Functional Analysis
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0 Preface

These are the classnotes for both MS310 (BSc) and MS320 (MSc) for 2005-2006. The difference between these module is the amount of material and the lesser emphasis on proofs. Parts of the notes that will not be examined in MS310 are denoted

$\diamondsuit$ with this sign and the wider margin.

For these notes, material has been drawn from the books:

- I. Stakgold, *Green's functions and boundary value problems,* Wiley, 1979
- E. Kreyszig, ,*Introductory Functional Analysis with Applications*, Wiley 1978.
- N. Young, *An introduction to Hilbert space,* Cambridge University Press, 2001.

1 Inner Products and Norms

Vector spaces: are spaces E in which you can add:

$$\forall\, x, y \in E,\ x + y \in E,$$

and multiply with a scalar:

$$\forall\, x \in E \text{ and } \lambda \in \mathbb{K},\ \text{ we have } \lambda x \in E.$$

Here $\mathbb{K}$ can be any field, but usually we take $\mathbb{K} = \mathbb{R}$ (*real vector space*) or $\mathbb{K} = \mathbb{C}$ (*complex vector space*). These operations satisfy a set of axioms for which we refer to a course in linear algebra. The dimension $\dim E = N$ if we can find a *basis* $\{e_1, \ldots, e_N\} \subset E$ such that each vector $x \in E$ can be written **uniquely** as a linear combination:

$$x = \lambda_1 e_1 + \cdots + \lambda_N e_N,$$

for some $\lambda_1, \ldots, \lambda_N \in \mathbb{K}$. The dimension $\dim(E) = \infty$ if no finite basis exists. Still it would be very nice to have an infinite basis; the properties of these bases are more involved and we come back to it later.

Inner product spaces: are spaces equipped with an inner product, i.e. a function $\langle\ ,\ \rangle : E \to E \to \mathbb{C}$ such that

1. $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for all $x, y \in E$. The bar denotes complex conjugate.
2. $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$ for all $x, y \in E$ and $\lambda \in \mathbb{C}$.
3. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ for all $x, y, z \in E$.
4. $\langle x, x\rangle > 0$ whenever $x \in E$, $x \neq 0$.

If E is a real vector space, then the inner product becomes more simple, as we can forget about the complex conjugate. Note that item 2 combined with 4 give that $\langle x, x\rangle = 0$ if and only if $x = 0$.

Examples: • $E = \mathbb{K}^n$ with standard inner product $\langle x, y\rangle = \sum_{i=1}^n x_i \overline{y_i}$.
(Remark: Some texts use $\langle x, y\rangle = \sum_{i=1}^N \overline{x_i} y_i$ as standard inner product here. This involves a slight change in item 2 of the definition of inner product, but as long as it is clear to everyone which inner product is used, it causes no problems.)
• $E = \mathbb{K}^n$ and $\langle x, y\rangle_A = \langle Ax, y\rangle$ for the standard inner product of above, and A a positive definite matrix.
• $E = C([a, b]) = \{f : [a, b] \to \mathbb{K} \mid f \text{ is continuous}\}$ and standard inner product $\langle f, g\rangle = \int_a^b f(t)\overline{g(t)}dt$.
• $E = M_{m\times n}(\mathbb{K})$, the space of $m \times n$ matrices with entries in $\mathbb{K}$, with standard inner product $\langle A, B\rangle = \mathrm{trace}(B^*A)$, where $B^* = \overline{(A^t)}$ is the complex conjugate of the transpose matrix.

Normed spaces: are vector spaces equipped with a norm $\| \; \| : E \to \mathbb{R}$, satisfying the following axioms:

1. $\|x\| > 0$ for all $x \in E$, $x \neq 0$.
2. $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in E$ and $\lambda \in \mathbb{K}$.
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in E$. This is the *triangle inequality.*

Note that item 1 and 2 together give $\|x\| = 0$ if and only if $x = 0$.

