Advanced Analysis. Revision

mock exam were MORE Difficult than the main one!!! (3 weeks is a bit more than 4 hours) - no essential technicalities are expected!

"Prove by first principles" - what does it mean?
- deduce the result from definitions.

Example: Prove or disprove that $x_n \to x_0$

for disprove you need to find $\ell \in X^*$ such that $\ell x_n \not\to \ell x_0$

for prove - check that $\forall \ell \in X^* \quad \ell x_n \to \ell x_0$.

not allowed: to use that conv. weakly seq must be bounded, etc.

4 Questions → you need to answer 3 of them (or your choice)

If "by 1st principles" is stated, you may use (Any) result from lectures without proving it.

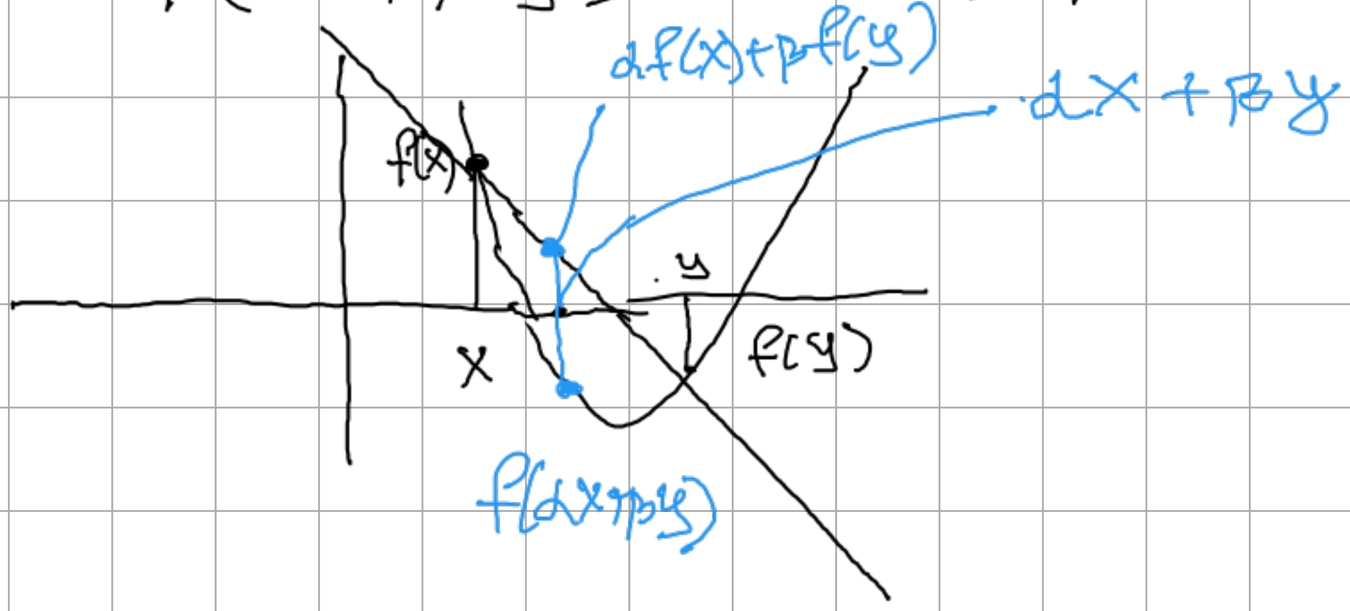
If you see "prove using Mazur theorem"
- you should use Mazur theorem.
If you use something NOT from lectures, you need to give the proof.

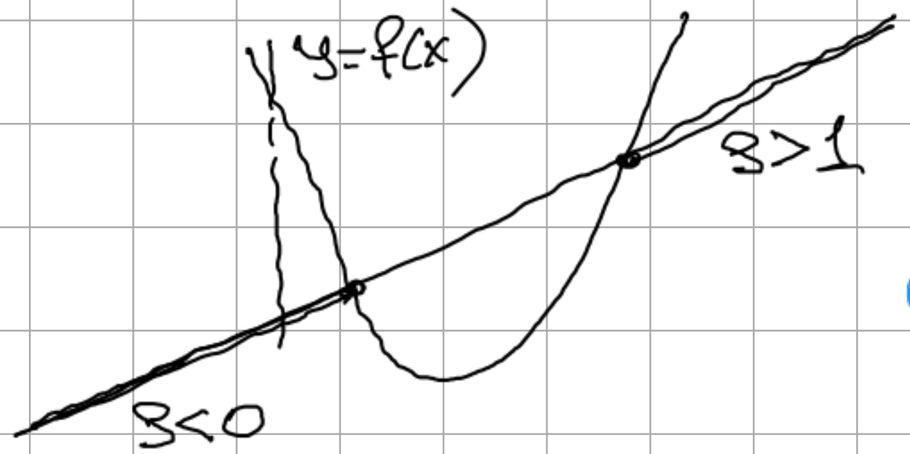
If there are hints, you may ignore them
correct solution without using hints will give you full mark.

