Lecture 13: Banach contraction theorem. (I)

Let (X, d) be a (metric) space and

$$f : X \to X$$

We are interested in fixed points of that map

Problem: find all $p \in X$ such that

$$p = F(p)$$

Why is it important?

Example: We intend to solve the ODE

(ODE) $$y'(x) = f(x, y(x)), \quad y(0) = y_0, \quad y(x) \in \mathbb{R}^n$$

(f is a given continuous function) on the interval

$$x \in [0, t_0]$$

Rewrite the equation in the integral form

$$y(x) = y_0 + \int_0^x f(s, y(s))\, ds$$

Introduce a map $T : C[0, t_0] \to C[0, t_0]$ by

$$(Ty)(x) := y_0 + \int_0^x f(s, y(s))\, ds$$

it is well-defined if f is continuous. Moreover

Solutions of (ODE) = Fixed points of the map T

Many other examples (problems) from ODEs, PDEs, differential geometry, ... can be reduced to the fixed point problem.

How to solve the fixed point problem?

Huge variety of methods!

The so-called FIXED POINT THEOREMS

Fixed point theorems

Class I: Global (topological)

Usually F is only continuous assumptions on the whole space only existence (without uniqueness)

Class II: Local

typically in a small neighborhood of a "good" point more regular f existence and uniqueness of a fixed points

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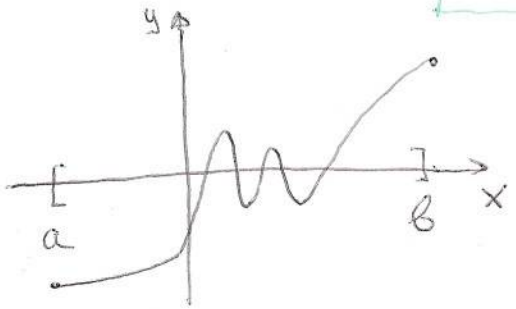
Simplest example for class I: Cauchy theorem

Let $f: [a, b] \to \mathbb{R}$ be continuous and let

$$f(a) \cdot f(b) < 0$$



Then $\exists\, x \in [a, b]$

$$f(x) = 0$$

may be a lot of solutions, but at least one always exists!

More complicated extension: Brauder theorem:

Let F maps a closed unit ball $\overline{B}_1(0)$ of $\mathbb{R}^n$ to itself $F: \overline{B}_1(0) \to \overline{B}_1(0)$

+ F is continuous $\Rightarrow$ $\exists$ at least one fixed point

Extensions to inf-dim case, degree theory, ...

Class II: exactly the Banach contraction theorem +

+various Implicit / Inverse Function theorems; ...

Def. Let (X, d) be a metric space and let $F: X \to X$ be a maps

F is a contraction if $\exists\, \varkappa < 1$ such that

$$d(F(x), F(y)) < \varkappa\, d(x, y)$$

for all $x, y \in X$, ($\varkappa$ is a contraction factor)

Theorem (Banach, 1922)

Let (X, d) be a complete metric space and let $F: X \to X$ be a contraction

Then, there exists a unique fixed point $p \in X$ (i.e. such that $p = F(p)$)

Some heuristics:

1) the theorem looks global, but typically the interesting maps F are not contractions on the whole domains of definitions, but only on the properly constructed "small" neighborhoods of "good" points

Namely, one usually has a map $F: H \to H$ defined on some Banach/metric space H and needs to find a closed set $X \subset H$ such that

1) $F: X \to X$ (F maps X to X)
2) F is a contraction on X.

most difficult part!

Then Banach theorem works.

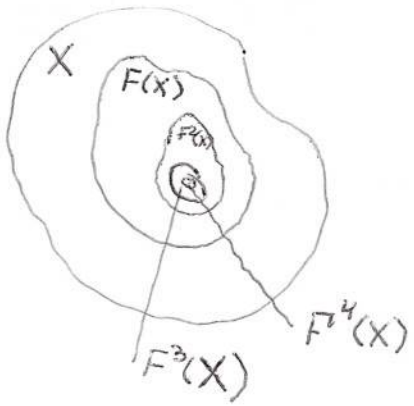
Some hints for the proof: Let $D(X) = \sup\{d(x, y), x, y \in X\}$ - diameter of X be finite. Then

$$D(F(X)) \leq \varkappa D(X)$$

$\varkappa < 1$

the set X is contracted by F!

Picture:



$F^n = \underbrace{F \circ F \circ \cdots \circ F}_{n}$ is nth iteration of F.

