delta,K}(B)$ is defined by

$$(1.3) \quad \|S_1 - S_2\|_{\mathbb{S}} := \sup_{h \in \mathcal{O}_\delta(B)} \|S_1 h - S_2 h\|_{H_1}.$$

For every $S \in \mathbb{S}_{\delta,K}(B)$, the associated semigroup $\{S(n), n \in \mathbb{N}\}$ is defined as the iterations of the map S .

The main result of this section is the following theorem which gives a Hölder continuous family of exponential attractors associated with the maps $S \in \mathbb{S}_{\delta,K}(B)$.

Theorem 1.1. *For every $S \in \mathbb{S}_{\delta,K}(B)$, there exists an exponential attractor \mathcal{M}_S which satisfies the following assumptions:*

1) *The set \mathcal{M}_S is a compact finite-dimensional subset of H_1 , i.e.,*

$$(1.4) \quad \dim_F(\mathcal{M}_S, H_1) \leq C_1,$$

where \dim_F is the fractal dimension in H_1 .

2) The set \mathcal{M}_S is semiinvariant with respect to S , i.e., $S\mathcal{M}_S \subset \mathcal{M}_S$.

3) This set enjoys the following exponential attraction property:

$$(1.5) \quad \text{dist}_{H_1}(S(n)B, \mathcal{M}_S) \leq C_2 e^{-\alpha n}, \quad n \in \mathbb{N},$$

where dist denotes the nonsymmetric Hausdorff distance between sets in H_1 .

4) The map $S \mapsto \mathcal{M}_S$ is Hölder continuous in the following sense:

$$(1.6) \quad \text{dist}_{H_1}^{\text{symm}}(\mathcal{M}_{S_1}, \mathcal{M}_{S_2}) \leq C_3 \|S_1 - S_2\|_{\mathbb{S}}^{\kappa},$$

where $\text{dist}^{\text{symm}}$ is the symmetric Hausdorff distance between sets and the positive constants C_i , $i = 1, 2, 3$, α and κ only depend on B , H , H_1 , δ and K , but are independent of the concrete choice of $S \in \mathbb{S}_{\delta, K}(B)$.

Proof. We first fix a finite covering of the set B by δ/K -balls in the space H (such a covering exists since the embedding $H_1 \subset H$ is compact). Let $V_0 := \{h_1, \dots, h_{N_0}\} \subset B$ be the centers of these balls.

We also fix some $S \in \mathbb{S}_{\delta, K}(B)$ and set $V_1 = V_1(S) := SV_0$. Then, according to estimate (1.2), the system of δ -balls in the space H_1 centered at the points of V_1 covers the image SB . Moreover, according to (1.1), every ball $B(\delta, h, H_1)$ centered at $h \in V_1$ belongs to $\mathcal{O}_{\delta}(B)$ (here and below, we denote by $B(r, h, V)$ the r -ball in V centered at h).

We now construct, for every $n \in \mathbb{N}$, a special covering of $S(n)B$ by $\delta 2^{-n+1}$ -balls centered at $V_n \subset B$ by using an inductive procedure. For $n = 1$, the required set V_1 has already been constructed. We assume that, for $n = k$, we already have the required system of H_1 -balls of radius $\delta 2^{-k+1}$ centered at $V_k \subset B$ which covers $S(k)B$. In order to construct the next set V_{k+1} , we first need to fix a model covering of the unit ball $B(1, 0, H_1)$ in the space H_1 by $1/(2K)$ -balls in the space H (such a covering exists due to the compactness of the embedding $H_1 \subset H$). Let $\mathbb{U} := \{u_1, \dots, u_N\} \subset B(1, 0, H_1)$ be the centers of these balls. Then, we cover every ball $B(\delta 2^{-k+1}, h, H_1)$ with $h \in V_k$ by N balls of radius $\delta 2^{-k+1}/(2K) = \delta 2^{-k}/K$ in the space H , defining the centers $V'_k(h)$ of these balls by the following formula:

$$(1.7) \quad V'_k(h) := h + \delta 2^{-k+1} \mathbb{U}, \quad h \in V_k$$

(it is essential for our construction to use always *the same* model covering \mathbb{U} which is also independent of the choice of $S \in \mathbb{S}_{\delta, K}(B)$). Thus, the balls in the space H with radius $\delta 2^{-k}/K$ centered at all the points of $V'_k := \cup_{h \in V_k} V'_k(h)$ cover $\cup_{h \in V_k} B(\delta 2^{-k+1}, h, H_1)$ and, consequently, they also cover $S(k)B$ by the induction assumption. We finally set $V_{k+1}(h) := SV'_k(h)$ and $V_{k+1} := \cup_{h \in V_k} V_{k+1}(h)$. Then, according to (1.2), the $\delta 2^{-k}$ -balls in the space H_1 centered at V_{k+1} cover $S(k+1)B$. Thus, by induction, the sets V_n are constructed for all $n \in \mathbb{N}$.

Furthermore, according to the above construction, we have

$$(1.8) \quad \begin{cases} 1. \quad \#V_k = N_0 \cdot N^{k-1}, \quad k \in \mathbb{N}, \\ 2. \quad \text{dist}_{H_1}(S(k)B, V_k) \leq \delta 2^{-k+1}, \\ 3. \quad \text{dist}_{H_1}^{\text{symm}}(V_{k+1}, SV_k) \leq \delta 2^{-k}. \end{cases}$$

Indeed, the first and the second assertions of (1.8) are straightforward and the third one follows from the fact that, by construction, $\text{dist}_{H_1}(V_{k+1}(h), Sh) \leq \delta 2^{-k}$ for all $h \in V_k$.

We now define the sets $E_k = E_k(S)$ by

$$(1.9) \quad E_1 := V_1, \quad E_{k+1} := V_{k+1} \cup SE_k, \quad k \in \mathbb{N}.$$

Then, obviously,

$$(1.10) \quad \begin{cases} 1. \ #E_k \leq kN_0 \cdot N^k, \\ 2. \ SE_k \subset E_{k+1}, \\ 3. \ \text{dist}_{H_1}(S(k)B, V_k) \leq \delta 2^{-k+1}. \end{cases}$$

We finally define the required exponential attractor $\mathcal{M} = \mathcal{M}_S$ as follows:

$$(1.11) \quad \mathcal{M} = [\mathcal{M}']_{H_1}, \quad \mathcal{M}' := \cup_{k=1}^{\infty} E_k,$$

where $[\cdot]_V$ denotes the closure in the space V . Thus, it only remains to verify that the set \mathcal{M} defined by (1.11) satisfies all the assertions of Theorem 1.1. To this end, we need to control the distance between E_k and $S(n)B$. We emphasize that, in contrast to [11] and [13], we do not "project" the sets E_k onto $S(k)B$ and, consequently, we do not have the embedding $E_k \subset S(k)B$. Nevertheless, instead of this embedding, we now have the following estimate.

Lemma 1.1. *Let the above assumptions hold. Then, there exist positive constants C and $\alpha < 1$, depending only on K , such that*

$$(1.12) \quad \text{dist}_{H_1}(E_k, S(n)B) \leq C\delta 2^{-\alpha k}$$

for all $k \in \mathbb{N}$ and $n \leq \alpha k$.

Proof. We first note that, due to the Lipschitz continuity of S provided by estimate (1.2), we have

$$(1.13) \quad \text{dist}_{H_1}^{symm}(S(m)A, S(m)C) \leq K^m \text{dist}_{H_1}^{symm}(A, C)$$

for all $A, C \subset B$ and all $m \in \mathbb{N}$ (without loss of generality, we may assume that $\|\cdot\|_H \leq \|\cdot\|_{H_1}$). Thus, iterating the third estimate of (1.8) and using (1.13), we obtain

$$(1.14) \quad \begin{aligned} \text{dist}_{H_1}^{symm}(S(n-1)V_{k+1-n}, S(n)V_{k-n}) &\leq \delta 2^{-k+n} K^{n-1}, \\ \text{dist}_{H_1}^{symm}(S(n-2)V_{k+1-(n-1)}, S(n-1)V_{k-(n-1)}) &\leq \delta 2^{-k+(n-1)} K^{n-2}, \\ &\vdots \\ \text{dist}_{H_1}^{symm}(V_{k+1}, SV_k) &\leq \delta 2^{-k}, \end{aligned}$$

and, consequently, using the triangle inequality, we have

$$(1.15) \quad \text{dist}_{H_1}^{symm}(S(m)V_{k+1-m}, S(n)V_{k-n}) \leq \sum_{l=0}^{n-1} \delta 2^{-k+l} K^l \leq C\delta 2^{-k} (2K)^n$$

for all $m \leq n - 1$, $n \leq k$ and $k \in \mathbb{N}$. We now recall that, due to the construction of V_k , we have $V_k \subset B$ and, due to assumption (1.1), $SB \subset B$. Thus, (1.15) implies that

$$(1.16) \quad \text{dist}_{H_1}(S(m)V_{k+1-m}, S(n)B) \leq C\delta 2^{-k}(2K)^n, \quad n \leq k, \quad m \leq n - 1.$$

We now restrict the possible values of n as follows:

$$(1.17) \quad n \leq \alpha k, \quad \alpha := \frac{1}{1 + \log_2(2K)}.$$

Then, (1.16) yields

$$(1.18) \quad \text{dist}_{H_1}(S(m)V_{k-m}, S(n)B) \leq 2C\delta 2^{-\alpha k}, \quad n \leq \alpha k, \quad m \leq n - 1.$$

Since, by definition, $E_k := \bigcup_{m=0}^{k-1} S(m)V_{k-m}$ and $S(m)V_{k-m} \subset S(m)B \subset S(n)B$ if $m \geq n$, then, estimate (1.18) implies (1.12), which finishes the proof of the lemma.