Any inner product space is also a normed space, if we define the norm as

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

Checking the first two axioms of the definition of a norm is straightforward. Checking the triangle inequality relies on the *Cauchy-Schwarz inequality*:

$$|\langle x, y\rangle| \leq \|x\| \, \|y\| \text{ for all } x, y \in E.$$

Proof. If $y = 0$, the inequality is obvious, so assume $y \neq 0$. Calculate

$$0 \leq \langle x - \lambda y, x - \lambda y\rangle = \|x\|^2 - \lambda\langle y, x\rangle - \overline{\lambda}\langle x, y\rangle + \lambda\overline{\lambda}\|y\|^2.$$

Now substitute $\lambda = \frac{\langle x,y\rangle}{\|y\|^2}$, then we get

$$0 \leq \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2}.$$

Multiply by $\|y\|^2$ and rearrange to $|\langle x, y\rangle|^2 \leq \|x\|^2\|y\|^2$ and finally take the square root. $\square$

To derive the triangle inequality from this, we compute

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle = \|x\|^2 + \langle x,y\rangle + \langle y,x\rangle + \|y\|^2 \\ &= \|x\|^2 + 2\mathrm{Re}\langle x,y\rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2 \qquad \text{(Use the Cauchy-Schwarz inequality)} \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\|+\|y\|)^2.\end{aligned}$$

Finally take the square root on either side.

Examples: Some examples for standard norms come straight from inner products:
- $E = \mathbb{K}^n$ with standard (= Euclidean) norm $\|x\| = \sqrt{\sum_{i=0}^n |x_i|^2}$.
- $E = C([a,b])$ and norm $\|f\| = \sqrt{\int_a^b |f(t)|^2 dt}$.
- $E = M_{m\times n}(\mathbb{K})$, with norm $\|A\| = \sqrt{\sum_{i=1,j=1}^{m,n} |a_{i,j}|^2}$.

There are however norms that are not related to inner products, such as:
- $C([a,b])$ with sup-norm $\|f\|_\infty = \sup\{|f(t)| \mid t \in [a,b]\}$.

Theorem 1 *On a normed space* $(E, \|\ \|)$, *an inner product compatible to the norm exists if and only if the* parallelogram law*:*

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

holds. In this case, the inner product can be defined by the polarisation identity

$$\langle x,y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right).$$

Metric spaces: are spaces (not necessarily vector spaces) equipped with a distance function, called *metric* $d : E \times E \to \mathbb{R}$, satisfying

1. $d(x,y) \geq 0$ for all $x,y \in E$, and $d(x,y) = 0$ if and only if $x = y$.
2. $d(x,y) = d(y,x)$ for all $x,y \in E$.
3. $d(x,y) \leq d(x,z) + d(z,y)$; the *triangle inequality.*

Any normed space is also a metric space, namely if we put $d(x,y) = \|x-y\|$. In fact, we get a metric that is *translation invariant*:

$$d(x+z, y+z) = d(x,y) \text{ for all } x,y,z \in E.$$

Since each normed space is also a metric space, notion such as continuity, open and closed sets and convergent sequences can be defined. We say that x_n *converges* to x in norm $\|\ \|$ if

$$\|x_n - x\| \to 0 \text{ as } n \to \infty.$$

Convergence of sequences therefore depends on the choice of norm.
Example: Let $E = C([0,1])$ and

$$f_n(x) = \begin{cases} nx & \text{if } x \in [0, \frac{1}{n}], \\ 1 & \text{if } x \in (\frac{1}{n}, 1]. \end{cases}$$

The pointwise limit of this sequence of functions is

$$f(x) = \begin{cases} 1 & \text{if } x \in (0,1], \\ 0 & \text{if } x = 0. \end{cases}$$

You can easily calculate that indeed $f_n \to f$ in the norm $\|g\|_2 = \sqrt{\int_0^1 |g(t)|^2 dt}$. However, in the sup-norm $\|g\|_\infty = \sup\{|g(t)| \mid t \in [0,1]\}$, the sequence f_n does not converge.