$D(F^n(X)) \leq \varkappa^n D(X)$ diameter of $F^n(X)$ tends to zero.

$+ \; F^{n+1}(X) \subset F^n(X)$

very similar to the Cauchy sequence

+ completeness, we expect that

$\bigcap_n F^n(X) \neq \emptyset$ and consist of a single point $p \in X$

then p necessarily a fixed point of F.

Rigorous proof:

Uniqueness: let p_1 and p_2 be two fixed points, then

$$d(p_1, p_2) = d(F(p_1), F(p_2)) \leq \alpha\, d(p_1, p_2)$$

$\alpha < 1 \Rightarrow d(p_1, p_2) = 0 \Rightarrow p_1 = p_2$.

Existence: Take $x_1 \in X$ arbitrarily and let

$$x_{n+1} = F(x_n)$$

sequence of iterations of F: $x_{n+1} = F^n(x_1)$

Need to prove that $\{x_n\}$ is a Cauchy sequence.

Indeed

$$d(x_{n+1}, x_n) = d(F(x_n), F(x_{n-1})) \leq \alpha\, d(x_n, x_{n-1}) = d(F(x_{n+1}), F(x_{n-2}))$$
$$\leq \alpha^2 d(x_{n-1}, x_{n-2}) \leq \dots \leq \alpha^{n-1} d(x_2, x_1)$$

Then, by triangle inequality

$$d(x_n, x_{n+m}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{n+m-1}, x_{n+m})$$
$$\leq \alpha^{n-1} d(x_2, x_1)\left(1 + \alpha + \alpha^2 + \alpha^{m-1}\right) \leq$$

(g.p. $\alpha < 1$)

$$\leq \frac{\alpha^{n-1}}{1-\alpha} d_2(x_2, x_1)$$

Thus, $d(x_n, x_{n+m}) \to 0$ as $n \to \infty \Rightarrow \{x_n\}$ is a Cauchy sequence

X is complete $\Rightarrow \exists\, p = \lim_{n\to\infty} x_n$

Prove that p is a fixed point:

1) F is continuous (even uniformly continuous) since it is a contraction.

2) Passing to the limit $n \to \infty$

$$p = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(p)$$

□

Lecture Application of the Banach contraction theorem to ODEs

Want to solve the ODE:

(ODE) $\frac{dy}{dx} = f(x, y(x)), \quad y|_{x=x_0} = y_0$ $\qquad y=(y^1,\dots y^n)\in\mathbb{R}^n$

Step 1: Reformulate it as a fixed point problem:
Integrating the equation over x, we have

$$y(x) = y_0 + \int_{x_0}^{x} f(x, y(x))\,dx$$

Let f - be continuous. Introduce the integral operator $T: C[-L,L] \to C[-L,L]$, $L>0$

$$(Ty)(x) := y_0 + \int_0^x f(s, y(s))\,ds$$

Lemma:

Fixed points of T in $X := C([-L,L], \mathbb{R}^n)$ $=$ solutions of (ODE) defined on $[-L,L]$

Proof: Indeed, integrating the ODE by $x \Rightarrow$ any solution of the ODE is a fixed point of T on X.
Let $y \in X$ be a fixed point of T. Then

$$y(x) = y_0 + \int_0^x f(s, y(s))\,ds$$

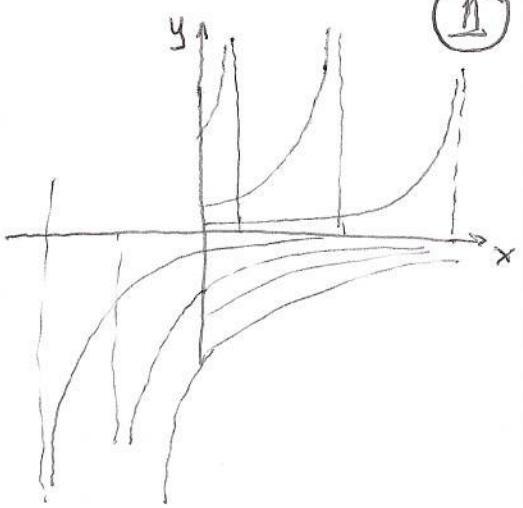
We see

1) $y(x_0) = y_0$
2) y is continuously differentiable (as integral from a continuous function)

Differentiating by $x \Rightarrow$ $\frac{dy}{dx} = f(x, y(x))$

□

Simple examples



1) Non-existence:

$$y = y^2, \quad y|_{x=0} = y_0$$

Solution:

$$y(x) = \frac{y_0}{1 - y_0 x}$$

for $y_0 > 0$ the solution exists for $x < 1/y_0$ only (it blows up (tends to ∞) at $x = 1/y_0$)

for $y_0 < 0$ the same happends for negative x.

