# THE ATTRACTOR FOR A NONLINEAR REACTION-DIFFUSION SYSTEM WITH A SUPERCRITICAL NONLINEARITY AND IT'S DIMENSION

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ABSTRACT. The nonlinear reaction-diffusion system of the type

$$\partial_t u = a\Delta_x u - f(u) + g(x), \quad x \in \Omega, \tag{1}$$

where $\Omega \subset\subset \mathbb{R}^n$, $u = (u^1, \cdots, u^k)$, $a + a^* > 0$ and the nonlinearity $f$ is not assumed to be subordinated to the Laplacian is considered.

The existence of a finite dimensional global attractor for the system (1) is proved under some natural regularity (but not growth) assumptions on the nonlinear term $f$. In order to obtain this result a new scheme of estimating the fractal dimension of invariant sets, which does not require the corresponding map to be differentiable is presented.

## INTRODUCTION

This paper is devoted to study the longtime behavior of solutions of the following reaction-diffusion system

$$\begin{cases} \partial_t u - a\Delta_x u + f(u) = g, \quad x \in \Omega \\ u\big|_{t=0} = u_0, \quad u\big|_{\partial\Omega} = 0 \end{cases} \tag{0.1}$$

It is assumed that $\Omega \subset\subset \mathbb{R}^n$ is a bounded domain in $\mathbb{R}^n$ with a sufficiently smooth boundary, $u = (u^1, \cdots, u^k)$ is an unknown vector function, $a$ is a given matrix with a positive symmetric part

$$a + a^* > 0, \tag{0.2}$$

the right-hand side $g \in L^2(\Omega)$ and $f(u) = (f_1(u^1, \cdots, u^k), \cdots, f_k(u^1, \cdots, u^k))$ is a given nonlinear function, which satisfies the conditions

$$\begin{cases} 1.\ f \in C^1(\mathbb{R}^k, \mathbb{R}^k) \\ 2.\ f(u).u \geq -C \\ 3.\ f'(u) \geq -K \end{cases} \tag{0.3}$$

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Here and below we denote by $u.v$ the inner product in $\mathbb{R}^k$.

The longtime behavior of solutions of (0.1) has been intensively studied by many authors (see [1], [4], [11] and references therein) but mainly under the additional *growth* restrictions to the nonlinear function $f$ like

$$|f(u)| \le C(1+|u|^q)\ ,\ \ q < q_{max} = \frac{n+2}{n-2} \tag{0.4}$$

which guarantee that the nonlinear term $f(u)$ is subordinated to the linear one $-\Delta_x u$ in the corresponding Sobolev space.

The fact that the equation (0.1) under the assumptions like (0.3) and (0.4) with an arbitrary $q \in \mathbb{R}_+$ possesses a global attractor in $L^2(\Omega)$ has been established in [1]. However, the restriction $q < q_{max}$ has been essentially exploited in order to obtain the quantitative or qualitative information about this attractor (particularly, in order to prove that it has a finite Hausdorff and fractal dimension).

The main aim of the present paper is to develop the attractor theory for the systems of the type (0.1) without any *growth* restrictions.

In Section 1 we prove that under the above assumptions the equation (0.1) possesses a unique solution $u(t) \in \mathbb{D}$ for every $u_0 \in \mathbb{D}$, where

$$\mathbb{D} = \{v \in H^2(\Omega):\ v\big|_{\partial\Omega} = 0,\ f(v) \in L^2(\Omega)\} \tag{0.5}$$

and consequently this equation generated a nonlinear semigroup

$$S_t : \mathbb{D} \to \mathbb{D},\quad S_t u_0 = u(t) \tag{0.6}$$

In Section 2 we show that (0.6) can be extended in a unique way to the Lipschitz continuous semigroup (which we also denote by $S_t$ for simplicity) acting in $L^2(\Omega)$ and prove that the semigroup thus obtained possesses a compact attractor $\mathcal{A} \subset L^2$.

Section 3 is devoted to study some regularity problems for the solutions $u(t) = S_t u_0$ belonging to the attractor $\mathcal{A}$ which will be used in the next Section.

Note that in contrast to the case when $f(u)$ is subordinated to $-\Delta_x u$ (e.g. when (0.4) is satisfied) in our case we construct only the Lipschitz continuous semigroup $S_t$ corresponding to the problem (0.1) and we do not have enough smoothness of solutions $u(t)$ to obtain the differentiability with respect to the initial value $u_0$. Consequently the classical methods of estimating the Hausdorff and fractal dimension of the attractor based on $k$-contraction maps and Liapunov exponents (see [11], [2], [5]) will not work.

That is why a new method of estimating the fractal dimension of the invariant sets which does not require the differentiability is presented in Section 4. Using this method we prove that under some additional regularity assumptions (but not the growth restrictions!) on the nonlinear term $f(u)$ the attractor $\mathcal{A}$, constructed in Section 2, has the finite fractal dimension in $L^2(\Omega)$.

As an example of the equation (0.1) for which all our assumptions are fulfilled we consider the generalized complex Ginzburg-Landau equation in $\Omega \subset \mathbb{R}^n$

$$\partial_t u = (1+i\alpha)\Delta_x u + Ru - (1+i\beta)u|u|^{2\sigma} + g(x),\quad u\big|_{\partial\Omega} = 0 \tag{0.7}$$

where $\alpha, \beta, R \in \mathbb{R}$, $u = u_1 + iu_2$, $g \in L^2(\Omega)$ and $\sigma > 0$. It is not difficult to verify that the quasimonotonicity assumption ($f'(u) \ge -K$) is fulfilled if

$$|\beta| < \frac{\sqrt{2\sigma+1}}{\sigma} \tag{0.8}$$

Thus, we have proved that if (0.8) holds then the attractor $\mathcal{A}$ of the Ginzburg-Landau equation has the finite fractal dimension for every $n \in \mathbb{N}$. To the best of our knowledge this result was known only if $n < 2 + \frac{2}{\sigma}$ (see [10], [7] for a more detailed study of this equation).

Our approach gives some new results even in the case of scalar Chafee-Infante equation in $\Omega \subset\subset \mathbb{R}^n$

$$\partial_t u - \Delta_x u + u|u|^{2\sigma} - Ru = g(x)\,, \quad u\big|_{\partial\Omega} = 0 \tag{0.9}$$

where $\sigma > 0$ and the right-hand side $g = g(x)$ belongs only to the space $L^2(\Omega)$ (and does not belong to $L^p(\Omega)$ if $p > 2$). Then the standard methods work only under the restriction $n < 2+2/\sigma$ (which follows from (0.4)) but we obtain the finite dimensionality of the attractor $\mathcal{A}$ of the equation (0.9) for an arbitrary $n$.

## §1 Existence of solutions.

In this Section we deduce a number of a priori estimates for the problem (0.1) and prove that for every $u_0 \in \mathbb{D}$ this problem has a unique solution. It is convenient for us to consider a little more general problem of the type (0.1) with the nonautonomous right-hand side $g = g(t)$:

$$\begin{cases} \partial_t u - a\Delta_x u + f(u) = g(t), \;\; x \in \Omega \\ u\big|_{t=0} = u_0, \;\; u\big|_{\partial\Omega} = 0 \end{cases} \tag{1.1}$$

It is assumed now that $g \in H^1([0,T], L^2(\Omega))$ for every $T \geq 0$. (Here and below we denote by $H^l$ the Sobolev space of functions whose derivatives up to the order $l$ inclusive belong to $L^2$.)

The main result of this Section is the following Theorem.

**Theorem 1.1.** *Let the above assumptions be valid. Then for every $u_0 \in \mathbb{D}$ the problem (1.1) has a unique solution $u \in C_w([0,T], \mathbb{D})$ and the following estimate holds:*

$$\|u(t)\|^2_{\mathbb{D}} \leq C_1(\|u(0)\|^2_{\mathbb{D}} + \|g(0)\|^2_{L^2})e^{2(K-\varepsilon)t} + \\ + C_2 \int_0^t e^{2(K-\varepsilon)(t-s)}(\|g(s)\|^2_{L^2} + \|g'(s)\|^2_{L^2})\, ds \tag{1.2}$$

*Here $K$ is the same as in (0.3), $\varepsilon > 0$ is small enough,*

$$\|v\|^2_{\mathbb{D}} \equiv \|v\|^2_{H^2} + \|f(v)\|^2_{L^2} \tag{1.3}$$

*and $u \in C_w([0,T], \mathbb{D})$ means by definition that $u \in C([0,T], H^2_w)$ and $f(u) \in C([0,T], L^2_w)$ (as usual the symbol 'w' denotes the weak topology).*

We give below only a formal proof of the estimate (1.2) which can be justified for instance using the Galerkin approximations method. To this end we need the following Lemmata.