This example is related to the following statement:

Theorem 2 *If (f_n) is a sequence of continuous functions from a metric space E to $\mathbb{K}$, converging in sup-norm (also called:* converging uniformly*) to f, then also f is continuous.*

Proof. Let us prove continuity in the point $x \in E$. Choose $\varepsilon > 0$ arbitrary. Since $\|f_n - f\|_\infty \to 0$, we can find N so that for all $y \in E$, $|f_N(y) - f(y)| < \frac{\varepsilon}{3}$. Since f_N is continuous, we can also find $\delta > 0$ such that if $d(x,y) < \delta$, then $|f_N(y) - f_N(x)| < \frac{\varepsilon}{3}$. Combining this, we obtain for $d(x,y) < \delta$:

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Remark: This proof also holds when f_n are functions from one metric space to another.
□
One way of comparing norms is the following:

Definition 3 *Two norms $\| \;\|$ and $\| \;\|_0$ on a vector space E are said to be* equivalent*, if there exist $m, M > 0$ such that*

$$m\|x\|_0 \leq \|x\| \leq M\|x\|_0 \text{ for all } x \in E.$$

Equivalent norms induce the same topology, i.e. the same open and closed sets, and if two norms are equivalent, then a sequence converges in the one norm if and only if it converges in the other norm. A special case, where all norms are equivalent, are the finite-dimensional spaces.

Theorem 4 *Let E be a finite-dimensional vector space. Then any two norms on E are equivalent.*

◇ **Proof.** The structure of the proof is as follows: we will construct a special norm ρ and show that any norm $\| \ \|$ is equivalent to it. As a consequence, any two norms are both equivalent to ρ and hence to each other.

Since $\dim(E) < \infty$, say $\dim(E) = n$, there is a basis $\{e_1, \ldots, e_n\}$ of E, and any vector $x \in E$ can be uniquely written as $x = \lambda_1 e_1 + \cdots + \lambda_n e_n$ for $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$. Define

$$\rho(x) := \sqrt{\sum_{i=1}^{n} |\lambda_i|^2}.$$

Check (yourself) that ρ is a norm. Now let $\| \ \|$ be any norm. Then,

$$\begin{aligned} \|x\| &\leq \|\sum_{i=1}^{n} \lambda_i e_i\| \leq \sum_{i=1}^{n} |\lambda_i| \ \|e_i\| \qquad \text{(by the Cauchy-Schwarz inequality)} \\ &\leq \sqrt{\sum_{i=1}^{n} |\lambda_i|^2} \sqrt{\sum_{i=1}^{n} \|e_i\|^2} \leq M\rho(x), \end{aligned}$$

where $M = \sqrt{\sum_{i=1}^{n} \|e_i\|^2}$.

Now for the other inequality, we state (without proof) that

$$f : (\mu_1, \ldots, \mu_n) \mapsto \|\sum_{i=1}^{n} \mu_i e_i\|$$

is a continuous map from $\mathbb{K}^n$ to $\mathbb{R}$. Moreover, the unit sphere

$$S = \{(\mu_1, \ldots, \mu_n) \mid \sqrt{\sum_{i=1}^{n} |\mu_i|^2} = 1\}$$

is a compact subset of $\mathbb{K}^n$. Therefore f assumes its infimum on S: there is a $(\tilde{\mu}_1, \ldots, \tilde{\mu}_n) \in S$ such that

$$m := \inf\{\|\sum_{i=1}^{n} \mu_i e_i\| \mid (\mu_1, \ldots, \mu_n) \in S\} = \|\sum_{i=1}^{n} \tilde{\mu}_i e_i\|.$$

