CLASSIFICATION OF POSITIVE SOLUTIONS OF SEMILINEAR LINEAR ELLIPTIC EQUATIONS IN A RECTANGLE. TWO DIMENSIONAL CASE.

M.Efendiev and S.Zelik

Introduction.

It is well known that positive solutions of semilinear second order elliptic problems have symmetry and monotonicity properties which reflects the symmetry of the operator and of the domain, see e.g.,[GNN81] and [BeN91] for the case of bounded domains and [BeN90,BeN92,BCN97,BCN98] and [GNN81a,BeL83,BuF01] for the case of unbounded domains.

In particular, the symmetry and monotonicity results for the case of semispace have been considered in [BCN97,BCN98] and the analogous results (including the existence and uniqueness of a nontrivial positive solution) for the case of whole space have been obtained in [GNN81a,BeL83,Kwo89,BuF01], see also the references therein.

The goal of the present paper is to give a description of all bounded nonnegative solutions of the following elliptic boundary value problem in a two dimensional rectangle $\Omega_+ := \{(x,y)\in\mathbb{R}^2,\ x\ge 0,\ y\ge 0\}$:

$$\begin{cases} \Delta_{x,y}u = f(u), \quad (x,y)\in\Omega_+,\\ u\big|_{\partial\Omega_+} = 0, \quad u(x,y)\ge 0, \end{cases} \tag{0.1}$$

where we assume that $u\in C_b(\Omega)$ and the nonlinearity f is smooth enough ($f\in C^1(\mathbb{R})$) and $f(0)=0$.

It is known (see [BCN97]) that, under the above assumptions, every solution $u(x,y)$ of (0.1) (if it exists) should be monotonic with respect to x and y and, consequently, there exist the following limits

$$\lim_{x\to\infty} u(x,y) = \psi_u(y), \quad \lim_{y\to\infty} u(x,y) = \phi_u(x). \tag{0.2}$$

Moreover functions ψ_u and ϕ_u bounded solutions of one dimensional analogue of problem (0.1)

$$\Psi'' = f(\Psi), \quad \Psi(0)=0, \quad \Psi(z)\ge 0, \quad z\ge 0. \tag{0.3}$$

We recall, that every solution of (0.3) stabilizes as $z\to\infty$ to some $c\ge 0$ such that $f(c)=0$ and, for fixed c there exists not greater than one solution $\Psi(z)=\Psi_c(z)$ of this problem. Consequently, the functions ψ_u and ϕ_u in (0.2) should coincide:

Typeset by $\mathcal{AMS}$-TEX

$\psi_u(z) = \phi_u(z) = \Psi_c(z)$, where the constant $c = c_u > 0$ satisfies $f(c) = 0$. Thus, we can rewrite (0.2) in the following form:

$$\lim_{(x,y)\to\infty} |u(x,y) - \Psi_c(x,y)| = 0, \quad \text{where} \quad \Psi_c(x,y) := \min\{\Psi_c(x), \Psi_c(y)\}. \tag{0.4}$$

The aime of this notes is to verify the existence and uniqueness of a solution $u(x,y)$ satisfying (0.4). We estasblish this fact under the following nondegeneracy assumption that

$$f'(c) \neq 0 \tag{0.5}$$

(in a fact, the existence of a solution $\Psi_c(z)$ of equation (0.3) and (0,5) imply that $f'(c) > 0$, see [BEZ01]). Thus, the main result of the paper is the following theorem.

Theorem 1. *Let the nonlinearity f satisfy the above assumptions, Ψ_c be a solution of (0.3) such that $f'(c) > 0$. Then, there exists a unique solution $u(x,y)$ of (0.1) which satisfies (0.4).*

The following corollary shows that, generically, equation (0.1) has only finite numberof different positive solutions.

Corollary 1. *Let the above assumptions hold and let, in addition, inequality (0.5) hold, for every solution $c > 0$ of equation $f(c) = 0$. Then, problem (0.1) has the finite number of different positive bounded solutions.*

SKETCH OF THE PROOF OF THEOREM 1.

For the proof, we need the following lemma.