Q1 - convexity : $f: \mathbb{R} \to \mathbb{R}$ - smooth (C^2)

convex $\Leftrightarrow$ $f''(x) \geq 0$

$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$, $\alpha + \beta = 1$, $\alpha, \beta \geq 0$



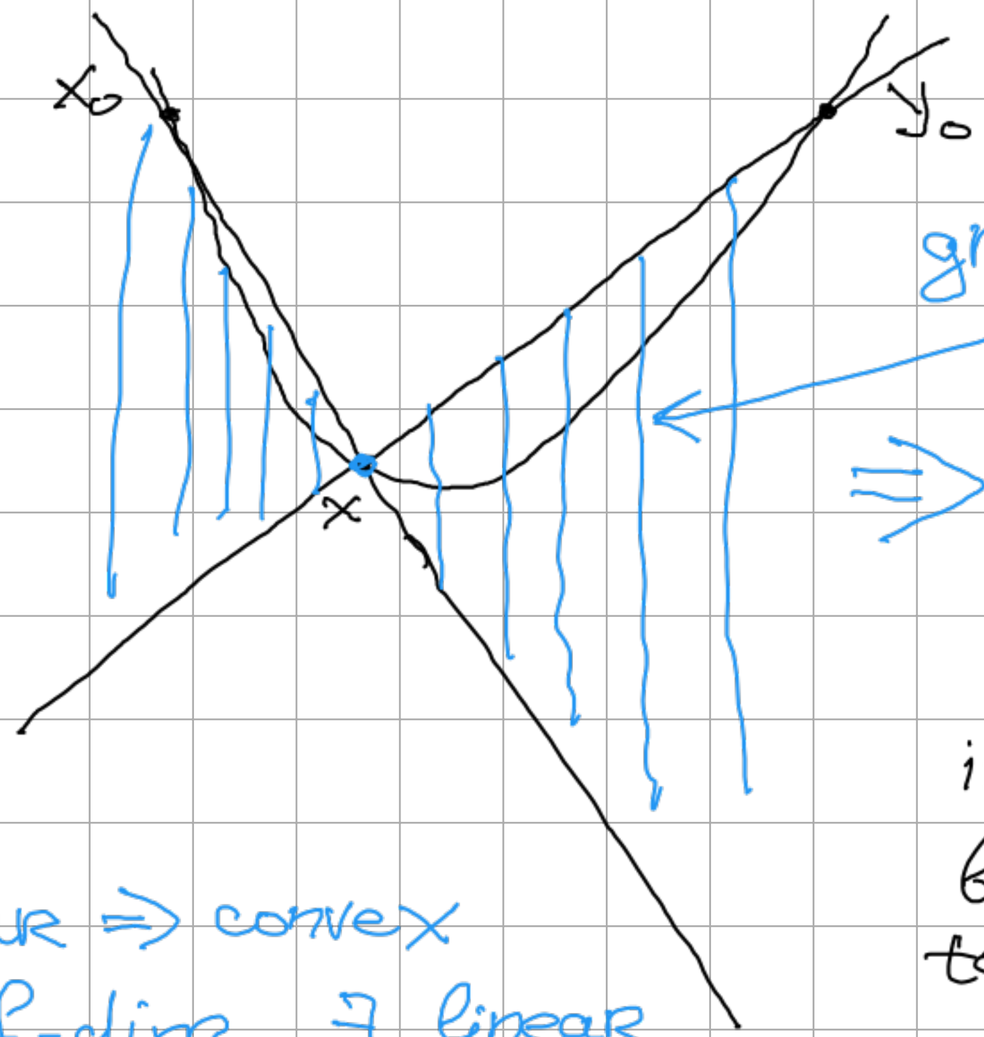


$$sx + (1-s)y \quad , \quad s \in [0,1]$$

$$f(sx+(1-s)y) \leq s f(x) + (1-s) f(y)$$

$$f(sx+(1-s)y) \geq s f(x) + (1-s) f(y)$$

if $s<0$ or $s>1$.



linear ⇒ convex

in inf-dim ∃ linear discont f-ls - may be unbounded.

graph of f(x) lives here

⇒ f is Lipschitz continuous

in finite dimensions

in inf-dim → you need boundedness of f in order to get Lip.

Q2 – Sobolev and interpolation ≤
weak derivatives + verification
$f \in W^{e,p}$?

- you are given some basic ≤ and asked to derive a bit more advanced one.

$f \in W^{e,p}$?

a) find classical derivatives, check that they belong to the desired space

b) $f'_{cl} = f'_w$ by checking integration by parts

Do not forget about δ-functions and about δ-like sequences !!

$f'_w \in L^1_{loc}(\mathbb{R})$ ∫ $f'_w(x)\,\varphi(x)\,dx$ is well-def $\forall \varphi \in C_0^\infty(\mathbb{R})$

$\int f\,\varphi'\,dx = -\int f'_w \cdot \varphi$ – is also well def as Lebesgue sense.

$f \in L^1_{loc}$ $\quad$ $f'_w \in L^1_{loc}$

if Ω is <u>unbounded</u> $\Omega = \mathbb{R}$

$f(x) = x$ - extremely nice!

$f'_{cc}(x) = 1 \Rightarrow f'_{dist} = 1$

$\Rightarrow f'_w = 1 \notin L^1(\mathbb{R})$ But in $L^1_{loc}(\mathbb{R})$.

<u>Q3</u> - <u>Functional Analysis</u>: linear f-cs, their norms, Riesz rep. ths, weak convergence, existence of a minimizer, Mazur th, etc.

- calculation of the norm of a given (simple!) functional - very likely
- likely whether or not the norm is attained.
- very likely, weak convergence (check $x_n \to x_0$ or does not converge)
- likely mazur theorem and consequences.

Rather straightforward Qs.

Q4: PDEs

- classical and weak solutions - very likely
- be able to use Riesz rep. Lax-Milgram, or existence of a min. th to find a weak solution.
- be able to find explicitly a weak solution of some simple ODE.
- do not forget about fundamental solutions (they solve $Lu = \delta(x)$, not $Lu = 0$!)