**Lemma 1.1.** *Let $u \in C_w([0,T],\mathbb{D})$ be a solution of the equation* (1.1). *Then the following estimate holds*

$$\|u(t)\|^2_{H^1} \le C_1\|u(0)\|^2_{H^1}e^{2(K-\varepsilon)t} + C_2\int_0^t e^{2(K-\varepsilon)(t-s)}\|g(s)\|^2_{L^2}\,ds \tag{1.4}$$

*where $K$ is the same as in* (0.3) *and $\varepsilon > 0$ is small enough.*

*Proof.* Indeed, multiplying the equation (1.1) by $-\Delta_x u$ and integrating over $\Omega$ we obtain after the standard integration by parts that

$$\partial_t\|\nabla_x u(t)\|^2_{L^2} + ((a+a^*)\Delta_x u(t), \Delta_x u(t)) + \\ + 2\,(f'(u(t))\nabla_x u(t), \nabla_x u(t)) + 2\,(g(t), \Delta_x u(t)) = 0 \tag{1.5}$$

Using the facts that $a + a^* > 0$, $f'(u) \ge -K$ and using Friedrichs and Holder inequality we deduce from (1.5) that

$$\partial_t\|\nabla_x u(t)\|^2_{L^2} + 2\varepsilon\|\nabla_x u(t)\|^2_{L^2} - 2K\|\nabla_x u(t)\|^2_{L^2} \le C_\varepsilon\|g(t)\|^2_{L^2} \tag{1.6}$$

where $\varepsilon > 0$ is a sufficiently small positive number. Applying the Gronewal inequality to the estimate (1.6) we obtain the assertion of the lemma.

**Lemma 1.2.** *Let $u \in C_w([0,T],\mathbb{D})$ be a solution of the problem* (1.1). *Then the following estimate is valid:*

$$\|\partial_t u(t)\|^2_{L^2} \le C_1(\|u(0)\|^2_{\mathbb{D}} + \|g(0)\|^2_{L^2})e^{2(K-\varepsilon)t} + C_2\int_0^t e^{2(K-\varepsilon)(t-s)}\|g'(s)\|^2_{L^2}\,ds \tag{1.7}$$

*where $K$ is the same as in* (0.3) *and $\varepsilon > 0$ is small enough.*

*Proof.* Differentiating the equation (1.1) by $t$ and denoting $\theta(t) = \partial_t u(t)$ we will derive

$$\begin{cases} \partial_t\theta(t) - a\Delta_x\theta(t) + f'(u(t))\theta(t) = g'(t) \\ \theta\big|_{t=0} = a\Delta_x u(0) - f(u(0)) + g(0)\,, \quad \theta\big|_{\partial\Omega} = 0 \end{cases} \tag{1.8}$$

Multiplying this equation by $\theta(t)$, integrating over $x \in \Omega$ and arguing as in the proof of previous Lemma we deduce that

$$\partial_t\|\theta(t)\|^2_{L^2} + 2\varepsilon\|\theta(t)\|^2_{L^2} - 2K\|\theta(t)\|^2_{L^2} \le C_\varepsilon\|g'(t)\|^2_{L^2} \tag{1.9}$$

Applying the Gronewal inequality to the estimate (1.9) we obtain the assertion of the lemma

**Lemma 1.3.** *Let $u \in C_w([0,T],\mathbb{D})$ be a solution of the problem* (1.1). *Then the following estimate is valid:*

$$\|u(t)\|^2_{H^2} \le C_1(\|u(0)\|^2_{\mathbb{D}} + \|g(0)\|^2_{L^2})e^{2(K-\varepsilon)t} + \\ + C_2\int_0^t e^{2(K-\varepsilon)(t-s)}(\|g(s)\|^2_{L^2} + \|g'(s)\|^2_{L^2})\,ds \tag{1.10}$$

*where $\varepsilon > 0$ is small enough.*

*Proof.* Let us rewrite the equation (1.1) in the form of elliptic boundary problem

$$(1.11)\qquad a\Delta_x u(t) - f(u(t)) = \partial_t u(t) - g(t) \equiv h_u(t), \quad u(t)\big|_{\partial\Omega} = 0$$

Multiplying (1.11) by $\Delta_x u(t)$ and integrating over $x \in \Omega$ we obtain arguing as in the proof of Lemma 1.1 that

$$(1.12)\qquad \|\Delta_x u(t)\|_{L^2}^2 \le C_1 K \|\nabla_x u(t)\|_{L^2}^2 + C_2(\|\partial_t u(t)\|_{L^2}^2 + \|g(t)\|_{L^2}^2)$$

According to $(L^2, H^2)$-regularity of solutions of the Laplace equation (see [12])

$$(1.13)\qquad \|u(t)\|_{H^2}^2 \le C\|\Delta_x u(t)\|_{L^2}^2$$

Inserting the inequalities (1.4) and (1.7) into the right-hand side of (1.12) and using (1.13) we obtain (1.10) after simple calculations. Lemma 1.3 is proved.

Now we are in position to complete the proof of the estimate (1.2). Indeed, according to (1.3) we should estimate the $H^2$-norm of $u(t)$ and the $L^2$-norm of $f(u(t))$. The $H^2$-norm is already estimated in Lemma 1.3, so it remains to estimate only $\|f(u(t))\|_{L^2}$. But it follows immediately from the equation (1.1) that

$$(1.14)\qquad \|f(u(t))\|_{L^2}^2 \le C\|u(t)\|_{H^2}^2 + C\|\partial_t u(t)\|_{L^2}^2 + C\|g(t)\|_{L^2}^2$$

Inserting the inequalities (1.7) and (1.10) into the right-hand side of (1.14) we obtain the estimate of the $L^2$-norm of $f(u(t))$. The estimate (1.2) is proved.

The existence of a solution $u \in C_w([0,T], \mathbb{D})$ for the problem (1.1) can be derived in a standard way using the a priori estimate (1.2) and the Galerkin approximations method (see for example [1], [8]). So it remains to prove the uniqueness.

**Lemma 1.4.** *Let $u_1, u_2 \in C_w([0,T], \mathbb{D})$ be two solutions of the equation* (1.1) *with the initial values $u_1(0)$ and $u_2(0)$ respectively. Then*

$$(1.15)\quad \|u_1(T)-u_2(T)\|_{L^2}^2 + \int_T^{T+1} \|u_1(t)-u_2(t)\|_{H^1}^2\, dt \le \|u_1(0)-u_2(0)\|_{L^2}^2 e^{2(K-\varepsilon)T}$$

*for some positive $\varepsilon > 0$. Particularly the problem* (1.1) *has the unique solution for every $u_0 \in \mathbb{D}$.*

*Proof.* Let $w(t) = u_1(t) - u_2(t)$. Then the function $w$ satisfies the equation

$$(1.16)\qquad \begin{cases} \partial_t w(t) - a\Delta_x w(t) = f(u_2(t)) - f(u_1(t)) \equiv h_{u_1,u_2}(t) \\ w\big|_{t=0} = u_1(0) - u_2(0) \end{cases}$$

Note, that $h_{u_1,u_2} \in C_w([0,T], L^2)$. Moreover, since $f'(v) \ge -K$ then

$$(1.17)\qquad (f(\xi_1) - f(\xi_2)).(\xi_1 - \xi_2) \ge -K|\xi_1 - \xi_2|^2$$

for every $\xi_1, \xi_2 \in \mathbb{R}^k$. Thus,

$$(1.18)\qquad (h_{u_1,u_2}(t), w(t)) \le K\|w(t)\|_{L^2}^2$$

Multiplying now the equation (1.16) by $w(t)$, integrating over $x \in \Omega$ and using the estimate (1.18) we deduce that

$$\partial_t \|w(t)\|_{L^2}^2 + \varepsilon \|w(t)\|_{H^1}^2 - 2K\|w(t)\|_{L^2}^2 \le 0 \tag{1.19}$$

Applying the Gronewal inequality to (1.19) we obtain the estimate (1.15). Lemma 1.4 is proved. Theorem 1.1 is proved.

**Remark 1.1.** Note, that the dissipativity assumption $f(u).u \ge -C$ has not been used in the proof of Theorem 1.1, consequently this theorem remains valid without this assumption. However the dissipativity assumption will be essentially used in the next Section in order to prove the existence of the absorbing set for the semigroup, generated by the equation (0.1).