We are now ready to verify that the attractor \mathcal{M} defined by (1.11) satisfies all the assertions of Theorem 1.1. Indeed, the semiinvariance and the exponential attraction property (1.5) are straightforward consequences of (1.10) and (1.11).

Let us verify the finite-dimensionality. Let $\varepsilon > 0$ be sufficiently small. We need to construct a covering of \mathcal{M} (or, equivalently, of \mathcal{M}') by a finite number of ε -balls in H_1 . We fix n such that the $\varepsilon/2$ -balls centered at V_n cover $S(n)B$. According to (1.8)₂, we then have $n \sim n(\varepsilon) := \log_2 \frac{2\delta}{\varepsilon}$. Moreover, according to Lemma 1.1, the ε -balls centered at V_n cover every E_k with $k \geq k(\varepsilon) := \alpha^{-1} \log_2 \frac{4C\delta}{\varepsilon}$. Thus, the minimal number $N_\varepsilon(\mathcal{M}, H_1)$ of ε -balls which are necessary to cover \mathcal{M} can be estimated as follows:

$$(1.19) \quad N_\varepsilon(\mathcal{M}, H_1) \leq \#V_{n(\varepsilon)} + \sum_{k \leq k(\varepsilon)} \#E_k \leq N_0(k(\varepsilon) + 1)^2 \cdot N^{k(\varepsilon)+1},$$

where we have used (1.8) and (1.10) in order to estimate $\#E_k$. Consequently,

$$(1.20) \quad \dim_F(\mathcal{M}, H_1) := \limsup_{\varepsilon \rightarrow 0} \frac{\log_2 N_\varepsilon(\mathcal{M}, H_1)}{\log_2 \frac{1}{\varepsilon}} \leq \alpha^{-1} \log_2 N < \infty.$$

Thus, the compactness and the finite-dimensionality of \mathcal{M} are also verified and it only remains to verify the Hölder continuity of the function $S \mapsto \mathcal{M}_S$. To this end, we need one more lemma.

Lemma 1.2. *Let the above assumptions hold. Then, for every $S_1, S_2 \in \mathbb{S}_{\delta, K}(B)$,*

$$(1.21) \quad \text{dist}_{H_1}^{\text{symm}}(E_k(S_1), E_k(S_2)) \leq CK^k \|S_1 - S_2\|_{\mathbb{S}}, \quad k \in \mathbb{N},$$

where the constant C only depends on K .

Proof. We first verify (1.21) for the sets $V_k(S_i)$ by induction. We denote the left-hand side of (1.21) by d_k . Then, since V_0 is the same for all $S \in \mathbb{S}_{\delta, K}(B)$, we have $d_1 \leq \|S_1 - S_2\|_{\mathbb{S}}$. We now assume that the required estimate is already verified for $k = n$. Then, for every $h^1 \in V_n(S_1)$, there exists $h^2 \in V_n(S_2)$ such that $\|h^1 - h^2\|_{H_1} \leq d_n$, and vice versa. Furthermore, according to our construction

of the sets V'_k (and, more precisely, due to the fact that our model covering \mathbb{U} is independent of S), we conclude that, for every $\bar{h}^1 \in V'_k(h^1)$, there exists $\bar{h}^2 \in V'_k(h^2)$ such that

$$(1.22) \quad \|\bar{h}^1 - \bar{h}^2\|_{H^1} \leq d_n.$$

Thus, it only remains to estimate $S_1\bar{h}^1 - S_2\bar{h}^2$:

$$(1.23) \quad \begin{aligned} \|S_1\bar{h}^1 - S_2\bar{h}^2\|_{H_1} &\leq \\ &\leq \|S_1\bar{h}^1 - S_1\bar{h}^2\|_{H_1} + \|S_2\bar{h}^2 - S_1\bar{h}^2\|_{H_1} \leq K\|h^1 - h^2\|_{H_1} + \|S_1 - S_2\|_{\mathbb{S}}. \end{aligned}$$

Consequently, (1.22) and (1.23) imply that

$$(1.24) \quad d_{n+1} \leq Kd_n + \|S_1 - S_2\|_{\mathbb{S}}, \quad d_1 \leq \|S_1 - S_2\|_{\mathbb{S}}.$$

Solving (1.24), we infer

$$(1.25) \quad \text{dist}_{H_1}^{\text{symm}}(V_k(S_1), V_k(S_2)) \leq CK^k \|S_1 - S_2\|_{\mathbb{S}}, \quad k \in \mathbb{N}.$$

The required estimate for the sets E_k is a straightforward consequence of (1.25) and of the obvious estimate

$$(1.26) \quad \text{dist}_{H_1}^{\text{symm}}(S_1A, S_2C) \leq K \text{dist}_{H_1}^{\text{symm}}(A, C) + \|S_1 - S_2\|_{\mathbb{S}},$$

which is valid for every $S_i \in \mathbb{S}_{\delta, K}(B)$, $i = 1, 2$, and every $A, C \subset B$. This finishes the proof of Lemma 1.2.

We are now ready to verify the required Hölder continuity of the map $S \mapsto \mathcal{M}_S$. Indeed, let $S_1, S_2 \in \mathbb{S}_{\delta, K}(B)$. Our aim is to prove that

$$(1.27) \quad \text{dist}_{H_1}(\mathcal{M}_{S_1}, \mathcal{M}_{S_2}) \leq C_1 \|S_1 - S_2\|_{\mathbb{S}}^{\kappa}$$

for some positive constants C_1 and κ (the estimate for the Hausdorff distance between \mathcal{M}_{S_2} and \mathcal{M}_{S_1} follows immediately from (1.27) by changing $S_1 \rightarrow S_2$ and $S_2 \rightarrow S_1$). Moreover, in order to prove (1.27), it is sufficient to verify it for every $E_k(S_1)$, $k \in \mathbb{N}$ (see (1.11)).

We assume that $k \in \mathbb{N}$ and $u_0 \in E_k(S_1)$ are arbitrary. Then, according to Lemma 1.2, we have

$$(1.28) \quad \text{dist}_{H_1}(u_0, \mathcal{M}_{S_2}) \leq CK^k \|S_1 - S_2\|_{\mathbb{S}}.$$

On the other hand, according to Lemma 1.1, for every $n \leq \alpha k$, there exists $u'_0 \in S_1(n)B$ such that

$$(1.29) \quad \|u_0 - u'_0\|_{H_1} \leq C\delta 2^{-\alpha k}.$$

Let $\bar{u} \in B$ be such that $u'_0 = S_1(n)\bar{u}$ and set $u''_0 := S_2(n)\bar{u}$. Then, obviously,

$$(1.30) \quad \|u'_0 - u''_0\|_{H_1} \leq CK^n \|S_1 - S_2\|_{\mathbb{S}}.$$

Moreover, due to estimate (1.5), we have

$$(1.31) \quad \text{dist}_{H_1}(u_0'', \mathcal{M}_{S_2}) \leq \delta 2^{-n+1}.$$

Combining now (1.29)–(1.31) and using the triangle inequality, we deduce that

$$(1.32) \quad \text{dist}_{H_1}(E_k(S_1), \mathcal{M}_{S_2}) \leq C_1(2^{-\alpha k} + 2^{-n} + K^n \|S_1 - S_2\|_{\mathbb{S}}),$$

where C_1 only depends on K and δ . We shall use estimate (1.28) for $k \leq n/\alpha$ and estimate (1.32) otherwise, where $n = n(\|S_1 - S_2\|_{\mathbb{S}})$ will be fixed below. Then, we have

$$(1.33) \quad \begin{aligned} \text{dist}_{H_1}(E_k, \mathcal{M}_{S_2}) &\leq C_1(2 \cdot 2^{-n} + K^n \|S_1 - S_2\|_{\mathbb{S}}) + \\ &\quad + CK^{n/\alpha} \|S_1 - S_2\|_{\mathbb{S}} \leq C_2(2^{-n} + K^{n/\alpha} \|S_1 - S_2\|_{\mathbb{S}}). \end{aligned}$$

Fixing now n in the right-hand side of (1.33) in an optimal way, i.e.,

$$n \sim \frac{\alpha}{\alpha + \log_2 K} \log_2 \frac{1}{\|S_1 - S_2\|_{\mathbb{S}}},$$

we obtain estimate (1.27) with $\kappa := \frac{\alpha}{\alpha + \log_2 K}$ and finish the proof of Theorem 1.1.

We now discuss several relaxations of the assumptions of Theorem 1.1.

Remark 1.1. We first note that, according to Definition 1.1, the smoothing property (1.2) should hold in a δ -neighborhood $\mathcal{O}_\delta(B)$ of the set B . However, very often in applications, one can verify (1.2) only for *smoother* initial data belonging to a compact absorbing (exponentially attracting) set in H_1 . Thus, this assumption can be a rather essential restriction. In order to overcome this restriction, we note that we have factually used properties (1.1) and (1.2) in the "neighborhood"

$$(1.34) \quad \mathcal{O}'_\delta(B) := B + \cup_{\nu \in [0, \delta]} \nu \mathbb{U},$$

where \mathbb{U} consists of the centers of the model covering of the unit ball of H_1 by $1/(2K)$ -balls in the space H_1 . So, Theorem 1.1 remains valid if we replace the neighborhood $\mathcal{O}_\delta(B)$ by $\mathcal{O}'_\delta(B)$. In particular, if B is a bounded set in a stronger space $H_2 \subset H_1$ and the centers \mathbb{U} can be taken in H_2 , then the δ -neighborhood of B can also be chosen in the metric of H_2 . Moreover, the corresponding exponential attractors \mathcal{M}_S will belong to H_2 .