Lemma 1. *Let the assumptions of Theorem 1 hold and let*

$$\Psi_c^M(x,y) := \begin{cases} c, & (x,y) \in [0,M]^2, \\ \Psi_c(x,y), & (x,y) \in \Omega_+ \backslash [0,M]^2, \end{cases} \tag{1.1}$$

where M is sufficiently large positive number. Then, the spectrum of the operator $\Delta_{x,y} - f'(\Psi_c^M(x,y))$ in Ω_+ (with the Dirichlet boundary conditions) is strictly negative:

$$\sigma(\Delta_{x,y} - f'(\Psi_c^M), L^2(\Omega_+)) \le -K < 0. \tag{1.2}$$

Indeed, estimate (1.2) can be easily deduced from the standard fact that

$$\sigma(\partial_z^2 - f'(\Psi_c(z)), L^2(\mathbb{R}_+)) \le -K \le 0 \tag{1.3}$$

(which is the corollary of the Perron-Frobenius theorem), the minimax principle and the special form of the function $\Psi_c(x,y)$.

The following two corollaries of Lemma 1.1 are of fundamental significance for what follows.

Corollary 1.1. *Let the assumptions of Theorem 1 hold and let $u(x,y)$ be a positive bounded solution of (0.1) which satisfies (0.4). Then:*

$$\sigma_{ess}(\Delta_{x,y} - f'(u(x,y)), L^2(\Omega)) \le -K < 0. \tag{1.4}$$

Indeed, due to (0.4) and (1.1) the operator $\Delta_{x,y} - f'(u(x,y))$ is a compact perturbation of $\Delta_{x,y} - f'(\Psi_c^M)$.

Corollary 1.2. *Let the assumptions of Corollary 1.1 hold. Then, the rate of decaying in* (0.4) *is exponential, i.e. there exist positive constants* $\varepsilon \geq 0$ *and* C *depending on* u *such that*

$$|u(x,y) - \Psi_c(x,y)| \leq Ce^{-\varepsilon(x+y)}, \quad (x,y) \in \Omega_+. \tag{1.5}$$

Indeed, estimate (1.5) is more or less standard corollary of (1.2), convergence (0.4) and the maximum principle, so we left its rigorous proof to the reader.

We are now ready to verify the existence of a solution $u(x,y)$. To this end, we consider the following sequence of auxiliary problems in the domains $\Omega_N := \{(x,y) \in \Omega_+, y \leq N\}$:

$$\begin{cases} \Delta_{x,y} u_N = f(u_N), \quad u(x,y) \geq 0, \\ u(0,y) = u(x,0) = 0, \quad u(x,N) = \Psi_c(x). \end{cases} \tag{1.6}$$

Obviously, for every $N \in \mathbb{N}$, this problem has at least one solution $u_N(x,y)$ satisfying

$$0 \leq u_N(x,y) \leq c \tag{1.7}$$

(which can be obtained using $u_- = 0$ and $u_+ = c$ as sub and super solutions respectively for problem (1.6), see e.g. [VoH85]). Moreover, this solution is also monotonic with respect to x and y and tends exponentially as $x \to \infty$ to $\Psi_c(y)$ (analogously to Corollary 1.2). We also note that, due to the elliptic regularity theorem, estimate (1.7) implies that

$$\|u_N\|_{C_b^2(\Omega_+)} \leq C \tag{1.8}$$

where the constant C is independent of N.

Thus, without loss of generality, we may assume that the sequence u_N tends in $C^2_{loc}(\overline{\Omega_+})$ to a some solution $u(x,y)$ of problem (0.1) as $N \to \infty$. As we have explained in the intoduction, this implies that there exists $0 \leq c' \leq c$ (may be $c' = 0$) such that $f(c') = 0$ and

$$\lim_{(x,y)\to\infty} |u(x,y) - \Psi_{c'}(x,y)| = 0. \tag{1.9}$$

We need to prove that, necessarily, $c' = c$. We prove this fact using the special integral identity. In order to derive it, we multiply equation (1.6) by $\partial_x u_N$. Then, we have

$$\partial_x(|\partial_x u_N|^2 - |\partial_y u_N|^2 - 2F(u_N)) = -2\partial_y(\partial_x u_N \cdot \partial_y u_N) \tag{1.10}$$

where $F(u)$ is a potential of $f(u)$. Integrating this formula over Ω_N and using the boundary conditions and the fact that $|\Psi_c'(0)|^2 = -2F(c) \geq 0$, we derive that

$$\int_0^N (|\Psi_c'(0)|^2 - |\partial_x u_N(0,y)|^2)\, dy = \\ = \int_0^N 2[F(c) - F(\Psi_c(y))] + |\Psi_c'(y)|^2\, dy - 2\int_0^\infty \Psi_c'(x) \cdot \partial_y u_N(x,N)\, dx. \tag{1.11}$$