## §2 The attractor.

In this Section we describe the longtime behavior of solutions of the autonomous equation (0.1) in terms of the attractor for the corresponding semigroup. Recall that, according to Theorem 1.1, the problem (0.1) generates a Lipschitz continuous semigroup $\{S_t,\, t \ge 0\}$ in $\mathbb{D}$:

$$S_t : \mathbb{D} \to \mathbb{D}, \quad S_t u_0 = u(t) \tag{2.1}$$

Moreover, (1.2) implies that

$$\|u(t)\|_{\mathbb{D}}^2 \le C\left(\|u(0)\|_{\mathbb{D}}^2 + \|g\|_{L^2}^2 + 1\right) e^{2(K-\varepsilon)t} \tag{2.2}$$

for a sufficiently small positive $\varepsilon$. But the right-hand side of (2.2) tends to $+\infty$ when $t \to \infty$ if $K - \varepsilon > 0$ (the case $K - \varepsilon < 0$ is not considered because in this case it is easy to prove that the attractor will consist of a unique exponentially attracting equilibria point). Hence, the estimate (2.2) does not guarantee us that $S_t$ will be bounded in $\mathbb{D}$ when $t \to \infty$. In fact under our assumptions we can prove that it will be bounded in $L^2$ or $H^1$ only and not in $\mathbb{D}$. To avoid this difficulty we extend by continuity the semigroup $S_t$, which is initially defined for $u_0 \in \mathbb{D}$ only to $\hat{S}_t : L^2 \to L^2$. Indeed, $\mathbb{D}$ dense in $L^2$ and according to (1.15) $S_t$ is uniformly continuous on $\mathbb{D}$ in $L^2$-metric for every fixed $t$. Consequently it can be extended in a unique way to the semigroup $\hat{S}_t$ on $L^2$ by expression:

$$\hat{S}_t u_0 = L^2\text{–}\lim_{n\to\infty} S_t u_0^n, \quad u_0^n \in \mathbb{D}, \quad u_0 = L^2\text{–}\lim_{n\to\infty} u_0^n \tag{2.3}$$

Moreover, the estimate (1.15) implies that

$$\|\hat{S}_t u_0^1 - \hat{S}_t u_0^2\|_{L^2}^2 + \int_T^{T+1} \|\hat{S}_t u_0^1 - \hat{S}_t u_0^2\|_{H^1}^2\, dt \le e^{2(K-\varepsilon)T}\|u_0^1 - u_0^2\|_{L^2} \tag{2.4}$$

for every $u_0^1, u_0^2 \in L^2$ and since $u_n(t) = S_t u_0^n \in C([0,T], L^2)$ if $u_0^n \in \mathbb{D}$ then $u(t) = \hat{S}_t u_0$ also belongs to $C([0,T], L^2)$ for every $u_0 \in L^2$.

Thus, we can naturally interpret the function $u(t) = \hat{S}_t u_0$ as a unique solution of the problem (0.1) for $u_0 \in L^2$ and study the longtime behavior of the semigroup $\hat{S}_t : L^2 \to L^2$. The following Theorem is of fundamental significance for these purposes.

**Theorem 2.1.** *Let $u_0 \in L^2$ and $u(t) = \hat{S}_t u_0$. Then $u \in C([0,T], L^2)$ for every $T \geq 0$ and*

$$\|u(T)\|^2_{L^2} + \int_T^{T+1} \|u(t)\|^2_{H^1}\, dt \leq C_1 \|u(0)\|^2_{L^2} e^{-\varepsilon t} + C_2(1 + \|g\|^2_{L^2}) \tag{2.5}$$

*for a sufficiently small positive $\varepsilon > 0$. Moreover, for every $t > 0$ $u(t) \in H^1$, $u \in C([t,T], H^1_w)$, and the following estimate is valid*

$$\|u(T)\|^2_{H^1} + \int_T^{T+1} \|u(t)\|^2_{H^2}\, dt \leq C\frac{t+1}{t}\left(\|u(0)\|^2_{L^2} e^{-\varepsilon t} + 1 + \|g\|^2_{L^2}\right) \tag{2.6}$$

*Proof.* According to (2.3) it is sufficient to deduce the estimates (2.5) and (2.6) only for $u_0 \in \mathbb{D}$. Let us prove firstly the estimate (2.5).

Multiplying the equation (0.1) by $u(t)$ and integrating over $x \in \Omega$ we obtain that

$$\partial_t \|u(t)\|^2_{L^2} + 2\left((a + a^*)\nabla_x u(t), \nabla_x u(t)\right) = -2\left(f(u(t)), u(t)\right) + 2\left(g(t), u(t)\right) \tag{2.7}$$

Using the fact that $a + a^* > 0$, the dissipativity assumption (0.3) on $f(u)$, Holder and Friedrichs inequalities we derive that

$$\partial_t \|u(t)\|^2_{L^2} + \varepsilon\|u(t)\|^2_{L^2} + \varepsilon\|u(t)\|^2_{H^1} \leq C\left(1 + \|g\|^2_{L^2}\right) \tag{2.8}$$

Applying the Gronewal inequality to (2.8) we obtain the estimate (2.5).

Let us prove now the estimate (2.6). We give below only a formal deriving of it which can be justified by the Galerkin approximations method.

Multiplying the equation (0.1) by $t\Delta_x u(t)$ integrating over $x \in \Omega$ we obtain after integration by parts that

$$\begin{aligned} \partial_t\left(t\|\nabla_x u(t)\|^2_{L^2}\right) - \|u(t)\|^2_{H^1} + t\left((a + a^*)\Delta_x u(t), \Delta_x u(t)\right) = \\ = -2t\left(f'(u)\nabla_x u(t), \nabla_x u(t)\right) - 2t(g, \Delta_x u(t)) \end{aligned} \tag{2.9}$$

Using now the quasimonotonicity assumption (0.3) on $f(u)$ and the fact that $a + a^* > 0$ we obtain as in the proof of previous estimate that

$$\begin{aligned} \partial_t\left(t\|u(t)\|^2_{H^1}\right) + \varepsilon(t\|u(t)\|^2_{H^1}) + \varepsilon(t\|u(t)\|^2_{H^2}) \leq \\ \leq C\left((t+1)\|u(t)\|^2_{H^1} + t\|g\|^2_{L^2}\right) \end{aligned} \tag{2.10}$$

for a sufficiently small positive $\varepsilon$. Applying the Gronewal inequality to (2.10) and using the estimate (2.5) for the integral of $\|u(t)\|^2_{H^1}$ we obtain after simple calculations the estimate (2.6).

Thus, it remains to prove the continuity of $u(t)$ with respect to $t$. Indeed, the fact $u \in C([0,T], L^2)$ follows from (2.3) and from the continuity of solutions $u_n$ for (0.1) with $u_n(0) \in \mathbb{D}$ proved in Theorem 1.1 ($u_n \in C([0,T], L^2)$).

The weak continuity in $H^1$ for $t > 0$ follows from the fact that $u \in C([0,T], L^2) \cap L^\infty([t,T], H^1)$ (see [9] for instance). Theorem 2.1 is proved.

Now we are in a position to construct a compact attractor for the semigroup $\hat{S}_t$ in $L^2$. Let us remind that a set $\mathcal{A} \subset L^2$ is called the attractor for $\hat{S}_t : L^2 \to L^2$ if

1. The set $\mathcal{A}$ is compact in $L^2$.
2. The set $\mathcal{A}$ is strictly invariant with respect to $\hat{S}_t$, i.e.

$$\hat{S}_t\mathcal{A} = \mathcal{A} \text{ for } \ t \geq 0 \tag{2.11}$$

3. $\mathcal{A}$ is an attracting set for $\hat{S}_t$ in $L^2$. The latter means that for every neighborhood $\mathcal{O}(\mathcal{A})$ of the set $\mathcal{A}$ in $L^2$ and for every bounded subset $B \subset L^2$ there exists $T = T(B, \mathcal{O})$ such that

$$\hat{S}_t B \subset \mathcal{O}(\mathcal{A}) \ \text{ for every } t \geq T \tag{2.12}$$

(See [1], [4], [11] for details).

**Theorem 2.2.** *Let the assumptions* (0.2) *and* (0.3) *be valid and let* $g \in L^2$. *Then the semigroup* $\hat{S}_t$, *defined by* (2.3), *possesses a compact attractor* $\mathcal{A} \subset\subset L^2$ *(*$\mathcal{A} \subset H^1$*) which has the following structure*

$$\mathcal{A} = \Pi_0\mathcal{K} \tag{2.13}$$

*where by* $\mathcal{K}$ *we denote the set of all complete bounded trajectories of the semigroup* $\hat{S}_t$*:*

$$\mathcal{K} = \{\hat{u} \in C_b(\mathbb{R}, L^2) \ : \hat{S}_h u(t) = u(t+h) \ \ \textit{for } t \in \mathbb{R}, \ h \geq 0, \|u(t)\|_{L^2} \leq C_u\} \tag{2.14}$$

*and* $\Pi_0 u \equiv u(0)$.

*Proof.* According to the abstract attractor's existence theorem (e.g. see [1] ) it is sufficient to verify that

1. The operators $\hat{S}_t : L^2 \to L^2$ are continuous for every fixed $t \geq 0$.
2. The semigroup $\hat{S}_t$ possesses a compact attracting set $\mathbb{K}$ in $L^2$.

The continuity is an immediate corollary of (2.4). So it remains only to verify the existence of the attracting set.

The estimate (2.6) implies that the $H^1$-ball

$$\mathbb{K} \equiv \{v \in H^1(\Omega) : \|v\|_{H^1} \leq R\}$$

will be the attracting (and even the absorbing) set for the semigroup $\hat{S}_t$ in $L^2$ if $R$ is large enough. Since $H^1 \subset\subset L^2$ then $\mathbb{K}$ is compact in $L^2$ and consequently the semigroup $\hat{S}_t$ possesses the attractor $\mathcal{A} \subset \mathbb{K} \subset H^1$. Theorem 2.2 is proved.