Obviously $m \geq 0$, and if $m = 0$, then $\sum_{i=1}^{n} \tilde{\mu}_i e_i = 0$. Because $\{e_1, \ldots, e_n\}$ is a basis (and therefore linearly independent), this would mean that $\tilde{\mu}_1 = \cdots = \tilde{\mu}_n = 0$, contradicting that $(\tilde{\mu}_1, \ldots, \tilde{\mu}_n) \in S$. Therefore $m > 0$. Now to conclude, we have

$$\begin{aligned}
\|x\| &= \|\sum_{i=1}^{n} \lambda_i e_i\| \\
&= \sqrt{\sum_{j=1}^{n} |\lambda_j|^2} \cdot \|\sum_{i=1}^{n} \frac{\lambda_i}{\sqrt{\sum_{j=1}^{n} |\lambda_j|^2}} e_i\| \qquad (\text{Call } \mu_i = \frac{\lambda_i}{\sqrt{\sum_{j=1}^{n} |\lambda_j|^2}}) \\
&= \sqrt{\sum_{j=1}^{n} |\lambda_j|^2} \cdot \|\sum_{i=1}^{n} \mu_i e_i\| \qquad (\text{because } (\mu_1, \ldots, \mu_n) \in S) \\
&= \sqrt{\sum_{j=1}^{n} |\lambda_j|^2} \cdot m = m\rho(x).
\end{aligned}$$

□

Some notation, that is commonly used, and that we will use in these notes.

- $\ell^\infty = \{x = (x_n)_{n=1}^\infty \mid x_n \in \mathbb{K},\ \sup_n |x_n| < \infty\}$ comes with its natural norm: $\|x\|_\infty = \sup_n |x_n|$. This norm is not compatible with any inner product.
- For $p \geq 1$: $\ell^p = \{x = (x_n)_{n=1}^\infty \mid x_n \in \mathbb{K},\ \sum_n |x_n|^p < \infty\}$ comes with its natural norm: $\|x\|_p = \sqrt[p]{\sum_n |x_n|^p}$. Only for $p = 2$ is this space compatible with an inner product: $\langle x, y\rangle = \sum x_n \overline{y_n}$.
- Analogous to ℓ^∞ we have $L^\infty([a,b]) = \{f : [a,b] \to \mathbb{K} \mid \sup_{t\in[a,b]} |f(t)| < \infty\}$ with sup-norm: $\|f\|_\infty = \sup_{t\in[a,b]} |f(t)|$. This norm is not compatible with any inner product.
- $L^p([a,b]) = \{f : [a,b] \to \mathbb{K} \mid \int_a^b |f(t)|^p dt < \infty\}$ with p-norm: $\|f\|_p = \sqrt[p]{\int_a^b |f(t)|^p dt}$. This space compatible with an inner product only for $p = 2$: $\langle f, g\rangle = \int_a^b f(t)\overline{g}(t)dt$.

In fact, there are some subtleties with L^p-spaces that have to do with measure theory. For example, think of the functions $f : [0,1] \to \mathbb{R}$, $f(x) = 0$ for all x and $g : [0,1] \to R$, $g(x) = 0$ for $x \neq \frac{1}{2}$ and $g(\frac{1}{2}) = 1$. Both f and g belong to L^p, and $\|f\|_p = \|g\|_p = 0$, but f is the 0-function and g is not! This violates condition 1. in the definition of the norm. For this reason, $\|\ \|_p$ is called a *pseudo-norm*. In practice we tend to say that f and g is the same whenever f and g are different only on a set of Lebesgue measure 0, or equivalently: $\int |f(t) - g(t)|\ dt = 0$. Any of the p-norms, $p \in [1, \infty]$, can be defined, without problem, on

- $C([a,b]) = \{f : [a,b] \to \mathbb{K} \mid f \text{ is continuous}\}$

The proof that ℓ^p is indeed a normed space is easy, except for the verification of the triangle inequality. For this, we need some inequalities that are interesting on their own right.

Definition 5 *For each $p > 1$, the* conjugate exponent *$q > 1$ is defined by*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and for $p = 1$, we say that $q = \infty$ is the conjugate exponent.

Obvious consequences are: $p + q = pq$, $(p-1)(q-1) = 1$, $\frac{1}{p-1} = q - 1$. Furthermore, $p = q$ if and only if $p = q = 2$.