Remark 1.1'. It is also worth noting that we have factually verified a slightly stronger than (1.4) inequality. Indeed, let us introduce the so-called Kolmogorov's ε -entropy $\mathbb{H}_\varepsilon(\mathcal{M}_S, H_1) := \log_2 N_\varepsilon(\mathcal{M}_S, H_1)$ of the exponential attractor constructed in Theorem 1.1. Then, according to (1.19), we have

$$(1.34') \quad \mathbb{H}_\varepsilon(\mathcal{M}_S, H_1) \leq C_1 \log_2 \frac{1}{\varepsilon} + C_2, \quad \varepsilon \rightarrow 0,$$

where the constants C_1 and C_2 are independent of the concrete choice of S . Thus, not only the fractal dimension of the exponential attractors, but also the minimal number of ε -balls which are necessary to cover them for every fixed positive ε , are uniformly bounded.

We now recall that the smoothing property (1.2) is typical for parabolic equations in bounded domains, but it is not usually satisfied for more general (e.g., hyperbolic) problems. In order to overcome this restriction, we need to generalize the class $\mathbb{S}_{\delta,K}(B)$ of admissible maps.

Definition 1.2. Let H and H_1 be two Banach spaces such that H_1 is compactly embedded into H and let B be a closed subset of H_1 . An operator $S : H_1 \rightarrow H_1$ belongs to the class $\mathbb{S}_{\delta,\varepsilon,K}(B)$ for some positive constants ε , K and δ if

- 1) Condition (1.1) is satisfied.
- 2) The following generalized version of (1.2):

$$(1.35) \quad \|Sh_1 - Sh_2\|_{H_1} \leq (1 - \varepsilon)\|h_1 - h_2\|_{H_1} + K\|h_1 - h_2\|_H$$

holds for every $h_1, h_2 \in \mathcal{O}_\delta(B)$.

The following theorem is the analogue of Theorem 1.1 for the class $\mathbb{S}_{\delta,\varepsilon,K}(B)$.

Theorem 1.2. *We assume that the set B can be covered by a finite number of δ -balls in the space H_1 with centers $V_0 \subset B$. Then, for every $S \in \mathbb{S}_{\delta,\varepsilon,K}(B)$, there exists an exponential attractor $\mathcal{M}_S \subset H_1$ which satisfies all the assertions of Theorem 1.1 (with constants C_i , $i = 1, \dots, 4$, α and κ depending only on H , H_1 , B , δ , ε and K).*

Proof. The proof of this theorem is very similar to that of Theorem 1.1. We thus only indicate below how to construct the sets $V_k(S)$, leaving the details to the reader. Indeed, the initial set $V_0 = V_0(S) \subset B$, which gives a δ -covering of B , is already constructed by assumption. We now assume that we have already constructed the set $V_k \subset B$ for some $k = n$ such that the $\delta_n := \delta(1 - \varepsilon/2)^n$ -balls centered at V_n cover $S(n)B$. We introduce the model covering of the unit ball $B(1, 0, H_1)$ by a finite number of $\varepsilon/(2K)$ -balls in the space H and let $\mathbb{U} \subset B(1, 0, H)$ be the centers in this covering. We cover (as in the proof of Theorem 1.1) every ball $B(\delta_n, h, H_1)$, $h \in V_n(S)$, by a finite number of $\delta_n\varepsilon/(2K)$ -balls in the space H centered at

$$(1.36) \quad V'_n(h) := h + \delta_n\mathbb{U}$$

and set $V_{n+1}(h) := SV'_n(h)$, $V_{n+1} := \cup_{h \in V_n} V_{n+1}(h)$. Then, formula (1.35) implies that the system of δ_{n+1} -balls (where $\delta_{n+1} = (1 - \varepsilon)\delta_n + K\delta_n\varepsilon/(2K) = \delta(1 - \varepsilon/2)^{n+1}$) covers $S(n+1)B$. Thus, the system of sets $V_n(S)$ is constructed for every $n \in \mathbb{N}$. The rest of the proof repeats word by word that of Theorem 1.1 (with very minor changes related with the fact that the exponent 2^{-n} is now replaced by $(1 - \varepsilon/2)^n$).

To conclude, we give a slightly different form of the asymptotic smoothing property (1.35) which gives exponential attractors in the weaker space H and which is useful for the exponential attractors' theory of hyperbolic equations (see [13]).

Definition 1.3. Let H and H_1 be two Banach spaces such that H_1 is compactly embedded into H and let B be a bounded subset of the space H . An operator $S : H \rightarrow H$ belongs to the class $\mathbb{S}'_{\delta,\varepsilon,K}(B)$ if

- 1) Assumption (1.1) holds for the δ -neighborhood $\mathcal{O}_\delta(B)$ in the space H .
- 2) For every $h_1, h_2 \in \mathcal{O}_\delta(B)$, the difference $v := Sh_1 - Sh_2$ can be split into a sum $v = v_1 + v_2$ such that

$$(1.37) \quad \|v_1\|_H \leq (1 - \varepsilon)\|h_1 - h_2\|_H, \quad \|v_2\|_{H_1} \leq K\|h_1 - h_2\|_H.$$

In contrast to (1.3), the distance between two maps $S_1, S_2 \in \mathbb{S}'_{\delta, \varepsilon, K}(B)$ is now defined by using the space H instead of H_1 .

The following theorem is the analogue of Theorems 1.1 and 1.2 for the class $\mathbb{S}'_{\delta, \varepsilon, K}(B)$.

Theorem 1.3. *We assume that the set B can be covered by a finite number of δ -balls in the space H with centers $V_0 \subset \mathcal{O}_\delta(B)$. Then, for every $S \in \mathbb{S}'_{\delta, \varepsilon, K}(B)$, there exists an exponential attractor $\mathcal{M}_S \subset \mathcal{O}_\delta(B)$ which satisfies all the assertions of Theorem 1.1 (in which the space H_1 is replaced by H).*

The proof of this theorem is completely analogous to those of theorems 1.1 and 1.2 and we leave it to the reader (see also [11] and [13]).

Remark 1.2. The idea of replacing the initial neighborhood $\mathcal{O}_\delta(B)$ by the special "neighborhood" $\mathcal{O}'_\delta(B)$ (see (1.34)) described in Remark 1.1 remains valid for the classes $\mathbb{S}_{\delta, \varepsilon, K}(B)$ and $\mathbb{S}'_{\delta, \varepsilon, K}(B)$ as well.

§2 A CONSTRUCTION OF EXPONENTIAL ATTRACTORS: THE NONAUTONOMOUS CASE.

In this section, we extend the results of Section 1 to nonautonomous dynamical systems. We first recall that, in the nonautonomous case, we should consider, instead of semigroups, the so-called dynamical processes. By definition, a dynamical process U in the phase space Φ is a two-parametrical family of maps $\{U(l, m) : \Phi \rightarrow \Phi, l, m \in \mathbb{Z}, l \geq m\}$ such that

$$(2.1) \quad U(l, k) \circ U(k, m) = U(l, m), \quad U(m, m) = Id, \quad l, k, m \in \mathbb{Z}, \quad l \geq k \geq m.$$

We set $U(n) := U(n+1, n)$. Then, every dynamical process U is uniquely determined by the one-parametrical family of maps $\{U(n)\}_{n \in \mathbb{Z}}$ by

$$(2.2) \quad U(n+k, n) = U(n+k-1, n) \circ U(n+k-2, n) \circ \cdots \circ U(n, n), \quad n \in \mathbb{Z}, \quad k \in \mathbb{N},$$

and, vice versa, every such family generates the associated dynamical process U defined by (2.2). So, one can identify the dynamical process U with the one-parametrical family $\{U(n)\}_{n \in \mathbb{Z}}$. We also recall that the case where the maps $U(n)$ are independent of n , $U(n) \equiv S$, $n \in \mathbb{Z}$, corresponds to the autonomous case considered in the previous section. Indeed, in that case, obviously,

$$U(n+k, n) = S(k),$$

where $S(k)$ is the semigroup generated by the map S .

The following theorem seems to be a natural generalization of Theorem 1.1 to the nonautonomous case.

Theorem 2.1. *Let the spaces H_1 and H and the set B be the same as in Definition 1.1. Then, for every dynamical process U in H_1 such that $U(n) \in \mathbb{S}_{\delta, K}(B)$ for every $n \in \mathbb{Z}$ (for some δ and K which are independent of n), there exists a nonautonomous exponential attractor $n \mapsto \mathcal{M}_U(n)$, $n \in \mathbb{Z}$, which satisfies the following properties: 1) The attractor $\mathcal{M}_U(n) \subset B$ for every $n \in \mathbb{Z}$ and its fractal dimension is finite, i.e.,*

$$(2.3) \quad \dim_F(\mathcal{M}_U(n), H_1) \leq C_1,$$

where the constant C_1 is independent of n .

2) The family $\mathcal{M}_U(n)$ is semiinvariant with respect to U , i.e.,

$$(2.4) \quad U(k, m)\mathcal{M}_U(m) \subset \mathcal{M}_U(k)$$

for all $k, m \in \mathbb{Z}$, $k \geq m$.