## §3 The regularity of solutions.

Let us remind that in Section 1 we have proved that the problem (0.1) has the unique solution $u(t) = S_t u_0$ for every $u_0 \in \mathbb{D}$. Then in Section 2 we have extended by continuity the semigroup $S_t$ from $\mathbb{D}$ to $\hat{S}_t : L^2 \to L^2$ and proved that the semigroup thus obtained possesses the attractor $\mathcal{A}$ in $L^2$. This Section studies the following three problems which are naturally arise after proving the above results:

1. In what sense the 'solution' $u(t) = \hat{S}_t u_0$ satisfies the equation (0.1) if $u_0$ only from $L^2$ (but not from $\mathbb{D}$).
2. Whether the attractor $\mathcal{A}$ belongs to the space $\mathbb{D}$.

3. Under what assumptions on $f$ the semigroup $\hat{S}_t$ possesses the smoothing property in the following form:

$$\hat{S}_t : L^2 \to \mathbb{D} \ \text{ for every } t > 0 \tag{3.1}$$

Note also that these problems occurs to be closely connected with the problem of the finite dimension of the attractor $\mathcal{A}$ which will be considered in the next Section.

We start here with the most simple case where the nonlinear term $f$ satisfies the following *growth* restriction:

$$|f(u)| \le C(1 + |u|^p) \ \text{ where } p \le p_{max} \equiv 1 + \frac{4}{n-4} \tag{3.2}$$

if $n > 4$ and $p$ is arbitrary if $n = 4$ (for $n \le 3$ we need not *any* growth restriction!). In this case one can easily verify (using Sobolev embedding theorem) that $f(v) \in L^2$ if $v \in H^2$. Thus,

$$\mathbb{D} = H^2(\Omega) \cap \{v\big|_{\partial\Omega} = 0\} \tag{3.3}$$

and therefore the nonlinearity $f(u)$ is subordinated by the linear term $\Delta_x u$.

**Theorem 3.1.** *Let the assumption* (3.2) *holds. Then the semigroup* $\hat{S}_t$ *possesses the smoothing property in the form of* (3.1) *and consequently for every* $u_0 \in L^2$ $u(t) = \hat{S}_t u_0$ *satisfies* (0.1) *in the sense of distributions. Moreover,*

$$\|u(1)\|^2_{\mathbb{D}} \le Q(\|u_0\|^2_{L^2} + \|g\|^2_{L^2}) \tag{3.4}$$

*for a certain monotonic function* $Q$ *depending on* $f$ *and therefore*

$$\mathcal{A} \subset \mathbb{D} \tag{3.5}$$

*Proof.* Indeed, according to (2.6) $u \in L^2([s,T], H^2)$ for every $s > 0$ hence due to Fubini theorem $u(t) \in H^2$ for almost all $t \in \mathbb{R}_+$. Then, according to (3.3), $u(t) \in \mathbb{D}$ for almost all $t \in \mathbb{R}_+$. But Theorem 1.1 implies that $\hat{S}_t : \mathbb{D} \to \mathbb{D}$, therefore $u(t) \in \mathbb{D}$ for every $t > 0$. Let us prove now the estimate (3.4).

Indeed, according to (2.6),

$$\int_{1/2}^{1} \|u(t)\|^2_{H^2} dt \le C(\|u(0)\|^2_{L^2} + 1 + \|g\|^2_{L^2})$$

The latter means that there exists a point $t_0 \in [1/2, 1]$, such that

$$\|u(t_0)\|^2_{H^2} \le 2C(\|u(0)\|^2_{L^2} + 1 + \|g\|^2_{L^2}) \tag{3.6}$$

and hence, according to (3.2) and the embedding theorem

$$\|u(t_0)\|^2_{\mathbb{D}} \equiv \|u(t_0)\|^2_{H^2} + \|f(u(t_0))\|^2_{L^2} \le Q(\|u(t_0)\|^2_{H^2}) \tag{3.7}$$

for a certain monotonic function $Q$. The estimate (3.4) follows now from the inequality (2.2) with $u_0 = u(t_0)$ applied in the point $t = 1 - t_0$ and from the estimates (3.6) and (3.7).

Thus it remains to prove the embedding (3.5). But this fact is an immediate corollary of the estimates (2.5) and (3.4). Theorem 3.1 is proved.

**Remark 3.1.** Let $n \le 3$. Then Theorem 3.1 and the embedding theorem imply that under the assumptions of Section 2

$$\hat{S}_t : L^2(\Omega) \to C(\Omega) \quad \text{for } t > 0 \text{ and } \quad \mathcal{A} \subset C(\Omega) \tag{3.8}$$

Assume now that $n \ge 4$, (3.2) holds with $p < p_0$ and the right-hand side $g \in L^r(\Omega)$ for a some $r > \frac{n}{2}$. Then using the $L^q$-regularity theory for the heat equations (see [6]) one can derive that the assertions (3.8) remains valid in this case as well. Moreover the space $C$ in (3.8) can be replaced by $H^{2,r} \subset\subset C$.

Note also that the growth condition (3.2) is essentially less restrictive then (0.4).

**Remark 3.2.** If $a$ is a scalar matrix and $g \in L^r$ with $r > \frac{n}{2}$ then due to the maximum principle one can construct the compact attractor for (0.1) in $H^{2,r} \subset\subset C$ without any growth restrictions (and even without the monotonicity assumption $f' \ge -K$!) (see [3] and [13] for instance).

Let us consider now the case when the nonlinearity $f$ is not subordinated by the linear part $\Delta_x u$.

**Theorem 3.2.** *Let the assumptions of Theorem 2.2 holds and let* $u(t) = \hat{S}_t u_0$ *with* $u_0 \in L^2$. *Then for every* $\psi \in C^1(\mathbb{R}^k, \mathbb{R}^k)$ *with the compact support in* $\mathbb{R}^k$ *the function* $\psi(u(t))$ *belongs to the space* $H^1([s,T], L^2(\Omega))$ *for* $s > 0$, *satisfies the equation*

$$\partial_t \psi(u) = \psi'(u) a \Delta_x u - \psi'(u) f(u) + \psi'(u) g \tag{3.9}$$

*in the sense of distributions and the following estimate is valid for* $T \ge 1$:

$$\int_T^{T+1} \|\partial_t \psi(u(t))\|^2_{L^2} + \|\psi(u(t))\|^2_{L^2} dt \le C_\psi \left( \|u_0\|^2_{L^2} e^{-\varepsilon t} + \|g\|^2_{L^2} + 1 \right) \tag{3.10}$$

*for a certain* $C_\psi$ *depending on* $\psi$ *and* $\varepsilon > 0$.

*Moreover, the function* $f(u(t)).u(t)$ *belongs to* $L^1([0,T], L^1(\Omega))$ *and satisfies the estimate*

$$\int_T^{T+1} \|f(u(t)).u(t)\|_{L^1} dt \le C\|u_0\|^2_{L^2} e^{-\varepsilon T} + C(1 + \|g\|^2_{L^2}) \tag{3.11}$$

*which is valid for every* $T \ge 0$.

*Proof.* Indeed, let $u(t) = L^2\text{–}\lim_{n\to\infty} u_n(t)$ where $u_n$ be the solutions of (0.1) with the initial condition $u_n(0) \in \mathbb{D}$. Then multiplying the equation (0.1) (with $u$ replaced by $u_n$) by the matrix $\psi'(u_n)$ we obtain that the functions $\psi(u_n)$ satisfy the equation

$$\partial_t \psi(u_n) = \psi'(u_n) a \Delta_x u_n - \psi'(u_n) f(u_n) + \psi'(u_n) g \tag{3.12}$$

Recall, that the function $\psi'(v)$, and therefore the function $\psi'(v)f(v)$ have the finite support in $\mathbb{R}^k$. It means that

$$\psi'(u_n) \to \psi(u) \text{ and } \psi'(u_n) f(u_n) \to \psi(u) f(u) \tag{3.13}$$

in the space $C([0,T],L^2)$ when $n\to\infty$. Moreover, according to the estimate (2.6)

$$a\Delta_x u_n \rightharpoondown a\Delta_x u \tag{3.14}$$

weakly in $L^2([s,T],L^2)$ for every $s>0$. The assertions (3.13) and (3.14) imply that

$$\psi'(u_n)a\Delta_x u_n \rightharpoondown \psi(u)a\Delta_x u \tag{3.15}$$

weakly in $L^1([s,T],L^1)$. Thus, we can pass to the limit $n\to\infty$ in the sense of distributions in the equations (3.12) and obtain that (3.9) is valid. The assertion $\psi(u)\in H^1([s,T],L^2)$ now is an immediate corollary of Theorem 2.1 and the equality (3.9). Let us prove the estimate (3.10). Note, that

$$|\psi(v)|+|\psi'(v)|+|\psi'(v)f(v)|\le C(\psi)$$

for every $v\in\mathbb{R}^k$ since these functions are continuos and have a finite support. Then, according to (3.9),

$$\begin{gathered}\int_T^{T+1}\|\partial_t\psi(u(t))\|^2_{L^2}+\|\psi(u(t))\|^2_{L^2}\,dt\le\|g\|^2_{L^2}+C\int_T^{T+1}\|\psi'(u(t))\|^2_{L^2}+\\ +\|\psi'(u(t))\|^2_{L^2}+\|\psi'(u(t))f(u(t))\|^2_{L^2}+\|\Delta_x u(t))\|^2_{L^2}\,dt\le\\ \le C_1(\psi)\left(\|g\|^2_{L^2}+\|u\|^2_{L^2([T,T+1],H^2)}+1\right)\end{gathered}\tag{3.16}$$

Inserting the estimate (2.6) into the estimate (3.16) and using the fact that $T\ge1$ we obtain (3.10).

