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A remark on the weakly damped wave equation

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Abstract. In this short note we present a direct method to establish the optimal regularity of the attractor for the semilinear weakly damped wave equation with a nonlinearity of critical growth

We consider the semilinear weakly damped wave equation on a bounded domain $\Omega \subset \mathbb{R}3$ with smooth boundary $\partial\Omega$

$$\begin{cases} \partial_{tt}u + \partial_t u - \Delta u + \varphi(u) = f, \\ u(0) = u_0, \quad \partial_t u(0) = u_1, \\ u_{|\partial\Omega} = 0. \end{cases} \tag{1}$$

Here, $f \in L2(\Omega)$ is independent of time and $\varphi \in C2(\mathbb{R})$, with $\varphi(0) = 0$, satisfies the growth and the dissipation conditions

$$|\varphi''(r)| \leq c\big(1 + |r|\big), \tag{2}$$

$$\liminf_{|r|\to\infty} \frac{\varphi(r)}{r} > -\lambda_1, \tag{3}$$

$$\varphi'(r) \geq -\ell, \tag{4}$$

for every $r \in \mathbb{R}$, where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ on $L2(\Omega)$ with Dirichlet boundary conditions and $\ell \geq 0$.

The asymptotic behavior of solutions to equation (1) has been the object of extensive studies (see, e.g. [1]–[5], [7] and [9]–[18]). In particular, denoting

$$\mathcal{H} = H_0 1(\Omega) \times L2(\Omega) \qquad \text{and} \qquad \mathcal{V} = \big[H2(\Omega) \cap H1_0(\Omega)\big] \times H1_0(\Omega),$$

the following result holds.

Theorem 1. *Problem* (1) *generates a C_0-semigroup $S(t)$ on the phase space $\mathcal{H}$ which possesses a compact global attractor $\mathcal{A}$. Besides, $\mathcal{A}$ is a bounded subset of $\mathcal{V}$.*

Theorem 1 was first proved by Babin and Vishik [3]. We mention that the result is still valid if one removes condition (4), which is however very reasonable. In that case, the existence of the attractor was shown in [1], whereas its $\mathcal{V}$-regularity first appeared in the papers [9, 10, 17]. In particular, the argument presented in [17] allows also to treat the nonautonomous case.

In all the preceding works, the $\mathcal{V}$-regularity of the attractor is achieved by means of rather complicated and long procedures, requiring multiplications by fractional operators and bootstrap arguments. The aim of this note is to show how to obtain this result in a very direct way, exploiting only quite simple energy estimates. This approach can be applied to treat more complicated boundary conditions such as, for example, dynamic boundary conditions (where the use of fractional operators may be problematic), as well as to deal with stabilization problems. The key step of our proof is a suitable decomposition of the solution u to (1), which has been already successfully employed in the recent works [6, 8, 18].

A new proof of Theorem 1. In what follows, we will often make use without explicit mention of the Sobolev embeddings and of the Young, the Hölder and the

Poincaré inequalities. As usual, we will perform formal estimates that can be justified in a proper Galerkin approximation scheme. Finally, for any function $z(t)$, we will write for short $\xi_z(t) = (z(t), \partial_t z(t))$.

We begin recalling a basic estimate.

Lemma 2. *For every $t \geq 0$, there holds*

$$\|\xi_u(t)\|_{\mathcal{H}}2 + \int_t^\infty \|\partial_t u(\tau)\|^2 d\tau \leq Q(\|\xi_u(0)\|_{\mathcal{H}})e^{-\varepsilon t} + Q(\|f\|),$$

for some $\varepsilon > 0$ and some positive increasing function Q.

The proof may be found, for instance, in [3], and it is carried out by multiplying the equation by $\partial_t u + \varepsilon u$, for some $\varepsilon > 0$ suitably small. In particular, this result yields the existence of a bounded absorbing set $\mathbb{B}_0 \subset \mathcal{H}$ for the semigroup $S(t)$.

In view of (2) and Lemma 2, we choose $\theta \geq \ell$ large enough such that the inequality

$$\frac{1}{2}\|\nabla z\|^2 + (\theta - 2\ell)\|z\|^2 - \langle \varphi'(u(t))z, z\rangle \geq 0 \tag{5}$$

holds for every $z \in H_0 1(\Omega)$, every $t \geq 0$ and every solution $u(t)$ with $\xi_u(0) \in \mathbb{B}_0$. Then, we set

$$\psi(r) = \varphi(r) + \theta r.$$

Clearly, condition (2) still holds with ψ in place of φ. Besides, on account of (4),

$$\psi'(r) \geq 0. \tag{6}$$

We now consider initial data $\xi_u(0) \in \mathbb{B}_0$, and we decompose the solution to (1) into the sum $u = v + w$, where v and w solve the equations

$$\begin{cases} \partial_{tt} v + \partial_t v - \Delta v + \psi(u) - \psi(w) = 0, \\ \xi_v(0) = \xi_u(0), \\ v_{|\partial\Omega} = 0, \end{cases} \tag{7}$$

and

$$\begin{cases} \partial_{tt} w + \partial_t w - \Delta w + \psi(w) = \theta u + f, \\ \xi_w(0) = (0, 0), \\ w_{|\partial\Omega} = 0. \end{cases} \tag{8}$$