3) This family enjoys a uniform exponential attraction property of the following form:

$$(2.5) \quad \text{dist}_{H_1}(U(n+k, n)B, \mathcal{M}_U(n+k)) \leq C_2 e^{-\alpha k},$$

where the constants C_2 and α are independent of $n \in \mathbb{Z}$ and $k \in \mathbb{N}$.

4) The map $U \mapsto \mathcal{M}_U(n)$ is uniformly Hölder continuous in the following sense: for every dynamical processes U_1 and U_2 such that $U_i(n) \in \mathbb{S}_{\delta, K}(B)$, $n \in \mathbb{Z}$, $i = 1, 2$, we have

$$(2.6) \quad \text{dist}_{H_1}^{\text{symm}}(\mathcal{M}_{U_1}(n), \mathcal{M}_{U_2}(n)) \leq C_3 \sup_{l \in (-\infty, n)} \left\{ e^{-\beta(n-l)} \|U_1(l) - U_2(l)\|_{\mathbb{S}}^{\kappa} \right\},$$

where the positive constants C_i , $i = 1, 2, 3$, α , β and κ only depend on B , H , H_1 , δ and K , but are independent of n and of the concrete choice of the U_i .

Remark 2.1. Estimate (2.6) confirms that the attractor $\mathcal{M}_U(n)$ satisfies the causal principle, i.e., $\mathcal{M}_U(n)$ is independent of $U(k)$, $k \geq n$. Moreover, (2.6) also shows that the influence of the past decays exponentially with respect to the time, in complete agreement with our physical intuition.

Proof. The proof of this theorem is similar to that of Theorem 1.1 in the autonomous case, except that we now need to consider "time-dependent" analogues of V_k , E_k and \mathcal{M} . As in Theorem 1.1, we fix a finite covering of the set B by δ/K -balls in the space H and let $V_0 := \{h_1, \dots, h_{N_0}\} \subset B$ be the centers of these balls. We also fix an arbitrary dynamical process U satisfying the assumptions of the theorem.

We now set $V_1(n) = V_1^U(n) := U(n-1)V_0$, $n \in \mathbb{Z}$. Since all the maps $U(n)$ satisfy (1.2), then, the system of δ -balls in the space H_1 centered at $V_1(n)$ covers the set $U(n-1)B = U(n, n-1)B$ for all $n \in \mathbb{Z}$.

As in Theorem 1.1, our aim is to construct the family of sets $V_k(n) = V_k^U(n)$ by induction with respect to k such that the $\delta 2^{-k+1}$ -balls in the space H_1 centered at $V_k(n)$ cover $U(n, n-k)B$ (for all $n \in \mathbb{Z}$). For $k = 1$, these sets have already been constructed. We now assume that the required sets are already constructed for some $k = l$ and we let the model covering \mathbb{U} be the same as in the proof of Theorem 1.1. Then, for every $n \in \mathbb{Z}$, we cover every ball $B(\delta 2^{-l+1}, h, H_1)$ with $h \in V_l(n)$ by $N := \#\mathbb{U}$ balls of radius $\delta 2^{-l+1}/(2K)$ in the space H centered at $V_l'(h) := h + \delta 2^{-l+1}\mathbb{U}$ (see (1.7)). Thus, according to the assumption of induction, the system of $\delta 2^{-l+1}/(2K)$ -balls in the space H centered at $V_l'(n) := \cup_{h \in V_l(n)} V_l'(h)$ covers $U(n, n-l)B$, $n \in \mathbb{Z}$. We finally set $V_{l+1}(n+1) := U(n)V_l'(n)$. Since all the maps $U(n)$ satisfy (1.2), the $\delta 2^{-l}$ -balls in the space H_1 centered at $V_{l+1}(n+1)$ cover $U(n) \circ U(n, n-l)B = U(n+1, n-l)B$ ($n \in \mathbb{Z}$) and condition (1.1) for $U(n)$ guarantees that $V_{l+1}(n+1) \subset B$. Thus, the required "nonautonomous" sets $V_k(n)$ are constructed for every $n \in \mathbb{Z}$ and $k \in \mathbb{N}$.

Furthermore, the above construction gives the following analogue of (1.8):

$$(2.7) \quad \begin{cases} 1. & V_k(n) = N_0 \cdot N^{k-1}, \quad N_0 := \#V_0, \quad N := \#\mathbb{U}, \quad k \in \mathbb{N}, \quad n \in \mathbb{Z}, \\ 2. & \text{dist}_{H_1}(U(n, n-k)B, V_k(n)) \leq \delta 2^{-k+1}, \\ 3. & \text{dist}_{H_1}^{symm}(V_{k+1}(n+1), U(n)V_k(n)) \leq \delta 2^{-k}. \end{cases}$$

As in the autonomous case, we define the sets $E_k(n) = E_k^U(n)$ by

$$(2.8) \quad E_1(n) := V_1(n), \quad E_{k+1}(n+1) := V_{k+1}(n+1) \cup U(n)E_k(n), \quad k \in \mathbb{N}, \quad n \in \mathbb{Z}.$$

Then, analogously to (1.10), we have

$$(2.9) \quad \begin{cases} 1. & \#E_k(n) \leq kN_0 \cdot N^k, \\ 2. & U(n)E_k(n) \subset E_{k+1}(n+1), \\ 3. & \text{dist}_{H_1}(U(n, n-k)B, E_k(n)) \leq \delta 2^{-k+1}, \end{cases}$$

and the required attractor $\mathcal{M}(n) = \mathcal{M}_U(n)$ can be defined analogously to (1.11):

$$(2.10) \quad \mathcal{M}(n) = [\mathcal{M}'(n)]_{H_1}, \quad \mathcal{M}'(n) := \cup_{k=1}^{\infty} E_k(n), \quad n \in \mathbb{Z}.$$

It only remains to verify that the attractor $\mathcal{M}(n)$ defined by (2.10) satisfies all the assertions of the theorem. Indeed, the semiinvariance (2.4) is a straightforward consequence of (2.9)₂ and (2.10) and the uniform exponential attraction property follows from (2.9)₃. In order to verify the finite-dimensionality and the Hölder continuity, we need the following natural analogue of Lemma 1.1.

Lemma 2.1. *Let the above assumptions hold. Then, there exist positive constants C and $\alpha < 1$ depending only on K such that*

$$(2.11) \quad \text{dist}_{H_1}(E_k(n), U(n, n-l)B) \leq C\delta 2^{-\alpha k}$$

for all $k \in \mathbb{N}$, $l \leq \alpha k$ and $n \in \mathbb{Z}$.

The proof of estimate (2.11) is based on the obvious nonautonomous analogue of estimate (1.13), namely,

$$(2.12) \quad \text{dist}_{H_1}^{symm}(U(n, n-m)A, U(n, n-m)C) \leq K^m \text{dist}_{H_1}^{symm}(A, C)$$

(which holds for all $A, C \subset B$, $n \in \mathbb{Z}$ and $m \in \mathbb{N}$) and iterations of estimate (2.7)₃; it can be obtained by repeating word by word the proof of Lemma 1.1 and we thus leave it to the reader.

Having estimate (2.12), it is not difficult to verify (arguing exactly as in the proof of Theorem 1.1) that the sets $\mathcal{M}(n)$ are indeed finite-dimensional and satisfy estimate (2.3) with exactly the same constant C_1 as in the autonomous case. So, it only remains to verify the Hölder continuity (2.6). To this end, we need the following natural generalization of Lemma 1.2.

Lemma 2.2. *Let the above assumptions hold. Then, for every dynamical processes U_1 and U_2 satisfying the assumptions of the theorem and for every $k \in \mathbb{N}$ and $n \in \mathbb{Z}$,*

$$(2.13) \quad \text{dist}_{H_1}^{\text{symm}}(E_k^{U_1}(n), E_k^{U_2}(n)) \leq C \sup_{l \in (0, k]} \|U_1(n-l) - U_2(n-l)\|_{\mathbb{S}},$$

where the constant C is the same as in Lemma 1.2.

The proof of this estimate also repeats word by word that of Lemma 1.2 in the autonomous case and we also leave it to the reader (we note that only the maps $U(n-1), \dots, U(n-k)$ are involved in the construction of the sets $E_k^U(n)$). That is the reason why we have the supremum with respect to $l \in (0, k]$ in the right-hand side of (2.13).