Thus, it remains to prove the estimate (3.11). To this end we consider again the sequence $u_n(t)$ the same as before, multiply the equation (0.1) (with $u$ replaced by $u_n$) by $u_n(t)$ and integrate over $(t,x)\in[T,T+1]\times\Omega$. We will have after the evident transformations that

$$\begin{gathered}\int_T^{T+1}(f(u_n(t)),u_n(t))\,dt=1/2\left(\|u_n(T)\|^2_{L^2}-\|u_n(T+1)\|^2_{L^2}\right)-\\ -\int_T^{T+1}(a\nabla_x u_n(t),\nabla_x u_n(t))dt+\int_T^{T+1}(g,u_n(t))\,dt\end{gathered}\tag{3.17}$$

Inserting the estimate (2.5) into the right-hand side of (3.17) and using the fact that $f(u).u\ge -C$ we obtain

$$\int_T^{T+1}\|f(u_n(t)).u_n(t)\|_{L^1}dt\le C\|u_n(0)\|^2_{L^2}e^{-\varepsilon T}+C(1+\|g\|^2_{L^2})\tag{3.18}$$

Our aim now is to pass to the limit $n\to\infty$ in the estimate (3.18). Note, that we cannot do it directly because we do not know whether the limit function $f(u).u$ belongs to $L^1$. To avoid this difficulty we introduce a cut-off function $\phi(z):\mathbb{R}\to[0,1]$ in such a way that $\phi(z)=1$ if $|z|<1$, $\phi(z)=0$ if $|z|>2$, $z\psi'(z)\le0$ and consider the functions $\phi_L(v)=\phi(|v|^2/L^2)$, $v\in\mathbb{R}^k$.

The estimate (3.18) implies that

$$\begin{gathered}\int_T^{T+1}\|\phi_L(u_n(t))f(u_n(t)).u_n(t)\|_{L^1}dt\le\|f(u_n).u_n\|_{L^1([T,T+1]\times\Omega)}\le\\ \le C\|u_n(0)\|^2_{L^2}e^{-\varepsilon T}+C(1+\|g\|^2_{L^2})\end{gathered}\tag{3.19}$$

Since $\phi_L(v)f(v).v$ has the finite support we can easily pass to the limit $n \to \infty$ in (3.19) and obtain that

$$\int_{[T,T+1]\times\Omega} F_L(t,x)dtdx \le C\|u(0)\|^2_{L^2}e^{-\varepsilon T} + C(1+\|g\|^2_{L^2}) \tag{3.20}$$

Where $F_L(t,x) = \phi_L(u(t,x))f(u(t,x)).u(t,x)$. Note that the sequence $F_L(t,x)$ $L \in \mathbb{N}$ is non decreasing with respect to $L$ and nonnegative for every fixed $(t,x)$. Consequently, passing to the limit $L \to \infty$ in (3.20) we obtain the assertion $F_\infty = f(u).u \in L^1$ and the estimate (3.11). Theorem 3.2 is proved.

**Corollary 3.1.** *Let the assumptions of Theorem 2.2 hold and let the function $f(u)$ satisfy the inequality*

$$|f(v)| \le C\left(|f(v).v| + 1 + |v|^2\right) \quad \textit{for every } v \in \mathbb{R}^k \tag{3.21}$$

*Then 1. $f(u) \in L^1([0,T],L^1)$, 2. $\partial_t u \in L^1([s,T],L^1)$ for every $s > 0$ and $u(t) = \hat{S}_t u_0$ satisfies* (0.1) *in the sense of distributions.*

*Proof.* Indeed, the first assertion is an immediate corollary of (3.21) and (3.11). To prove the second one it is sufficient to consider (3.9) with $\psi(u) = \psi_L(u) = u\phi_L(u)$ where $\phi_L$ is the same as in the proof of Theorem 3.2 and pass to the limit $L \to \infty$ in the distribution sense.

Now we are going to study the smoothing properties of solutions for (0.1). We start our consideration with the case where the main part of the nonlinearity $f$ has a gradient structure.

**Theorem 3.3.** *Let the assumptions of Theorem 2.2 be valid and let the function $f$ have the structure*

$$f(v) = f_1(v) + f_2(v) \tag{3.22}$$

*where the function $f_1$ also satisfies* (0.3) *and $f_1(v) = \nabla_v F(v)$, and the function $f_2$ be subordinated to $f_1$ in the following sense*

$$|f_2(v)|^2 \le C\left(F(v) + 1 + |v|^2\right) \tag{3.23}$$

*Then the semigroup $\hat{S}_t$, defined by* (2.3)*, maps $L^2$ to $\mathbb{D}$ for every $t > 0$. Moreover,*

$$\|u(t)\|^2_{\mathbb{D}} \le C\frac{1+t^2}{t^2}\left(\|u(0)\|^2_{L^2}e^{-\varepsilon t} + 1 + \|g\|^2_{L^2}\right) \tag{3.24}$$

*and therefore $\mathcal{A} \subset \mathbb{D}$.*

The proof of this theorem is based on a number of lemmata.

**Lemma 3.1.** *Under the assumptions of Theorem 3.3 the following estimate is valid:*

$$-C\left(1 + \ln(|v|+1)\right) \le F(v) \le C\left(|f(v).v| + 1 + |v|^2\right) \quad \textit{for every } v \in \mathbb{R}^k \tag{3.25}$$

*and consequently*

$$\int_T^{T+1} \|f_2(u(t))\|^2_{L^2} + \|F(u(t))\|_{L^1}\, dt \le C\|u(0)\|^2_{L^2} e^{-\varepsilon t} + C(1+\|g\|^2_{L^2}) \tag{3.26}$$

*Proof.* Indeed, the estimate (3.26) is an immediate corollary of (3.25), (3.23) and (3.11). Thus, it remains to prove (3.25).

Since $f_1(v).v \ge -C$ and $f_1$ is continuous then

$$F(v) - F(0) = \int_0^1 f_1(sv).v\, ds = \int_0^{1/(|v|+1)} f_1(sv).v\, ds +$$
$$+ \int_{1/(|v|+1)}^1 f_1(sv).v\, ds \le -C_1(f)\frac{|v|}{|v|+1} - \int_{1/(|v|+1)}^1 \frac{C}{s} ds \le -C_1(f) - C\ln(|v|+1)$$

The left-hand side of (3.25) is proved. Let us prove the right-hand side of it. To this end we introduce a function

$$\Phi(v) = F(v) - f_1(v).v \tag{3.27}$$

Then, using the monotonicity assumption ($f_1'(v) \ge -K$) we obtain that

$$\Phi(v) - \Phi(0) = \int_0^1 \nabla_v \Phi(sv).v\, ds = -\int_0^1 f_1'(sv)v.v\, ds \le K|v|^2 \tag{3.28}$$

Consequently, according to (3.23)

$$F(v) \le f_1(v).v + K|v|^2 = f(v).v - f_2(v).v + K|v|^2 \le$$
$$\le f(v).v + \mu|f_2(v)|^2 + C_\mu|v|^2 \le 1/2F(v) + f(v).v + C(|v|^2+1)$$

The right-hand inequality of (3.25) is proved. Lemma 3.1 is proved.