In the following, $c \geq 0$ will stand for a generic constant depending (possibly) only on the size of $\mathbb{B}_0$ (but neither on the particular $\xi_u(0) \in \mathbb{B}_0$ nor of the time t).

Lemma 3. *For every $t \geq 0$, we have that $\|\xi_w(t)\|_{\mathcal{H}} \leq c$.*

Proof. The same argument of the proof of Lemma 2 applies to (8), since from Lemma 2 we know that the right-hand side belongs to $L^\infty(0, \infty; L2(\Omega))$. Observe also that here the initial data are null. □

Lemma 4. *For every $t \geq s \geq 0$ and every $\omega > 0$, there holds*

$$\int_s^t \|\partial_t w(\tau)\|^2 d\tau \leq \omega(t - s) + \frac{c}{\omega}.$$

Proof. Define the functional

$$\Lambda = \|\nabla w\|^2 + \|\partial_t w\|^2 + 2\langle \Psi(w), 1\rangle - 2\theta\langle u, w\rangle - 2\langle f, w\rangle,$$

where $\Psi(w) = \int_0^w \psi(y)dy$. Note that $\Lambda \leq c$, due to (2) and Lemma 3. Thus, multiplying (8) by $\partial_t w$, and applying once more Lemma 3, we obtain

$$\frac{d}{dt}\Lambda + 2\|\partial_t w\|^2 = -2\theta\langle \partial_t u, w\rangle \leq 2\omega + \frac{c}{\omega}\|\partial_t u\|^2,$$

and the claim is proved integrating in time on (s, t), exploiting the integral estimate furnished by Lemma 2. □

Collecting the above results, for all initial data $\xi_u(0) \in \mathbb{B}_0$ we have the bounds

$$\|\xi_u(t)\|_{\mathcal{H}} + \|\xi_w(t)\|_{\mathcal{H}} \leq c, \tag{9}$$

and

$$\int_s^t \left[\|\partial_t u(\tau)\|^2 + \|\partial_t w(\tau)\|^2\right] d\tau \leq \omega(t-s) + \frac{c}{\omega}, \qquad \forall \omega > 0. \tag{10}$$

In order to conclude, we need the following generalized version of the Gronwall lemma.

Lemma 5. *Let $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ be an absolutely continuous function satisfying*

$$\frac{d}{dt}\Lambda(t) + 2\varepsilon\Lambda(t) \leq h(t)\Lambda(t) + k,$$

where $\varepsilon > 0$, $k \geq 0$ and $\int_s^t h(\tau)d\tau \leq \varepsilon(t-s) + m$, for all $t \geq s \geq 0$ and some $m \geq 0$. Then,

$$\Lambda(t) \leq \Lambda(0)e^m e^{-\varepsilon t} + \frac{ke^m}{\varepsilon}, \qquad \forall t \geq 0.$$

We are now in a position to prove

Lemma 6. *For every $t \geq 0$ and some $\nu > 0$, there holds*

$$\|\xi_v(t)\|_{\mathcal{H}} \leq ce^{-\nu t}.$$

Proof. For $\varepsilon \in (0, 1)$ to be determined later, define the functional

$$\Lambda = \|\nabla v\|^2 + \|\partial_t v\|^2 + \varepsilon\|v\|^2 + 2\langle \psi(u) - \psi(w), v\rangle - \langle \psi'(u)v, v\rangle + 2\varepsilon\langle \partial_t v, v\rangle.$$

Note that, from (4) and (5),

$$2\langle \psi(u) - \psi(w), v\rangle - \langle \psi'(u)v, v\rangle \geq (\theta - 2\ell)\|v\|^2 - \langle \varphi'(u)v, v\rangle \geq -\frac{1}{2}\|\nabla v\|^2.$$