We are now ready to verify the Hölder continuity (2.6) and finish the proof of Theorem 2.1. Let U_1 and U_2 be two dynamical processes satisfying the assumptions of the theorem. Then, as in Theorem 1.1, it is sufficient to verify that, for every $k \in \mathbb{N}$ and every $n \in \mathbb{Z}$,

$$(2.14) \quad \text{dist}_{H_1}(E_k^{U_1}(n), \mathcal{M}^{U_2}(n)) \leq CK^k \sup_{l \in (-\infty, n)} e^{-\beta(n-l)} \|U_1(l) - U_2(l)\|_{\mathbb{S}}^{\kappa}.$$

Indeed, let $u_0 \in E_k^{U_1}(n)$ be arbitrary. Then, according to Lemma 2.2, we have

$$(2.15) \quad \text{dist}_{H_1}(u_0, \mathcal{M}_{U_2}(n)) \leq CK^k \sup_{l \in (0, k]} \|U_1(n-l) - U_2(n-l)\|_{\mathbb{S}}.$$

On the other hand, according to Lemma 2.1, for every $l \leq \alpha k$, there exists $u'_0 \in U_1(n, n-l)B$ such that

$$(2.16) \quad \|u_0 - u'_0\|_{H_1} \leq C\delta 2^{-\alpha k}.$$

Let $\bar{u} \in U_1(n, n-l)B$ be such that $u'_0 = U(n, n-l)\bar{u}$ and set $u''_0 := U_2(n, n-l)\bar{u}$. Then, obviously,

$$(2.17) \quad \|u'_0 - u''_0\|_{H_1} \leq CK^l \sup_{m \in (0, l]} \|U_1(n-m) - U_2(n-m)\|_{\mathbb{S}}.$$

We now recall that $\mathcal{M}^{U_2}(n)$ attracts exponentially the images of B with respect to U_2 and, consequently (see (2.9)₃),

$$(2.18) \quad \text{dist}_{H_1}(u''_0, \mathcal{M}^{U_2}(n)) \leq \delta 2^{-l+1}.$$

Estimating now the right-hand side of (2.17) as follows:

$$(2.19) \quad \begin{aligned} K^l \sup_{m \in (0, l]} \|U_1(n-m) - U_2(n-m)\|_{\mathbb{S}} &\leq \\ &\leq K^l e^{\beta l} \sup_{m \in (0, l]} e^{-\beta m} \|U_1(n-m) - U_2(n-m)\|_{\mathbb{S}} \leq \\ &\leq (Ke^{\beta})^l \sup_{m \in (-\infty, n)} e^{-\beta(n-m)} \|U_1(m) - U_2(m)\|_{\mathbb{S}} := K_1^l \cdot D_n(U_1, U_2) \end{aligned}$$

(which is valid for some positive β) and, using the triangle inequality, we have

$$(2.20) \quad \text{dist}_{H_1}(E_k^{U_1}(n), \mathcal{M}^{U_2}(n)) \leq C_1(2^{-\alpha k} + 2^{-l} + K_1^l \cdot D_n(U_1, U_2)),$$

where C_1 only depends on K and δ . As in the proof of Theorem 1.1, we use estimate (2.15) for $k \leq l/\alpha$ and estimate (2.20) otherwise. Then, analogously to (1.33), we deduce that

$$(2.21) \quad \text{dist}_{H_1}(E_k^{U_1}(n), \mathcal{M}_{U_2}(n)) \leq C_2(2^{-l} + K_1^{l/\alpha} \cdot D_n(U_1, U_2)).$$

Fixing now $l = l(D_n(U_1, U_2))$ in an optimal way, i.e.,

$$l \sim \frac{\alpha}{\alpha + \log_2 K_1} \log_2 \frac{1}{D_n(U_1, U_2)},$$

we obtain (2.14) and finish the proof of Theorem 2.1.

We now study the dependence of the exponential attractors $\mathcal{M}_U(n)$ on $n \in \mathbb{Z}$. To this end, we first introduce the group of temporal translations $\{T_k, k \in \mathbb{Z}\}$ acting on the space of all dynamical processes by

$$(2.22) \quad (T_k U)(m, n) := U(m + k, n + k), \quad k, m, n \in \mathbb{Z}, \quad m \geq n.$$

Then, obviously, $(T_k U)(n) = U(n + k)$. The following simple corollary gives the translation invariance of the exponential attractors $\mathcal{M}_U(n)$ constructed in the previous theorem.

Corollary 2.1. *Let the assumptions of Theorem 2.1 hold. Then, for every dynamical process U satisfying the assumptions of that theorem, the associated exponential attractor satisfies the following cocycle identity:*

$$(2.23) \quad \mathcal{M}_U(n + k) = \mathcal{M}_{T_k U}(n)$$

for all $k, n \in \mathbb{Z}$.

Indeed, identity (2.23) follows immediately from the explicit construction of $\mathcal{M}_U(n)$ given in Theorem 2.1.

We now note that, in the autonomous case, where $U(n) \equiv S$, $n \in \mathbb{Z}$, the corresponding exponential attractor $n \mapsto \mathcal{M}_U(n)$ is independent of n and, thus, coincides with the "autonomous" exponential attractor \mathcal{M}_S of the associated semigroup $S(n)$ constructed in Theorem 1.1. Indeed, according to (2.16) and (2.23),

$$\text{dist}_{H_1}^{\text{symm}}(\mathcal{M}_U(n + k), \mathcal{M}_U(n)) \leq C \sup_{m \in (-\infty, n)} \|U(m + k) - U(m)\|_S^\kappa = 0.$$

More generally, if the process U is time-periodic, i.e., $U(n + T) \equiv U(n)$ for all $n \in \mathbb{Z}$ and some period $T \in \mathbb{N}$, then, analogous reasonings show that the associated exponential attractor $n \mapsto \mathcal{M}_U(n)$ is also time-periodic with the same period. The next corollary addresses another important case, namely, the case where the process U is *asymptotically* autonomous.

Corollary 2.2. *Let U be a dynamical process satisfying the assumptions of Theorem 2.1. We assume, in addition, that the process U is a heteroclinic orbit joining two autonomous semigroups $S_1(n)$ and $S_2(n)$ ($S_1, S_2 \in \mathbb{S}_{\delta, K}(B)$), i.e.,*

$$(2.24) \quad \lim_{n \rightarrow -\infty} \|U(n) - S_1\|_{\mathbb{S}} = \lim_{n \rightarrow +\infty} \|U(n) - S_2\|_{\mathbb{S}} = 0.$$

Then, the associated exponential attractor $n \mapsto \mathcal{M}_U(n)$ is also a heteroclinic orbit joining the "autonomous" exponential attractors \mathcal{M}_{S_1} and \mathcal{M}_{S_2} associated with the limit semigroups:

$$\lim_{n \rightarrow -\infty} \text{dist}_{H_1}^{\text{symm}}(\mathcal{M}_U(n), \mathcal{M}_{S_1}) = \lim_{n \rightarrow +\infty} \text{dist}_{H_1}^{\text{symm}}(\mathcal{M}_U(n), \mathcal{M}_{S_2}) = 0.$$

Indeed, according to (2.6), we have

$$(2.25) \quad \text{dist}_{H_1}^{\text{symm}}(\mathcal{M}_U(n), \mathcal{M}_{S_i}) \leq C \sup_{l \in (-\infty, n)} \left\{ e^{-\beta(n-l)} \|U(l) - S_i\|_{\mathbb{S}}^{\kappa} \right\}.$$

Passing to the limit $n \rightarrow \pm\infty$ and using (2.24) and the obvious fact that $\|U(n) - S_i\|_{\mathbb{S}} \leq 2\|B\|_{H_1}$ is uniformly bounded, we deduce the assertion of the corollary.

Remark 2.2. If the dynamical process U is quasiperiodic or almost periodic with respect to the time (in the sense of Bochner-Amerio), then, it is not difficult to verify, using (2.6), that the associated exponential attractor, considered as a set-valued function $n \mapsto \mathcal{M}_U(n)$, will be also quasiperiodic or almost periodic with the same frequency basis (see [12] for analogous results for nonautonomous regular attractors).

We can now formulate the analogues of Theorems 1.2 and 1.3 in the non-autonomous case.

Theorem 2.2. *Let the spaces H and H_1 and the set B be the same as in Definition 1.2. We assume, in addition, that the set B can be covered by a finite number of δ -balls in the space H_1 with centers $V_0 \subset B$. Then, for every dynamical process U such that $U(n) \in \mathbb{S}_{\delta, \varepsilon, K}(B)$ for every $n \in \mathbb{Z}$, there exists an exponential attractor $n \mapsto \mathcal{M}_U(n) \subset B$, $n \in \mathbb{Z}$, which satisfies all the assertions of Theorem 2.1 (with constants C_i , $i = 1, \dots, 4$, α and κ depending only on H , H_1 , B , δ , ε and K).*

Theorem 2.3. *Let the spaces H and H_1 and the set B be the same as in Definition 1.3. We assume, in addition, that the set B can be covered by a finite number of δ -balls in the space H with centers $V_0 \subset \mathcal{O}_{\delta}(B)$. Then, for every dynamical process U such that $U(n) \in \mathbb{S}'_{\delta, \varepsilon, K}(B)$ for every $n \in \mathbb{Z}$, there exists an exponential attractor $n \mapsto \mathcal{M}_U(n) \subset \mathcal{O}_{\delta}(B)$, $n \in \mathbb{Z}$, which satisfies all the assertions of Theorem 2.1, in which the space H_1 is replaced by H .*

The proofs of these theorems are completely analogous to that of Theorem 2.1 (and to those of Theorems 1.2 and 1.3 in the autonomous case) and we leave them to the reader.

Remark 2.3. The idea of replacing the initial neighborhood $\mathcal{O}_{\delta}(B)$ by the special "neighborhood" $\mathcal{O}'_{\delta}(B)$ (see (1.34)) described in Remark 1.1 remains obviously valid in the nonautonomous case as well.

§3 EXPONENTIAL ATTRACTORS FOR A
NONAUTONOMOUS REACTION-DIFFUSION SYSTEM.