**Lemma 3.2.** *Let the assumptions of Theorem 3.3 hold. Then for $T > 0$*

$$\int_T^{T+1} \|\partial_t u(t)\|^2_{L^2}\, dt \le C\frac{T+1}{T}\left(\|u(0)\|^2_{L^2} e^{-\varepsilon t} + 1 + \|g\|^2_{L^2}\right) \tag{3.29}$$

*Proof.* Let us multiply the equation (0.1) by $t\partial_t u(t)$ and integrate over $t \in [0,2]$:

$$\int_0^2 t\|\partial_t u(t)\|^2_{L^2}\, dt = \int_0^2 (a\Delta_x u(t), t\partial_t u(t))\, dt - 2F(u(2)) + \tag{3.30}$$
$$+ \int_0^2 F(u(t))\, dt - \int_0^2 t(f_2(u(t)), \partial_t u(t))\, dt + \int_0^2 t(g, \partial_t u)\, dt$$

Applying the Holder inequality together with (2.5) and (3.26) to the right-hand side of (3.30) we deduce that

$$\int_0^2 t\|\partial_t u(t)\|^2_{L^2}\, dt \le C\left(\int_0^2 t\|\Delta_x u(t)\|^2_{L^2}\, dt + 1 + \|g\|^2_{L^2} + \|u_0\|^2_{L^2}\right)$$

Arguing as in the proof of estimate (2.6) one can easily derive that

$$\int_0^2 t\|\Delta_x u(t)\|_{L^2}^2\,dt \le C\left(\|u_0\|_{L^2}^2+1+\|g\|_{L^2}^2\right) \tag{3.31}$$

and therefore

$$\int_0^2 t\|\partial_t u(t)\|_{L^2}^2\,dt \le C_1\left(1+\|g\|_{L^2}^2+\|u_0\|_{L^2}^2\right) \tag{3.32}$$

Note that the estimate (3.32) implies (3.29). Indeed, for $T\le 1$ we derive from (3.32) that

$$T\int_T^{T+1} \|\partial_t u(t)\|_{L^2}^2\,dt \le C_1\left(1+\|g\|_{L^2}^2+\|u_0\|_{L^2}^2\right)$$

And if $T\ge 1$ then according to (3.32) and (2.5)

$$\int_{T-1}^{T} \|\partial_t u(t)\|_{L^2}^2\,dt \le C(\|u(T-1)\|_{L^2}^2+1+\|g\|_{L^2}^2) \le C_1\left(\|u(0)\|_{L^2}^2e^{-\varepsilon t}+1+\|g\|_{L^2}^2\right)$$

Lemma 3.2 is proved.

**Lemma 3.3.** *Let the assumptions of Theorem 3.3 hold. Then for $t>0$*

$$\|\partial_t u(t)\|_{L^2}^2 \le C\frac{1+t^2}{t^2}\left(\|u(0)\|_{L^2}^2e^{-\varepsilon t}+1+\|g\|_{L^2}^2\right) \tag{3.33}$$

*Proof.* Let us differentiate the equation (0.1) with respect to $t$ and denote $\theta(t)=\partial_t u(t)$. We will obtain the equation

$$\partial_t\theta(t)=a\Delta_x\theta(t)-f'(u(t))\theta(t) \tag{3.34}$$

Multiplying the equation (3.34) by $t^2\partial_t\theta$ and using the monotonicity assumption on $f$ we derive that

$$\partial_t(t^2\|\theta(t)\|_{L^2}^2)-2t\|\theta(t)\|_{L^2}^2 \le -t^2((a+a^*)\partial_t\theta,\partial_t\theta)+2Kt^2\|\theta(t)\|_{L^2}^2 \tag{3.35}$$

and therefore

$$\partial_t\left(t^2\|\theta(t)\|_{L^2}^2\right)+\varepsilon\left(t^2\|\theta(t)\|_{L^2}^2\right) \le Ct(t+1)\|\partial_t u(t)\|_{L^2}^2 \tag{3.36}$$

Applying the Gronewal inequality to the estimate (3.36) and using the estimate (3.32) for $\partial_t u(t)$ in the right-hand side of (3.36) we obtain the assertion of the lemma.

Now we are in a position to complete the proof of the Theorem. Indeed, the estimate (3.33) inserted in (1.12) gives us that

$$\|u(t)\|_{H^2}^2 \le C\frac{1+t^2}{t^2}\left(\|u(0)\|_{L^2}^2e^{-\varepsilon t}+1+\|g\|_{L^2}^2\right)$$

Inserting this estimate to (1.14) we derive the analogous estimate for the norm of $f(u(t))$. Theorem 3.3 is proved.

**Remark 3.3.** The model example of the nonlinearity $f(u)$ for which the assumptions of previous Theorem hold is the following:

$$f_1(u)=(a_1u_1|u_1|^{p_1},\cdots,a_ku_k|u_k|^{p_k}) \quad , \quad f_2(u)=Lu \tag{3.37}$$

where $a_i>0$, $p_i>0$ and $L$ is and arbitrary linear operator ($L\in\mathcal{L}(\mathbb{R}^k,\mathbb{R}^k)$).

In conclusion of this Section we consider another type of smoothing property which is well adopted to study the complex Ginzburg-Landau equations.

**Theorem 3.4.** *Let the nonlinear function $f$ can be represented in the following form*

$$f(v) = f_1(v) + f_2(v) \tag{3.38}$$

*where $f_1'(v) \geq 0$ and the function $f_2(v) \in C^1(\mathbb{R}^k, \mathbb{R}^k)$ has the finite support in $\mathbb{R}^k$. Then the attractor $\mathcal{A}$ of the corresponding equation* (0.1) *belongs to $\mathbb{D}$. Moreover, this attractor is bounded in $\mathbb{D}$.*

$$\|\mathcal{A}\|_{\mathbb{D}} \leq C \tag{3.39}$$

*Proof.* Let us represent the solution $u(t)$ of the equation (0.1) as a sum of two functions

$$u(t) = v(t) + w(t) \tag{3.40}$$

where $v(t)$ satisfies the equation

$$\partial_t v = a\Delta_x v - f_1(v) - f_2(u(t)) + g\ , \quad v\big|_{\partial\Omega} = 0\,, \quad v\big|_{t=0} = 0 \tag{3.41}$$

and $w(t)$ is a solution of the following equation

$$\partial_t w = a\Delta_x w - f_1(v+w) + f_1(v)\ , v\big|_{t=0} = u\big|_{t=0} \tag{3.42}$$

Since $f_1' \geq 0$ then evidently $(f_1(v+w) - f_1(v)).w \geq 0$ and consequently the solution $w(t)$ of the equation (3.42) satisfies the estimate

$$\|w(t)\|^2_{L^2} \leq Ce^{-\varepsilon t}\|u(0)\|^2_{L^2} \tag{3.43}$$

which holds now only for $u_0 \in \mathbb{D}$.

Let us study the equation (3.41). Since $\hat{S}_t : L^2 \to H^1$ then without loss of generality we may assume that $u_0 \in H^1$ (instead of starting with $u_0 \in L^2$). Note that $f_2$ has the compact support, therefore, arguing as in the proof of Theorem 3.2, we deduce that $f_2(u(t)) \in H^1([0,T], L^2)$ and satisfies the estimate

$$\int_T^{T+1} \|\partial_t f_2(u(t))\|^2_{L^2} + \|f_2(u(t))\|^2_{L^2} dt \leq C_{f_2}\left(\|u_0\|^2_{H^1}e^{-\varepsilon t} + \|g\|^2_{L^2} + 1\right)$$

Theorem 1.1 applied to the equation (3.41) (in which $-f_2(u(t)) + g$ is interpreted as the nonautonomous right-hand side) implies now that

$$\|v(t)\|^2_{\mathbb{D}} \leq C_1\|u(0)\|^2_{H^1}e^{-\varepsilon t} + C_2(1 + \|g\|^2_{L^2}) \tag{3.44}$$

Moreover, let $v_1(t)$ and $v_2(t)$ be two solutions of (3.41) which correspond to the solutions $u_1(t)$ and $u_2(t)$ of the initial problem (0.1). Then,

$$\|v_1(t) - v_2(t)\|^2_{L^2} \leq Ce^{2Kt}\|u_1(0) - u_2(0)\|^2_{L^2} \tag{3.45}$$

Indeed, since $f_2$ has the finite support then $f_2'$ is bounded and hence (2.4) implies that

$$\|f_2(u_2(t)) - f_1(u_1(t))\|^2_{L^2} \leq C\|u_1(t) - u_2(t)\|^2_{L^2} \leq C_1\|u_1(0) - u_2(0)\|^2 e^{2Kt} \tag{3.46}$$

Let $\tilde{v} = v_1 - v_2$. Then, evidently,

$$\partial_t \tilde{v} = a\Delta_x \tilde{v} - (f_1(v_1) - f_1(v_2)) - (f_2(u_1) - f_2(u_2)) \tag{3.47}$$

Multiplying the equation (3.47) by $\tilde{v}$ and integrating over $x \in \Omega$ we obtain after the evident transformations (which use (3.46) and the fact that $(f_1(v_1) - f_1(v_2)).\tilde{v} \geq 0$) that (3.45) is valid.

Note that the functions $v(t)$ and $w(t)$ are defined now only if $u_0 \in \mathbb{D}$. But due to the estimates (2.4) and (3.45) we can define by continuity (as in (2.2)) the decomposition (3.40) for every $u_0 \in L^2$ (and particularly for $u_0 \in H^1$). Moreover the estimates (3.43) and (3.44) remain valid for $u_0 \in H^1$ as well.

Thus, we have decomposed the operator $\hat{S}_t : H^1 \to H^1$ into a sum of two operators $\hat{S}_t^1 u_0 = v(t)$ and $\hat{S}_t^2 u_0 = w(t)$ one of them is smoothing ($\hat{S}_t^1 : H^1 \to \mathbb{D}$) and another is exponentially contracting in $L^2$-norm. Consequently, the attractor $\mathcal{A}$ belongs to $\mathbb{D}$ and

$$\|\mathcal{A}\|_{\mathbb{D}}^2 \leq C_2(1 + \|g\|_{L^2}^2) \tag{3.48}$$

where $C_2$ is the same as in (3.44). Theorem 3.4 is proved.