Hence, on account of (2) and (9), Λ satisfies the inequalities

$$\frac{1}{4}\|\xi_v\|^2_{\mathcal{H}} \leq \Lambda \leq c\|\xi_v\|^2_{\mathcal{H}}, \tag{11}$$

provided that ε is small enough. Multiplying (7) by $\partial_t v + \varepsilon v$, we find the equality

$$\frac{d}{dt}\Lambda + \varepsilon\Lambda + \frac{\varepsilon}{2}\|\nabla v\|^2 + \Gamma = 2\langle (\psi'(u) - \psi'(w))\partial_t w, v\rangle - \langle \psi''(u)\partial_t u, v2\rangle,$$

where we set

$$\Gamma = \frac{\varepsilon}{2}\|\nabla v\|^2 + (2-3\varepsilon)\|\partial_t v\|^2 + \varepsilon\langle\psi'(u)v, v\rangle - \varepsilon 2\|v\|^2 - 2\varepsilon 2\langle\partial_t v, v\rangle.$$

Using (2), (6) and (9), it is apparent that $\Gamma \geq 0$ if ε is small enough, and

$$\begin{aligned} 2\langle(\psi'(u) - \psi'(w)\partial_t w, v\rangle - \langle\psi''(u)\partial_t u, v2\rangle &\leq c\big(\|\partial_t u\| + \|\partial_t w\|\big)\|\nabla v\|^2 \\ &\leq \frac{\varepsilon}{2}\|\nabla v\|^2 + \frac{c}{\varepsilon}\big(\|\partial_t u\|^2 + \|\partial_t w\|^2\big)\Lambda, \end{aligned}$$

by means of (11). At this point, choosing $\varepsilon > 0$ such that the above conditions are all satisfied, we obtain the differential inequality

$$\frac{d}{dt}\Lambda + \varepsilon\Lambda \leq c\big(\|\partial_t u\|^2 + \|\partial_t w\|^2\big)\Lambda. \tag{12}$$

In view of (10), the desired conclusion follows from Lemma 5 and (11). □

Lemma 7. *For every $t \geq 0$, there holds*

$$\|\xi_w(t)\|_{\mathcal{V}} \leq c.$$

Proof. Setting $q = \partial_t w$, we differentiate (8) with respect to time, so to obtain

$$\partial_{tt} q + \partial_t q - \Delta q + \psi'(w)q = \theta\partial_t u.$$

Then, for $\varepsilon > 0$, we define the functional

$$\Lambda = \|\nabla q\|^2 + \|\partial_t q\|^2 + \varepsilon\|q\|^2 + \langle\psi'(w)q, q\rangle + 2\varepsilon\langle\partial_t q, q\rangle,$$

which, similarly to the previous lemma, satisfies the inequalities

$$\frac{1}{2}\|\xi_q\|^2_{\mathcal{H}} \leq \Lambda \leq c\|\xi_q\|^2_{\mathcal{H}},$$

when ε is small enough. Multiplying the above equation by $\partial_t q + \varepsilon q$, we are led to

$$\frac{d}{dt}\Lambda + \varepsilon\Lambda + \frac{\varepsilon}{2}\|\nabla q\|^2 + \|\partial_t q\|^2 + \Gamma = 2\theta\langle\partial_t u, \partial_t q\rangle + \langle\psi''(w)\partial_t w, q2\rangle + 2\varepsilon\theta\langle\partial_t u, \partial_t w\rangle,$$

where

$$\Gamma = \frac{\varepsilon}{2}\|\nabla q\|^2 + (1-3\varepsilon)\|\partial_t q\|^2 + \varepsilon\langle\psi'(w)q, q\rangle - \varepsilon 2\|q\|^2 - 2\varepsilon 2\langle\partial_t q, q\rangle.$$

Again, $\Gamma \geq 0$ provided that ε is small enough, whereas the right-hand side of the above differential equality is controlled as

$$\begin{aligned} &2\theta\langle\partial_t u, \partial_t q\rangle + 2\varepsilon\theta\langle\partial_t u, \partial_t w\rangle + 2\langle\psi''(w)\partial_t w, q2\rangle \\ &\leq \frac{\varepsilon}{2}\|\nabla q\|^2 + \|\partial_t q\|^2 + c\|\partial_t w\|^2\Lambda + c. \end{aligned}$$

Hence, fixing ε small, we end up with the differential inequality

$$\frac{d}{dt}\Lambda + \varepsilon\Lambda \leq c\|\partial_t w\|^2\Lambda + c,$$

and from Lemma 5, we get the bound

$$\|\nabla\partial_t w(t)\| + \|\partial_{tt} w(t)\| \leq c.$$

With this information, we recover from (8) the further control $\|\Delta w(t)\| \leq c$. □

Collecting Lemma 6 and Lemma 7, we learn that $S(t)\mathbb{B}_0$ is (exponentially) attracted by a bounded subset $\mathcal{C} \subset \mathcal{V}$. In other words, $\mathcal{C}$ is a compact attracting set. This, by standard arguments of the theory of attractors (see e.g. [3, 12, 16]), yields the existence of a compact global attractor $\mathcal{A} \subset \mathcal{C}$ for the semigroup $S(t)$. The proof of Theorem 1 is then completed.

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