In this concluding section, we apply the results obtained above to the following reaction-diffusion problem in a bounded domain $\Omega \subset \subset \mathbb{R}^n$:

$$(3.1) \quad \partial_t u = a \Delta_x u - f(u) + g(t), \quad u|_{t=\tau} = u_\tau, \quad u|_{\partial\Omega} = 0.$$

Here, $u = (u_1, \dots, u_k)$ is an unknown vector-valued function, a is a given constant diffusion matrix with positive symmetric part, $a + a^* > 0$, and $f \in C^2(\mathbb{R}^k, \mathbb{R}^k)$ is a given nonlinear interaction function which satisfies the following standard dissipativity and growth assumptions:

$$(3.2) \quad \begin{cases} 1. & f(u) \cdot u \geq -C, \\ 2. & f'(u) \geq -K, \\ 3. & |f(u)| \leq C(1 + |u|^p), \quad p < p_{max} := \frac{n+2}{n-2}, \end{cases}$$

where $u \cdot v$ denotes the standard inner product in \mathbb{R}^k and $f'(u) \geq -K$ means that $f'(u)v \cdot v \geq -K|v|^2$ for all $u, v \in \mathbb{R}^k$. Finally, we assume that the external forces g belong to the space $L^\infty(\mathbb{R}, L^2(\Omega))$ and satisfy

$$(3.3) \quad \|g\|_{L^\infty(\mathbb{R}, L^2(\Omega))} \leq M$$

for some given (possibly large) constant M .

It is well known (see, e.g., [3] or [6]) that, under the above assumptions, equation (3.1) possesses, for every $\tau \in \mathbb{R}$ and $u_\tau \in H_0^1(\Omega)$, a unique solution $u(t)$, $t \geq \tau$, which satisfies the following dissipative estimate:

$$(3.4) \quad \|u(t)\|_{H_0^1(\Omega)} \leq Q(\|u_\tau\|_{H_0^1(\Omega)})e^{-\alpha(t-\tau)} + C_K, \quad t \geq \tau,$$

where α and Q are a positive constant and a monotonic function depending only on a and f and where the positive constant C_K depends also on M (but is independent of the concrete choice of g). Thus, equation (0.1) defines a dynamical process $\{U_g(t, \tau), \tau \in \mathbb{R}, t \geq \tau\}$ in the phase space $H_0^1(\Omega)$ by

$$(3.5) \quad U_g(t, \tau)u_\tau := u(t), \quad \text{where } u(t) \text{ solves (3.1) with } u(\tau) = u_\tau.$$

Moreover, the following Lipschitz continuity and smoothing properties for the difference of two solutions $u_1(t)$ and $u_2(t)$ have been verified in [10]:

$$(3.5') \quad \begin{aligned} \|u_1(\tau + t) - u_2(\tau + t)\|_{H_0^1(\Omega)} &\leq \\ &\leq C e^{Kt} \|u_1(\tau) - u_2(\tau)\|_{H_0^1(\Omega)}, \quad t \in \mathbb{R}_+, \tau \in \mathbb{R}, \end{aligned}$$

and

$$(3.6) \quad \begin{aligned} \|u_1(\tau + t) - u_2(\tau + t)\|_{H_0^1(\Omega)} &\leq \\ &\leq C(1 + t^{-1/2})e^{Kt} \|u_1(\tau) - u_2(\tau)\|_{L^2(\Omega)}, \quad t \in \mathbb{R}_+, \tau \in \mathbb{R}, \end{aligned}$$

where the constants C and K depend on M and on the H^1 -norms of $u_1(\tau)$ and $u_2(\tau)$, but are independent of t and τ . Furthermore, the following natural Lipschitz

continuity holds for solutions of equation (3.1) with different external forces g_1 and g_2 satisfying (3.3), see, e.g., [6] and [10]:

$$(3.7) \quad \|U_{g_1}(\tau+t, \tau)u_\tau - U_{g_2}(\tau+t, \tau)u_\tau\|_{H_0^1(\Omega)}^2 \leq C e^{Kt} \int_\tau^{\tau+t} \|g_1(s) - g_2(s)\|_{L^2(\Omega)}^2 ds,$$

where C and K depend on M and the H^1 -norm of u_τ , but are independent of t and τ . We also recall that the trajectories of (3.1) possess a further smoothing property of the form

$$(3.8) \quad \|U_g(t + \tau, \tau)u_\tau\|_{H^{1+\delta}(\Omega)} \leq Q_T(\|u_\tau\|_{H_0^1(\Omega)}), \quad t \geq T > 0,$$

where $0 < \delta < 1$ and the monotonic function Q_T depends on M , δ and T , but is independent of t and τ (see [1], [6] and [10] for details).

The main aim of this section is to show that the above estimates are sufficient to construct a robust family of nonautonomous exponential attractors for problem (3.1). To be more precise, the main result of this section is the following theorem.

Theorem 3.1. *Let the nonlinear function f , the diffusion matrix a and the external forces g satisfy the above assumptions. Then, for every external forces g enjoying (3.3), there exists an exponential attractor $t \mapsto \mathcal{M}_g(t)$ of the dynamical process (3.5) which satisfies the following properties:*

1) *The sets $\mathcal{M}_g(t)$ are compact finite-dimensional subsets of $H_0^1(\Omega)$:*

$$(3.9) \quad \dim_F(\mathcal{M}_g(t), H_0^1(\Omega)) \leq C_1, \quad t \in \mathbb{R},$$

where the constant C_1 only depends on M (and is independent of t and g).

2) *These sets are semiinvariant with respect to $U_g(t, \tau)$ and translation-invariant with respect to time-shifts:*

$$(3.10) \quad 1. \ U_g(t, \tau)\mathcal{M}_g(\tau) \subset \mathcal{M}_g(t), \quad 2. \ \mathcal{M}_g(t + s) = \mathcal{M}_{T_s g}(t),$$

where $t, s, \tau \in \mathbb{R}$, $t \geq \tau$ and $\{T_h, h \in \mathbb{R}\}$ is the group of temporal shifts, $(T_h g)(t) := g(t + h)$.

3) *They satisfy a uniform exponential attraction property as follows: there exist a positive constant α and a monotonic function Q (both depending only on M) such that, for every bounded subset B of $H_0^1(\Omega)$, we have*

$$(3.11) \quad \text{dist}_{H_0^1(\Omega)}(U_g(\tau + t, \tau)B, \mathcal{M}_g(\tau + t)) \leq Q(\|B\|_{H_0^1(\Omega)})e^{-\alpha t}$$

for all $\tau \in \mathbb{R}$ and $t \geq 0$.

4) *The map $g \mapsto \mathcal{M}_g(t)$ is Hölder continuous in the following sense:*

$$(3.12) \quad \text{dist}_{H_0^1(\Omega)}^{\text{symm}}(\mathcal{M}_{g_1}(t), \mathcal{M}_{g_2}(t)) \leq C_2 \left(\int_{-\infty}^t e^{-\beta(t-s)} \|g_1(s) - g_2(s)\|_{L^2(\Omega)}^2 ds \right)^\kappa,$$

where the positive constants C , β and κ only depend on M and are independent of g_1 , g_2 and t . In particular, the function $t \mapsto \mathcal{M}_g(t)$ is uniformly Hölder continuous in the Hausdorff metric:

$$(3.13) \quad \text{dist}_{H_0^1(\Omega)}^{\text{symm}}(\mathcal{M}_g(t + s), \mathcal{M}_g(t)) \leq C_3 |s|^{\kappa_1},$$

where C_3 and κ_1 are also independent of g , t and s .

Proof. We first construct a family of exponential attractors for the discrete dynamical processes associated with equation (3.1). To this end, we note that, according to estimate (3.4), the ball $\mathbb{B} = B_R := \{v \in H_0^1(\Omega), \|v\|_{H_0^1(\Omega)} \leq R\}$, where R is a sufficiently large number depending only on M given in (3.3), is a uniform absorbing set for all the processes $U_g(t, \tau)$ generated by equation (0.1). Thus, it only remains to construct the required exponential attractors for initial data belonging to this ball. Moreover, it also follows from the above estimates that there exists $T = T(M)$ such that

$$U_g(\tau + T, \tau)\mathcal{O}_1(\mathbb{B}) \subset \mathbb{B}$$

for all $\tau \in \mathbb{R}$ and all g satisfying (3.3). This embedding, together with estimate (3.6), proves that, for a sufficiently large $K = K(M)$, we have

$$(3.14) \quad U_g(\tau + T, \tau) \in \mathbb{S}_{1,K}(\mathbb{B}) \quad \text{with } H := L^2(\Omega) \text{ and } H_1 := H_0^1(\Omega)$$

for all τ and g . Thus, we can apply Theorem 2.1 to the family of discrete dynamical processes $U_g^r(m, l) := U_g(\tau + mT, \tau + lT)$, $m, l \in \mathbb{Z}$, $m \geq l$. According to this theorem, these processes possess exponential attractors $l \mapsto \mathcal{M}_g(l, \tau)$, $l \in \mathbb{Z}$, which satisfy the following properties:

1) These sets are compact subsets of $H_0^1(\Omega)$ whose fractal dimension is uniformly bounded:

$$(3.15) \quad \mathbb{H}_\varepsilon(\mathcal{M}_g(l, \tau), H_0^1(\Omega)) \leq C_1 \log_2 \frac{1}{\varepsilon} + C_2,$$

where the positive constants C_1 and C_2 only depend on M .

2) They are semiinvariant with respect to the discrete processes: $U_g(\tau + mT, \tau + lT)\mathcal{M}_g(l, \tau) \subset \mathcal{M}_g(m, \tau)$.

3) They enjoy the following uniform exponential attraction property:

$$(3.16) \quad \text{dist}_{H_0^1(\Omega)}(U_g(\tau + mT, \tau + lT)\mathbb{B}, \mathcal{M}_g(m, \tau)) \leq C_3 e^{-\alpha(m-l)},$$

where the positive constants C_3 and α only depend on M and are independent of $m, l \in \mathbb{Z}$, $m \geq l$, $\tau \in \mathbb{R}$ and g satisfying (3.3).