Since $\Psi_c'(x) \geq 0$ and $\partial_y u_N(x, N) \geq 0$, then

$$\int_0^N (|\Psi_c'(0)|^2 - |\partial_x u_N(0,y)|^2)\, dy \leq C_{\Psi_c} \tag{1.12}$$

where the constant C_{Ψ_c} is independent of N. Moreover, obviously, the function $\partial_x u_N(0,y)$ is strictly increasing with respect to y and $\partial_x u_N(0,N) = \Psi_c'(0)$. Consequently, (1.12) implies that

$$\int_0^N |\Psi_c'(0)^2 - \partial_x u_N(0,y)^2|\, dy \leq C_{\Psi_c}. \tag{1.13}$$

We now note that $\partial_x u(0,y)$ is monotone increasing function (since $u(x,y)$ is monotone with respect to y and $u(0,y) = 0$) and

$$\partial_x u(0,y) < \partial_x u(0,\infty) = \Psi_{c'}'(0), \quad \forall y \in \mathbb{R}_+. \tag{1.14}$$

Since $\Psi_{c'}'(0) < \Psi_c'(0)$ if $c' < c$, see [BEZ01] and $u_N \to u$ in $C^2_{loc}(\overline{\Omega_+}))$ then estimates (1.13) and (1.14) imply that the limit function $u(x,y)$ satisfies (1.9) with $c = c'$. Thus, the existence of a solution is verified.

Let us now verify the uniqueness of the constructed solution $u(x,y)$. To this end, we need the following lemma which is of independent interest also.

Lemma 1.2. *Let $u(x,y)$ be an arbitrary solution of* (0.1) *which satisfies* (0.4). *Then the spectrum of the linearization of* (0.1) *on $u(t,x)$ is strictly negative, i.e.*

$$\sigma(\Delta_{x,y} - f'(u)) \leq -C_u, \tag{1.15}$$

for some positive constant C_u, depending on the solution u.

Proof. Indeed, assume that (1.15) is wrong. Then, according to (1.4), there exists a nonnegative eigenvalue $\lambda_0 \geq 0$ of this operator and the corresponding eigenvector $v \in L^2(\Omega_+)$. Moreover, it can be deduced in a standard way, using condition (1.2) and the exponential convergence (1.5) that

$$|v(x,y)| \leq C_v e^{-\varepsilon(x+y)}, \quad (x,y) \in \Omega_+, \tag{1.16}$$

for some positive constant C_v, depending on v. We may also assume, without loss of generality, then the eigenvalue $\lambda_0 \geq 0$ is maximal. Then, thanks to the Perron-Frobenius theory, function $v(x,y)$ is strictly positive inside of Ω_+.

We note that the function $v_1(x,y) := \partial_x u(x,y)$ is also strictly positive and satisfies the equation

$$\Delta_{x,y} v_1 - f'(u(x,y)) v_1 = 0. \tag{1.17}$$

Multiplying this equation by the eigenvector $v(x,y)$ and integrating over Ω_+, integrating by parts and using the boundary conditions, we derive that

$$\int_0^\infty v_1(0,y) \partial_x v(0,y)\, dy + \lambda_0 \int_{\Omega_+} v \cdot v_1 \, dx\, dy = 0. \tag{1.18}$$

We now recall that $v_1(x,y) := \partial_x u(x,y) \ge 0$, $v(x,y) \ge 0$ and $\partial_x v(0,y) > 0$ (due to the strict maximum principle). Consequently, (1.18) implies that

$$v_1(0,y) := \partial_x u(0,y) \equiv 0. \tag{1.19}$$

Since, $u(0,y) \equiv 0$ due to the boundary conditions, then (1.19) implies that $u(x,y) \equiv 0$ (due to the uniqueness theorem for elliptic equations). This contradiction proves estimate (1.15) and Lemma 1.2.

Now we are ready to verify the uniqueness. Indeed, let $u_1(x,y)$ and $u_2(x,y)$ be two solutions of problem (0.1) which satisfy (0.4). Then, without loss of generality, we may assume that

$$u_2(x,y) \ge u_1(x,y). \tag{1.20}$$

Indeed, if (1.20) is not satisfied, then, using the sub and supersolution method (parabolic equation method, see e.g., [VoH85]), we may construct the third solution $u_3(x,y)$ such that

$$c \ge u_3(x,y) \ge \max\{u_1(x,y), u_2(x,y)\} \tag{1.21}$$

which is not coincide with u_1 and u_2 and for which (1.20) is satisfied.