**Remark 3.4.** Let us consider the complex Ginzburg-Landau equation in the form of (0.7). It is not difficult to verify that the nonlinearity $f(u) = (1+i\beta)u|u|^{2\sigma} - Ru$, (written in a standard real form $u = (Re\, u, Im\, u)$) is quasimonotonic if and only if $\beta \leq \frac{\sqrt{2\sigma+1}}{\sigma}$. Moreover, if the inequality is strict then

$$\nabla_u \left((1 + i\beta)u|u|^{2\sigma}\right) \geq \varepsilon|u|^{2\sigma}$$

for a some positive $\varepsilon > 0$ and therefore $f'(u) \geq 0$ if $|u|$ is large enough. Thus, this nonlinearity possesses the decomposition (3.38) and consequently under the assumption (0.8) the attractor $\mathcal{A}_{G-L}$ of complex Ginzburg-Landau equation (0.7) satisfies the condition

$$\mathcal{A}_{G-L} \subset \mathbb{D} = H^2(\Omega) \cap \{u\big|_{\partial\Omega} = 0\} \cap L^{4\sigma+2}(\Omega) \tag{3.49}$$

Not that (3.49) holds for every $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\sigma > 0$ and $\beta$ satisfying (0.8).

## §4 The dimension of the attractor

In this Section we prove that under some additional assumptions on the nonlinear term $f(u)$ the attractor $\mathcal{A}$ of the equation (0.1) has a finite fractal dimension. Note that the usual way of estimating the fractal dimension of invariant sets involving the Liapunov exponents and $k$-contraction maps (see for instance [11]) requires the semigroup to be quasidifferentiable with respect to the initial data on the attractor. But in our case where $f(u)$ is not subordinated by the linear part $\Delta_x u$ (in the sense of (3.3)) we were able to prove only that $\mathcal{A} \subset \mathbb{D}$ (under the assumptions of previous Section) which is not sufficient to obtain the differentiability. To avoid this difficulty we present below a new scheme of estimating the dimension of invariant sets which works without the differentiability assumptions.

First of all we remind here the definition and the simplest properties of the fractal dimension (see [11] for further details).

**Definition 4.1.** Let $X$ be a metric space and let $M$ be a precompact set in $X$. Then, according to Hausdorff criteria the set $M$ can be covered by a finite number of $\varepsilon$-balls in $X$ for every $\varepsilon > 0$. Denote by $N_\varepsilon(M, X)$ the minimal number of $\varepsilon$-balls in $X$ which cover $M$. Then the Kolmogorov's entropy of the set $M$ in $X$ is defined to be the following number

$$\mathcal{H}_\varepsilon(M, X) \equiv \log_2 N_\varepsilon(M, X) \tag{4.1}$$

and the fractal (entropy, box-counting) dimension of $M$ can be defined in the following way

$$d_F(M) = d_F(M, X) = \limsup_{\varepsilon \to 0} \frac{\mathcal{H}_\varepsilon(M, X)}{\log_2 \frac{1}{\varepsilon}} \tag{4.2}$$

The following properties of the fractal dimension can be easily deduced from it's definition

**Proposition 4.1.** *1. Let $M$ be a compact $k$ dimensional Lipschitz manifold in $X$. Then $d_F(M, X) = k$.*

*2. Let $X$ and $Y$ be metric spaces $M \subset X$ and $L : X \to Y$. Assume that the map $L$ is globally Lipschitz continuous on $M$. Then*

$$d_F(L(M), Y) \le d_F(M, X) \tag{4.3}$$

*Particularly, the fractal dimension preserves under Lipschitz continuous homeomorphisms.*

The following Theorem is of fundamental significance in our study the dimension of attractors.

**Theorem 4.1.** *Let $H_1$ and $H$ be Banach spaces, $H_1$ be compactly embedded in $H$ and let $K \subset\subset H$. Assume that there exists a map $L : K \to K$, such that $L(K) = K$ and the following 'smoothing' property is valid*

$$\|L(k_1) - L(k_2)\|_{H_1} \le C\|k_1 - k_2\|_H \tag{4.4}$$

*for every $k_1, k_2 \in K$. Then the fractal dimension of $K$ in $H$ is finite and can be estimated in the following way:*

$$d_F(K, H) \le \mathcal{H}_{1/4C}(B(1, 0, H_1), H) \tag{4.5}$$

*where $C$ is the same as in* (4.4) *and $B(1, 0, H_1)$ means the unitary ball in the space $H_1$.*

*Proof.* Let $\{B(\varepsilon, k_i, H)\}_{i=1}^{N_\varepsilon}$, $k_i \in K$, be some $\varepsilon$-covering of the set $K$ (here and below we denote by $B(\varepsilon, k, V)$ the $\varepsilon$-ball in the space $V$, centered in $k$). Then, according to (4.4), the system $\{B(C\varepsilon, L(k_i), H_1)\}$ of $C\varepsilon$-balls in $H_1$ covers the set $L(K)$ and consequently (since $L(K) = K$) the same system covers the set $K$. Cover now every $H_1$-ball with radius $C\varepsilon$ by a finite number of $\varepsilon/4$-balls in $H$. By definition, the minimal number of such balls equals to

$$\begin{aligned} N_{\varepsilon/4}(B(C\varepsilon, L(k_i), H_1), H) = N_{\varepsilon/4}(B(C\varepsilon, 0, H_1), H) = \\ = N_{1/4C}(B(1, 0, H_1), H) \equiv \mathcal{N} \end{aligned} \tag{4.6}$$

Note, that the centers of $\varepsilon/4$-covering thus obtained not necessarily belongs to $K$ but we evidently can construct the $\varepsilon/2$-covering with centers in $K$ and with the same number of balls.

Thus, having the initial $\varepsilon$-covering of $K$ in $H$ with the number of balls $N_\varepsilon$ we have constructed the $\varepsilon/2$-covering with the number of balls $N_{\varepsilon/2} = \mathcal{N} N_\varepsilon$. Consequently, the $\varepsilon$-entropy of the set $K$ possesses the following estimate

$$\mathcal{H}_{\varepsilon/2}(K, H) \le \mathcal{H}_\varepsilon(K, H) + \log_2 \mathcal{N} \tag{4.7}$$

In fact the assertion of the theorem is a corollary of this recurrent estimate. Indeed, since $K \subset\subset H$ then there exists $\varepsilon_0$ such that $K \subset B(\varepsilon_0, k_0, H)$ and consequently

$$\mathcal{H}_{\varepsilon_0}(K, H) = 0$$

Iterating the estimate (4.7) $n$-times we obtain that

$$\mathcal{H}_{\varepsilon_0/2^n}(K, H) \le n \log_2 \mathcal{N} \tag{4.8}$$

Fix now an arbitrary $\varepsilon > 0$ and choose $n = n(\varepsilon)$ in such a way that

$$\frac{\varepsilon_0}{2^n} \le \varepsilon \le \frac{\varepsilon_0}{2^{n-1}} \tag{4.8}$$

Then

$$\mathcal{H}_\varepsilon(K) \le \mathcal{H}_{\varepsilon_0/2^n}(K) \le n \log_2 \mathcal{N} \le \log_2\left(\frac{2\varepsilon_0}{\varepsilon}\right) \log_2 \mathcal{N}$$

Theorem 4.1 is proved.

Now we are in a position to formulate the main result of this Section.

**Theorem 4.2.** *Let the assumptions of Theorem 2.2 hold and let $\mathcal{A}$ be the attractor of the equation (0.1). Assume that for a sufficiently small $\delta > 0$ the following regularity assumption is valid*

$$\|f'(\alpha u_0 + (1-\alpha)u_1)\|_{L^{2-\delta}(\Omega)} \le C \tag{4.9}$$

*uniformly with respect to $u_0, u_1 \in \mathcal{A}$ and $\alpha \in [0,1]$. Then the fractal dimension of the attractor $\mathcal{A}$ is finite.*

$$d_F(\mathcal{A}, L^2(\Omega)) < \infty \tag{4.10}$$

We are going to apply Theorem 4.1. In order to do so we need some estimates on a difference $v(t) = u_1(t) - u_2(t)$ between two solutions $u_1$ and $u_2$ belonging to the attractor.