For different g_1 and g_2 satisfying (3.3), we have

$$(3.17) \quad \begin{aligned} & \text{dist}_{H_0^1(\Omega)}^{\text{symm}}(\mathcal{M}_{g_1}(0, \tau), \mathcal{M}_{g_2}(0, \tau)) \leq \\ & \leq C_4 \sup_{n \in \mathbb{N}} \{e^{-\beta n} \|U_{g_1}(\tau - (n-1)T, \tau - nT) - U_{g_2}(\tau - (n-1)T, \tau - nT)\|_{\mathbb{S}}^\kappa\} \leq \\ & \leq C_5 \left(\sup_{n \in \mathbb{N}} e^{-2\beta n/\kappa} \int_{\tau - nT}^{\tau - (n-1)T} \|g_1(s) - g_2(s)\|_{L^2(\Omega)}^2 ds \right)^{\kappa/2} \leq \\ & \leq C_6 \left(\int_{-\infty}^{\tau} e^{-\beta'(\tau-s)} \|g_1(s) - g_2(s)\|_{L^2(\Omega)}^2 ds \right)^{\kappa'} , \end{aligned}$$

where all the constants are positive and only depend on M (here, we have used estimates (2.6) and (3.7)). Moreover, analogously to (2.23), we have the following translation-invariance:

$$(3.18) \quad 1. \mathcal{M}_g(l, \tau) = \mathcal{M}_g(0, lT + \tau), \quad 2. \mathcal{M}_{T_s g}(l, \tau) = \mathcal{M}_g(l, \tau + s).$$

We now verify the Hölder continuity of the function $\tau \mapsto \mathcal{M}_g(0, \tau)$. To this end, we need the following lemma which gives the Hölder continuity of the processes $U_g(t, \tau)$ with respect to the time.

Lemma 3.1. *Let the above assumptions on equation (3.1) hold. Then, for every $u_\tau \in H_0^1(\Omega)$,*

$$(3.19) \quad \|U_g(\tau + s + t, \tau)u_\tau - U_g(\tau + t, \tau)u_\tau\|_{L^2(\Omega)} \leq C|s|^{1/2},$$

where the constant C depends on M and the H^1 -norm of u_τ , but is independent of $t \geq 0$, $\tau \in \mathbb{R}$ and $s \geq 0$. Moreover, for every $T > 0$, we also have

$$(3.20) \quad \|U_g(t + \tau + s, \tau + s)u_\tau - U_g(t + \tau, \tau)u_\tau\|_{H_0^1(\Omega)} \leq \\ \leq C_T e^{Kt} |s|^\gamma, \quad t \geq T, \quad 0 \leq s \leq T/2,$$

where γ is a positive number and the positive constant C_T depends on T and on the H^1 -norm of u_τ , but is independent of t , τ and s .

Proof. It follows from estimate (3.4) and equation (3.1) that the function $u(t) := U_g(t, \tau)u_\tau$, $t \geq \tau$, satisfies

$$(3.21) \quad u \in L^\infty([\tau, +\infty), H_0^1(\Omega)), \quad \partial_t u \in L^\infty([\tau, +\infty), H^{-1}(\Omega)),$$

and is uniformly bounded in these spaces, which immediately implies the Hölder continuity (3.19). In order to verify (3.20), we note that, due to (3.19) and (3.6), for every $v \in H_0^1(\Omega)$, we have

$$(3.22) \quad \|U_g(t + s + \tau, \tau + s)v - U_g(\tau + t, \tau)v\|_{L^2(\Omega)} \leq \\ \leq \|U_g(t + s + \tau, t + \tau)(U_g(t + \tau, \tau + s)v) - U_g(t + \tau, \tau + s)v\|_{L^2(\Omega)} + \\ + \|U_g(t + \tau, \tau + s)v - U_g(t + \tau, \tau + s)(U_g(\tau + s, \tau)v)\|_{L^2(\Omega)} \leq \\ \leq C_T |s|^{1/2} + C_T e^{K(t-s)} \|v - U_g(\tau + s, \tau)v\|_{L^2(\Omega)} \leq C'_T e^{Kt} |s|^{1/2},$$

where all the constants depend on T , M and the H^1 -norm of v , but are independent of t , s and τ . Using now the smoothing property (3.8), we verify that $U_g(t + s + \tau, \tau + s)v$ and $U_g(t + \tau, \tau)v$ are uniformly bounded in $H^{1+\delta}(\Omega)$, with positive δ . The obvious interpolation inequality $\|\cdot\|_{H_0^1(\Omega)} \leq C \|\cdot\|_{L^2(\Omega)}^{\delta/(1+\delta)} \|\cdot\|_{H^{1+\delta}(\Omega)}^{1/(1+\delta)}$ now finishes the proofs of estimate (3.20) and Lemma 3.1.

We are now ready to verify the Hölder continuity of the attractors $\mathcal{M}_g(0, \tau)$ with respect to τ . Indeed, according to (2.6) and (3.20), we have

$$(3.23) \quad \text{dist}_{H_0^1(\Omega)}^{\text{symm}}(\mathcal{M}_g(0, \tau + s), \mathcal{M}_g(0, \tau)) \leq \\ \leq C_7 \sup_{l \in \mathbb{N}} e^{-\beta l} \|U_g(\tau - (l-1)T, \tau - lT) - U_g(\tau + s - (l-1)T, \tau + s - lT)\|_S \leq C_8 |s|^\gamma.$$

We then define the required exponential attractors for continuous times by the following natural formula:

$$(3.24) \quad \mathcal{M}_g(\tau) := \cup_{s \in [0, T]} U_g(\tau, \tau - T - s) \mathcal{M}_g(0, \tau - T - s).$$

We claim that the exponential attractor $\tau \mapsto \mathcal{M}_g(\tau)$ thus defined satisfies all the assertions of Theorem 3.1. We first verify the semiinvariance (3.10)₁. To this end, we first note that, due to the semiinvariance of $\mathcal{M}_g(0, \tau)$ with respect to the

discrete processes $U_g(\tau + lT, \tau + mT)$ (i.e., $U_g(\tau + lT, \tau)\mathcal{M}_g(0, \tau) \subset \mathcal{M}_g(l, \tau) = \mathcal{M}_g(0, \tau + lT)$), it is sufficient to verify (3.10)₁ for $t - \tau := \alpha \in [0, T]$ only. Then, we have

$$\begin{aligned}
U_g(t + \alpha, t)\mathcal{M}_g(t) &= \cup_{s \in [0, T]} U_g(t + \alpha, t - T - s)\mathcal{M}_g(0, t - T - s) = \\
\cup_{s \in [\alpha, T]} U_g(t + \alpha, t - T - s)\mathcal{M}_g(0, t - T - s) &\cup \cup_{s \in [0, \alpha]} U_g(t + \alpha, t - T - s)\mathcal{M}_g(0, t - T - s) \\
&= \cup_{s' \in [0, T - \alpha]} U_g(t + \alpha, t + \alpha - T - s')\mathcal{M}_g(0, t + \alpha - T - s') \cup \\
\cup_{s' \in [T - \alpha, T]} U_g(t + \alpha, t + \alpha - T - s') &\circ U_g(t + \alpha - T - s', t + \alpha - 2T - s')\mathcal{M}_g(0, t + \alpha - 2T - s') \\
&\subset \cup_{s' \in [0, T - \alpha]} U_g(t + \alpha, t + \alpha - T - s')\mathcal{M}_g(0, t + \alpha - T - s') \cup \\
\cup_{s' \in [T - \alpha, T]} U_g(t + \alpha, t + \alpha - T - s') &\mathcal{M}_g(0, t + \alpha - T - s') = \mathcal{M}_g(t + \alpha).
\end{aligned}$$

Thus, the semiinvariance is verified. The translation-invariance with respect to time-shifts (3.10)₂ is an immediate consequence of (3.18)₂ and the obvious translation identity $U_{T_s g}(t, \tau) = U_g(t + s, \tau + s)$. Estimate (3.11) follows in a standard way from (3.16), (3.4) and from the fact that the ball \mathbb{B} is chosen as a uniformly absorbing set for all the processes $U_g(t, \tau)$. The Hölder continuity (3.12) follows easily from its analogue (3.17) for discrete times, definition (3.24) and the Lipschitz continuities (3.6) and (3.7) and the Hölder continuity (3.13) follows immediately from its discrete analogue (3.23) and the Hölder continuity (3.20) (we have introduced the additional time shift T in (3.24) in order to be able to apply (3.20)). We also note that the sets $\mathcal{M}_g(t)$ are closed in $H_0^1(\Omega)$, since the set-valued function $\tau \mapsto \mathcal{M}_g(0, \tau)$ is (Hölder) continuous and all the maps $U_g(t, t - T - s)$ are also continuous. Thus, it only remains to verify the finiteness of the fractal dimension of $\mathcal{M}_g(t)$. In order to prove this, we first note that, according to the Hölder continuities (3.23) and (3.20), we have

$$\begin{aligned}
(3.25) \quad \text{dist}_{H_0^1(\Omega)}^{\text{symm}}(U_g(t, t - T - s_1)\mathcal{M}_g(0, t - T - s_1), \\
U_g(t, t - T - s_2)\mathcal{M}_g(0, t - T - s_2)) \leq C|s_1 - s_2|^{\kappa''}
\end{aligned}$$

for all $s_1, s_2 \in [0, T]$ and $t \in \mathbb{R}$ and for some positive constants C and κ'' . Thus, for a given $\varepsilon > 0$, the minimal number $N_\varepsilon(\mathcal{M}_g(t), H_0^1(\Omega))$ of ε -balls which are necessary to cover $\mathcal{M}_g(t)$ can be estimated as follows:

$$\begin{aligned}
(3.26) \quad N_\varepsilon(\mathcal{M}_g(t), H_0^1(\Omega)) &\leq \\
&\leq \sum_{l=0}^{[(\frac{2C}{\varepsilon})^{1/\kappa''}]} N_{\varepsilon/2}(U_g(t, t - T - l(\varepsilon/2C)^{1/\kappa''})\mathcal{M}_g(0, t - T - l(\varepsilon/2C)^{1/\kappa''}), H_0^1(\Omega)).
\end{aligned}$$