Let us now consider the parabolic boundary value problem in Ω_+

$$\partial_t U = \Delta_{x,y} U - f(U), \quad U\big|_{\partial\Omega_+} = 0, \quad U\big|_{t=0} = U_0 \tag{1.22}$$

with the phase space

$$W_0 := \{U_0 \in L^\infty(\Omega_+), \quad u_1(x,y) \le U_0(x,y) \le u_2(x,y)\}. \tag{1.23}$$

Then, this problem generates a semiflow on the phase space W_0:

$$S_t : W_0 \to W_0, \quad S_t U_0 := U(t) \tag{1.24}$$

which (according to the general theory, see [BaV92], [Tem88] and [EfZ01]) possesses a global attractor $\mathcal{A}_0 \subset W_0$. Moreover, due to (1.5) and (1.23), we have the following Lyapunov function on W_0:

$$L(U_0) := \int_{\Omega_+} |\nabla(U_0 - u_1)|^2 + 2F_{u_1}(U_0 - u_1, x, y)\, dx\, dy \tag{1.25}$$

where $F_{u_1}(z,x,y) := \int_0^z f(u_1(x,y) + z) - f(u_1(x,y))\, dz$.

Thus, the attractor $\mathcal{A}_0$ should consist of heteroclinic orbits to the appropriate equilibria, belonging to W_0 (see [Bav92]), but as proved in Lemma 1.2, all of these equilibria are exponentially stable which is possible only in the case $u_1 \equiv u_2$. Therefore, the uniqueness is also proven and Theorem 1 is proven.

References

[BaV92] A.Babin and M.Vishik, *Attractors of evolutionary equations*, Stud. Math. Appl., 25, North Holland, Amsterdam, 1992.

[BEZ01] H.Berestycki, M.Efendiev and S.Zelik, *Dynamical approach for positive solutions of semilnear elliptic problems in unbounded domains*, Preprint 01-11, Universitat Stuttgart, Mathematishes Institut A (2001).

[BeL83] H.Berestycki and P.Lions, *Nonlinear scalar field equations I. Existence of a ground state*, Arch. Rational Mech. Anal. **82** (1983), 313–345.

[BeN90] H.Berestycki and L.Nirenberg, *Some qualitative properties of solutions of semilinear elliptic equations in cylindrical domains*, Anal. et cetera, Academic press, Boston, MA (1990), 115–164.

[BeN91] H.Berestycki and L.Nirenberg, *On the method of moving planes and the sliding method*, Bol. Soc. Brasil Mat. (N.S.) **22** (1991), no. 1, 1–37.

[BeN92] H.Berestycki and L.Nirenberg, *Travelling fronts in cylinders*, Ann. Inst. Poincare NA **9** (1992), no. 5, 497–572.

[BCN97] H.Berestycki, L.Caffarelli and L.Nirenberg, *Monotonicity for elliptic equations in unbounded Lipschitz domains*, Comm. Pure Appl. Math. **50** (1997), no. 11, 1089–1111.

[BCN98] H.Berestycki, L.Caffarelli and L.Nirenberg, *Furhter quantitative properties for elliptic equations in unbounded domains*, Ann. Scuola Norm Sup. Pisa Cl Sci. **25** (1998), no. 1-2, 69–94.

[BuF01] J.Busca and P.Felmer, *Qualitative properties of some bounded positive solutions to scalar field equations*, Calc. Var and PDE **13**, no. 2, 191–211.

[EfZ01] M.Efendiev and S.Zelik, *The attractor for a nonlinear reaction-diffusion system in the unbounded domain*, Comm. Pure Appl. Math. **54** (2001), no. 6, 625–688.

[GNN81] B.Gidas, W.Ni and L.Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **6** (1981), 883–901.

[GNN81a] B.Gidas, W.Ni and L.Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in* $\mathbb{R}^n$, Math. Anal and Appl. Part A, Academic Press, New York (1981), 369–402.

[Kwo89] M.Kwong, *Uniqueness of positive solutions of* $\Delta_x u - u + u^p = 0$ *in* $\mathbb{R}^n$, Arch. Rational Mech. Anal. **105** (1983), 243–266.

[Tem88] R.Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Springer-Verlag, New-York, 1988.

[VoH85] A.Volpert and S.Khudyaev, *Analysis in classes of discontinuous functions and equations of mathematical physics*, Nijhoff, Dordrecht, 1985.