Conclusion: a) Even for smooth f, we may expect only local existence (for x close to x_0)

b) The existence interval depends on the initial data y_0.

2) Non-uniqueness:

$$y' = 3y^{2/3}, \quad y(0) = 0$$

Has 2 different solutions

$$y_1(x) \equiv 0, \quad y_2(x) = x^3$$

In a fact, it has infinitely many solutions

Ex. find all of them!

Conclusion: only continuity of f is not sufficient for the uniqueness.

Remark: It is sufficient for the (local) existence. But it cannot be proved by Banach contraction theorem (why?) and we do not consider it here.

$\Rightarrow$ Some more regularity of f is necessary for existence and uniqueness

Theorem (local existence and uniqueness):
Let there exist a neighborhood of (x_0, y_0) in the extended phase space $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$

$V := [-L_0 + x_0, L_0 + x_0] \times \overline{B}^{\varepsilon}_{y_0}$ such that

1) f is continuous on V (w.r.t. both x and y arguments)

2) f is uniformly Lipschitz continuous w.r.t. y:

$$|f(x, y_1) - f(x, y_2)| \le K \cdot |y_1 - y_2|, \quad x \in [-L_0 + x_0, L_0 + x_0], \quad y_i \in \overline{B}^{\varepsilon}_{y_0}$$

(usually can be verified by the mean value theorem if f_y exists and continuous)

Then: there exists $L < L_0$, $L > 0$ such that the (ODE) has a unique solution $y(x)$ defined for $x \in [-L + x_0, L + x_0]$.

Proof: We want to apply the B-theorem to the integral map

$$(Ty)(x) := y_0 + \int_{x_0}^{x} f(s, y(s))\, ds$$

on the properly defined subset

$$X \subset C([-L + x_0, L + x_0], \mathbb{R}^n) = C([-L+x_0, L+x_0], \mathbb{R}^n)$$

Need to find X such that

1) X is closed (then it will be complete)

2) $TX \subset X$

3) T must be a contraction on X.

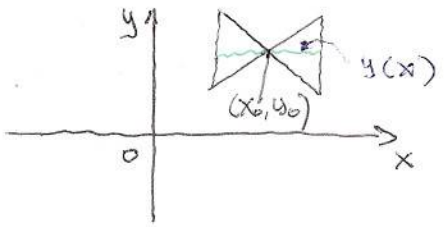
Then the existence and uniqueness will follow from the Banach contraction theorem.

Construction of X: Let $P := \max_{(x,y) \in V} |f(x, y)|$ and let

$L < \varepsilon / P$ and

$$X := \left\{ y \in C([-L + x_0, L + x_0], \mathbb{R}^n), \quad |y(x) - y_0| \le P|x - x_0| \right\}$$

$$|y(x) - y_0| \le P|x - x_0|$$



All functions from X belong to the <u>cone</u> centered at (x_0, y_0).

(standard construction in applications of the Banach theorem)

Then 1) X is <u>closed</u> in $C([-L+x_0, L+x_0], \mathbb{R}^n)$ with the sup-metric – <u>obvious</u>

2) $TX \subset X$ Indeed

$$|(Ty)(x) - y_0| \le \left| \int_0^x f(s, y(s))\,ds \right| \le p \left| \int_{x_0}^x |dx| \right| = p|x|_{x_0}$$

<u>Moreover</u> ALL possible solutions of (ODE) on the interval $[-L, L]$ must belong to X.

(the derivative $\left|\frac{dy}{dx}\right| \le \underline{P}$

3) Let us check that T is a contraction

$$|(Ty_1)(x) - (Ty_2)(x)| \le \left| \int_{x_0}^x |f(x, y_1(x)) - f(x, y_2(x))|\,dx \right| \le$$

$$\le \left| \int_{x_0}^x k\,|y_1(x) - y_2(x)|\,dx \right| \le kL\,\|y_1 - y_2\|_{sup}$$

use Lipschitz continuity

$\Rightarrow$

$$\|Ty_1 - Ty_2\|_{sup} \le (kL)\,\|y_1 - y_2\|_{sup}$$

It will be a <u>contraction</u> on X if in addition L is chosen in such way that $(kL) < 1$

+ Banach contraction theorem $\exists!$ fixed point of T in X = unique solution of the (ODE) on the interval

$$x \in [-L + x_0, L + x_0]$$

□