**Lemma 4.1.** *Let the assumptions of the theorem hold and let $\varepsilon > 0$ and $\delta > 0$ satisfies the condition*

$$0 < k(\varepsilon, \delta) \equiv \frac{4\varepsilon + \delta - \varepsilon\delta}{1 - (\varepsilon + \delta)} \le \frac{4}{n-2}$$

*Then the following estimate is valid:*

$$\|\partial_t v\|_{L^{1+\varepsilon}([1,2],L^{1+\varepsilon}(\Omega))} + \|v\|_{L^2([1,2],H^1(\Omega))} \le C\|v(0)\|_{L^2} \tag{4.11}$$

*Proof.* Recall, that the function $v(t)$ satisfies the equation

$$\partial_t v(t) = a\Delta_x v - l(t)v \ , v\big|_{\partial\Omega} = 0 \tag{4.12}$$

with $l(t) = \int_0^1 f'(su_1(t) + (1-s)u_2(t))\, ds$. Since $u_1(t), u_2(t) \in \mathcal{A}$ then the assumption (4.9) implies that

$$\|l(t)\|_{L^{2-\delta}} \le C_1 \tag{4.13}$$

Let us estimate the $L^{1+\varepsilon}$-norm of the function $h_v(t) = l(t)v(t)$ using Holder inequality, the estimate (4.13), and Sobolev embedding theorem $H^1 \subset L^p$ if $p \le 2 + \frac{4}{n-2}$:

$$\|h_v(t)\|_{L^{1+\varepsilon}} \le \|l(t)\|_{L^{2-\delta}}\|v(t)\|_{L^{k(\varepsilon,\delta)}} \le C_2\|v(t)\|_{H^1} \tag{4.14}$$

It follows from the estimates (2.4) and (4.14) that

$$\|h_v\|_{L^{1+\varepsilon}([0,2],L^{1+\varepsilon})} \le C_3\|v(0)\|_{L^2} \tag{4.15}$$

Let us rewrite (4.12) as the linear nonhomogeneous parabolic problem in $\Omega$

$$\partial_t v = a\Delta_x v - h_v(t) \tag{4.16}$$

Then according to the $L^{1+\varepsilon}$-regularity theorem for the linear parabolic equation (4.16) and using the smoothing property for the corresponding homogeneous problem (see for instance [6]) we derive that

$$\begin{aligned} \|\partial_t v\|_{L^{1+\varepsilon}([1,2],L^{1+\varepsilon})} + \|\Delta_x v\|_{L^{1+\varepsilon}([1,2],L^{1+\varepsilon})} \le \\ \le C\left(\|v(0)\|_{L^{1+\varepsilon}} + \|h_v\|_{L^{1+\varepsilon}([0,2],L^{1+\varepsilon})}\right) \le C_4\|v(0)\|_{L^2} \end{aligned} \tag{4.17}$$

The estimate (4.17) together with (2.4) completes the proof of Lemma 4.1.

**Lemma 4.2.** *Let the assumptions of previous Lemma hold. Then*

$$\|v(1)\|_{L^2}^2 \le C\int_0^1 \|v(t)\|_{L^2}^2\, dt \tag{4.18}$$

*Proof.* Indeed, multiplying the equation (4.12) by $tv(t)$ and integrating over $x \in \Omega$ we obtain using the fact that $l(t) \ge -K$

$$\partial_t(t\|v(t)\|_{L^2}^2) - 2K\left(t\|v(t)\|_{L^2}^2\right) \le \|v(t)\|_{L^2}^2 \tag{4.19}$$

Applying the Gronewal inequality to the estimate (4.19) we obtain the assertion of the lemma.

Thus, combining the results of lemmata 4.1 and 4.2 we derive that

$$\|\partial_t v\|_{L^{1+\varepsilon}([2,3],L^{1+\varepsilon}(\Omega))} + \|v\|_{L^2([2,3],H^1(\Omega))} \le C\|v\|_{L^2([0,1],L^2)} \tag{4.20}$$

Now we are in the position to complete the proof of the theorem. To this end we introduce a space

$$\mathcal{W} = \{u \in L^2([0,1],H^1) : \partial_t u \in L^{1+\varepsilon}([0,1],L^{1+\varepsilon})\} \tag{4.21}$$

It is known (see [8]) that the space $\mathcal{W}$ is compactly embedded in $L^2([0,1],L^2)$.

Let us consider the restriction $\mathcal{K}\big|_{[0,1]}$ of the kernel $\mathcal{K}$, defined by (2.14) and the map

$$L : \mathcal{K}\big|_{[0,1]} \to \mathcal{K}\big|_{[0,1]}\ , (Lu)(t) = \hat{S}_2 u(t) \tag{4.22}$$

Since the attractor is strictly invariant with respect to $\hat{S}_t$ then

$$L\left(\mathcal{K}\big|_{[0,1]}\right) = \mathcal{K}\big|_{[0,1]}$$

and due to (4.20)

$$\|L(u_1) - L(u_2)\|_{\mathcal{W}} \le C\|u_1 - u_2\|_{L^2([0,1],L^2)}$$

Consequently, according to Theorem 4.1,

$$d_F\left(\mathcal{K}\big|_{[0,1]}, L^2([0,1],L^2(\Omega))\right) < \infty \tag{4.23}$$

The finite dimensionality of $\mathcal{A}$ in $L^2(\Omega)$ is an immediate corollary of (4.23), (4.18) and the second assertion of Proposition 4.1. Theorem 4.2 is proved.

Thus, we have proved that the attractor is finite dimensional under the regularity assumption (4.9). But it is still not clear how to verify this condition in applications. The following corollary gives an answer on this question.

**Corollary 4.1.** *Let the attractor $\mathcal{A}$ be bounded in $\mathbb{D}$ (for instance let the assumptions of Theorem 3.3 or 3.4 be valid). Let us assume also that there exists a convex function $\Psi : \mathbb{R}^k \to \mathbb{R}_+$ such that*

$$K_2\Psi(v) - C_2 \le \|f'(v)\|_{\mathcal{L}(\mathbb{R}^k,\mathbb{R}^k)} \le K_1\Psi(v) + C_1\ ,\ \forall v \in \mathbb{R}^k \tag{4.24}$$

*where $K_i > 0$. Moreover, it is assumed that the derivative $f'$ satisfies the estimate*

$$\|f'(v)\|_{\mathcal{L}(\mathbb{R}^k,\mathbb{R}^k)} \le C(|f(v)|^{1+\beta} + 1) \tag{4.25}$$

*for a sufficiently small $\beta > 0$ and every $v \in \mathbb{R}^k$. Then the assumption* (4.9) *is satisfied and consequently the attractor $\mathcal{A}$ has the finite fractal dimension.*

Indeed, since the function $\Psi$ is convex then

$$\begin{aligned}\|f'(\alpha v_1 - (1-\alpha)v_2)\|_{\mathcal{L}(\mathbb{R}^k,\mathbb{R}^k)} &\le K_1\alpha\Psi(v_1) + K_1(1-\alpha)\Psi(v_2) + C_2 \le \\ &\le \frac{K_1}{K_2}\left(\alpha\|f'(v_1)\|_{\mathcal{L}(\mathbb{R}^k,\mathbb{R}^k)} + (1-\alpha)\|f'(v_2)\|_{\mathcal{L}(\mathbb{R}^k,\mathbb{R}^k)} + C\right)\end{aligned} \tag{4.26}$$

for every $v_1, v_2 \in \mathbb{R}^k$ and $\alpha \in [0,1]$. Thus, (4.9) is fulfilled if

$$\|f'(u_0)\|_{L^{2-\delta}(\Omega)} \le C \quad \text{for every } u_0 \in \mathcal{A} \tag{4.27}$$

In order to verify the assumption (4.27) we use the estimate (4.25). Indeed, according to (4.27) and due to the fact that $\mathcal{A}$ is bounded in $\mathbb{D}$

$$\|f'(u_0)\|^{2-\delta}_{L^{2-\delta}} \le C(\|f(v)\|^2_{L^2} + 1) \le C(\|u\|^2_{\mathbb{D}} + 1) \le C_1$$

for $\delta = 2 - \frac{2}{1+\beta}$. Corollary 4.1 is proved.

**Remark 4.1.** Since the solutions of the equation $y' = y^{1+\beta}$ blow up in finite time then (4.25) is not a growth restriction but only a some kind of regularity assumption.

**Example 4.1.** Let us consider the Chaffee-Infante equation (0.9) in $\Omega \subset\subset \mathbb{R}^n$. Then all assumptions of Corollary 4.1 are evidently satisfied and therefore the attractor $\mathcal{A}$ of this equation has a finite dimension for an arbitrary $n \in \mathbb{N}$.

**Example 4.2.** All assumptions of Corollary 4.1 are fulfilled also for the complex Ginzburg-Landau equation (0.7) under the assumption (0.8). Consequently the corresponding attractor has the finite fractal dimension.

**Example 4.3.** Assume now that the nonlinear function $f(u)$ has the form $f(u) = f_1(u) + f_2(u)$ and (3.37) is satisfied. Then the assumptions of Corollary 4.1 are also satisfied and consequently for every $a \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^k)$ such that $a + a^* > 0$ and for an arbitrary $n \in \mathbb{N}$ the system (0.1) with such nonlinearity has a finite dimensional attractor.

**Remark 4.2.** Recall, that in this Section we have primarily considered the case where the nonlinearity is not subordinated by the linear term $\Delta_x u$. In the case where this subordination assumption is fulfilled (for instance if the assumptions of Remark 3.1 is valid) the differentiability of $\hat{S}_t$ with respect to the initial data $u_0$ can be verified directly and therefore the finite dimensionality of the attractor can be obtained by standard methods without any additional restrictions on $f$.

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