Moreover, since the maps $U_g(t, t - T - s)$ are uniformly Lipschitz continuous, estimate (3.15) implies that

$$\begin{aligned}
(3.27) \quad \mathbb{H}_\varepsilon(U_g(t, t - T - l(\varepsilon/2C)^{1/\kappa''})\mathcal{M}_g(0, t - T - l(\varepsilon/2C)^{1/\kappa''}), H_0^1(\Omega)) &\leq \\
&\leq \mathbb{H}_{\varepsilon/L}(\mathcal{M}_g(0, t - T - l(\varepsilon/2C)^{1/\kappa''}), H_0^1(\Omega)) \leq C'_1/\kappa'' \log_2 \frac{1}{\varepsilon} + C'_2.
\end{aligned}$$

Combining (3.26) and (3.27), we finally obtain

$$(3.28) \quad \mathbb{H}_\varepsilon(\mathcal{M}_g(t), H_0^1(\Omega)) \leq \frac{C}{\kappa''} \log_2 \frac{1}{\varepsilon} + C'$$

for some constants C and C' which are independent of t . Theorem 3.1 is proven.

To conclude, we compare the nonautonomous exponential attractor $t \mapsto \mathcal{M}_g(t)$ obtained above with the so-called infinite-dimensional (uniform) exponential attractor constructed in [10]. To this end, we first briefly recall the most important objects related with the uniform attractor's approach for nonautonomous dynamical systems, using our reaction-diffusion system as a model example (see [3] and [6] for detailed expositions).

Let $g \in L^\infty(\mathbb{R}, L^2(\Omega))$ be some external forces. The hull $\mathcal{H}(g) \subset L^\infty(\mathbb{R}, L^2(\Omega))$ is defined as the following set:

$$(3.29) \quad \mathcal{H}(g) := \left[T_s g, s \in \mathbb{R} \right]_{L_{loc}^\infty(\mathbb{R}, L^2(\Omega))},$$

where the closure is taken in the *local* topology of $L^\infty(\mathbb{R}, L^2(\Omega))$. Obviously, the group of temporal shifts $\{T_h, h \in \mathbb{R}\}$ acts on the hull of g ,

$$(3.30) \quad T_h \mathcal{H}(g) = \mathcal{H}(g), \quad h \in \mathbb{R},$$

which is usually endowed with the local topology $L_{loc}^\infty(\mathbb{R}, L^2(\Omega))$. Using the standard skew-product technique, see [3] and [6], we can embed, for every external forces g satisfying (3.3), the associated dynamical process $U_g(t, \tau)$ into the autonomous dynamical system \mathbb{S}_t acting on the extended phase space $\Phi := H_0^1(\Omega) \times \mathcal{H}(g)$ via

$$(3.31) \quad \mathbb{S}_t(u_0, h) := (U_h(t, 0)u_0, T_t h), \quad u_0 \in H_0^1(\Omega), \quad h \in \mathcal{H}(g), \quad t \geq 0.$$

It is well known that (3.31) is indeed a semigroup. If this semigroup possesses the global attractor $\mathbb{A} = \mathbb{A}(g) \subset \Phi$, then, its projection $\mathcal{A}^{un}(g) := \Pi_1 \mathbb{A}(g)$ onto the first component of the Cartesian product is called the uniform attractor associated with problem (3.1).

It is also known that the uniform attractor $\mathcal{A}^{un}(g)$ exists under the relatively weak assumption that the hull $\mathcal{H}(g)$ is compact in $L_{loc}^\infty(\mathbb{R}, L^2(\Omega))$, but, unfortunately, for more or less general external forces g , its Hausdorff and fractal dimensions are infinite. Instead, the following estimate for its Kolmogorov's ε -entropy holds, see [5] and [10].

Proposition 3.1. *Let the above assumptions hold and let, in addition, the hull $\mathcal{H}(g)$ of the initial external forces be compact. Then, equation (3.1) possesses the uniform attractor $\mathcal{A}^{un}(g)$ and its ε -entropy can be estimated in terms of the ε -entropy of the hull $\mathcal{H}(g)$ as follows:*

$$(3.32) \quad \mathbb{H}_\varepsilon(\mathcal{A}^{un}(g), H_0^1(\Omega)) \leq C_1 \log_2 \frac{1}{\varepsilon} + \\ + \mathbb{H}_{\varepsilon/L}(\mathcal{H}(g) \Big|_{[0, T_0 \log_2 \frac{1}{\varepsilon}] \times \Omega}, L^\infty([0, T_0 \log_2 \frac{1}{\varepsilon}], L^2(\Omega)))$$

for some positive constants C_1 , L and T_0 depending on a and f .

Since an exponential attractor always contains the global attractor, a uniform exponential attractor $\mathcal{M}^{un}(g)$ should necessarily be infinite-dimensional if the uniform attractor $\mathcal{A}^{un}(g)$ has infinite dimension. Thus, following [5], it is natural to use the Kolmogorov's entropy in order to control the "size" of an exponential attractor. To be more precise, the following object has been introduced in [10].

Definition 3.1. A set $\mathcal{M}^{un}(g)$ is an (uniform) exponential attractor of equation (3.1) if the following properties are satisfied:

- 1) Entropy estimate: $\mathcal{M}^{un}(g)$ is a compact subset of the phase space $H_0^1(\Omega)$ which satisfies estimate (3.32) (possibly, for larger constants C_1 , L and T_0).
- 2) Semiinvariance: for every $u_0 \in \mathcal{M}^{un}(g)$, there exists $h \in \mathcal{H}(g)$ such that $U_h(t, 0)u_0 \subset \mathcal{M}^{un}(g)$ for all $t \geq 0$.
- 3) Uniform exponential attraction property: there exists a positive constant α and a monotonic function Q such that, for every $h \in \mathcal{H}(g)$ and every bounded subset $B \subset H_0^1(\Omega)$, we have

$$(3.33) \quad \text{dist}_{H_0^1(\Omega)}(U_h(t + \tau, \tau)B, \mathcal{M}^{un}(g)) \leq Q(\|B\|_{H_0^1(\Omega)})e^{-\alpha t}$$

for all $\tau \in \mathbb{R}$ and $t \geq 0$.

The existence of a (infinite-dimensional) uniform exponential attractor for problem (3.1) has been verified directly in [10]. In the present article, we show that a uniform exponential attractor $\mathcal{M}^{un}(g)$ can be easily constructed if the (nonautonomous) exponential attractor $t \mapsto \mathcal{M}_g(t)$ has already been constructed.

Corollary 3.1. *Let the assumptions of Theorem 3.1 hold and let, in addition, the hull $\mathcal{H}(g)$ of some external forces satisfying (3.3) be compact. Then, there exists a uniform exponential attractor $\mathcal{M}^{un}(g)$ for problem (3.1) which can be constructed as follows:*

$$(3.34) \quad \mathcal{M}^{un}(g) := \left[\bigcup_{t \in \mathbb{R}} \mathcal{M}_g(t) \right]_{H_0^1(\Omega)} = \bigcup_{h \in \mathcal{H}(g)} \mathcal{M}_h(0).$$

Indeed, the second equality in (3.34) is an immediate consequence of the Hölder continuity (3.12) and the definition of the hull $\mathcal{H}(g)$. The entropy estimate (3.32) for $\mathcal{M}^{un}(g)$ is also a standard consequence of this Hölder continuity and of the uniform entropy estimates (3.28) for the nonautonomous attractors. The semiinvariance of $\mathcal{M}^{un}(g)$ follows from equality (3.34) and from the semiinvariance of $\mathcal{M}_h(t)$ and the exponential attraction property (3.33) is an immediate consequence of the analogous property (3.11) for nonautonomous exponential attractors.

Remark 3.1. To conclude, it is worth noting that, although our article is mainly devoted to the *nonautonomous* attractors' approach, Corollary 3.1 allows to make essential improvements in the alternative *uniform* theory as well. In particular, formula (3.34), together with the Hölder continuity (3.12), allows to develop a perturbation theory for the uniform exponential attractors $\mathcal{M}^{un}(g)$. Indeed, it is not difficult to deduce from (3.12) and (3.34) that, for two different external forces satisfying the assumptions of Corollary 3.1, we have

$$(3.35) \quad \text{dist}_{H_0^1(\Omega)}^{symm}(\mathcal{M}^{un}(g_1), \mathcal{M}^{un}(g_2)) \leq C \left(\text{dist}_{L^\infty(\mathbb{R}, L^2(\Omega))}^{symm}(\mathcal{H}(g_1), \mathcal{H}(g_2)) \right)^\kappa,$$

where the positive constants C and κ only depend on M .

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