Theorem 6 *If $p > 1$ and $q > 1$ are conjugate exponents, then for each $x \in \ell^p$ and $y \in \ell^q$:*

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{\infty} |y_i|^q \right)^{\frac{1}{q}}$$

This formula is called the Hölder inequality*.*

If $p = q = 2$, the Hölder inequality simplifies to the Cauchy-Schwarz inequality.

♢ **Proof.** We start with an auxiliary inequality. From the fact that $u = t^{p-1}$ and $t = u^{q-1}$ are each other inverse function, we get

$$a \cdot b \leq \int_0^a t^{p-1} dt + \int_0^b u^{q-1} du = \frac{a^p}{p} + \frac{b^q}{q}. \qquad (1)$$

for all $a, b \geq 0$ (Make a picture). Let $x \in \ell^p$ and $y \in \ell^q$ be arbitrary. Scale

$$\tilde{x}_i = \frac{x_i}{(\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}}} \quad \text{and} \quad \tilde{y}_i = \frac{y_i}{(\sum_{k=1}^{\infty} |y_k|^q)^{\frac{1}{q}}}.$$

Then $\sum |\tilde{x}_i|^p = 1$ and $\sum |\tilde{y}_i|^q = 1$. By (1), we get

$$\sum_i |\tilde{x}_i \tilde{y}_i| \leq \sum_i \left(\frac{|\tilde{x}_i|^p}{p} + \frac{|\tilde{y}_i|^q}{q} \right) = \frac{1}{p} + \frac{1}{q} = 1.$$

For the unscaled x_i and y_i, this gives:

$$\sum_i |x_i y_i| = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{\infty} |y_i|^q \right)^{\frac{1}{q}} \cdot \sum_i |\tilde{x}_i \tilde{y}_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{\infty} |y_i|^q \right)^{\frac{1}{q}}.$$

□

Theorem 7 *For each $p \geq 1$ and $x, y \in \ell^p$:*

$$\left(\sum_{i=1}^{\infty} |x_i + y_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} |y_i|^p \right)^{\frac{1}{p}}.$$

This formula is called the Minkovski inequality*.*

The Minkovski inequality is precisely the triangle inequality for the space ℓ^p. Analogous Hölder and Minkovski equalities hold for L^p.

◇ **Proof.** The inequality is clear for $p = 1$, so assume $p > 1$. Write $z_i = x_i + y_i$. Then $|z_i|^p = |x_i + y_i|\ |z_i|^{p-1} \leq (|x_i| + |y_i|)|z_i|^{p-1}$. Taking the sum over all i we get

$$\sum_i |z_i|^p \leq \sum_i |x_i|\ |z_i|^{p-1} + \sum_i |y_i|\ |z_i|^{p-1}.$$

Apply the Hölder inequality to the first term at the right hand side.

$$\sum_i |x_i|\ |z_i|^{p-1} \leq \left(\sum_i |x_i|^p\right)^{\frac{1}{p}} \left(\sum_i |z_i|^{(p-1)q}\right)^{\frac{1}{q}} = \left(\sum_i |x_i|^p\right)^{\frac{1}{p}} \left(\sum_i |z_i|^p\right)^{\frac{1}{q}}.$$

Do the same to the second term and combine:

$$\sum_i |z_i|^p \leq \left\{\left(\sum_i |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_i |y_i|^p\right)^{\frac{1}{p}}\right\} \cdot \left(\sum_i |z_i|^p\right)^{\frac{1}{q}}.$$

Now divide out the rightmost factor:

$$\left(\sum_i |z_i|^p\right)^{1-\frac{1}{q}} \leq \left(\sum_i |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_i |y_i|^p\right)^{\frac{1}{p}},$$

and remember that $1 - \frac{1}{q} = \frac{1}{p}$. □

2 Banach and Hilbert Spaces

The big advantage of $\mathbb{R}$ over $\mathbb{Q}$ is that it is completeness: sequences that seem to converge actually have limits. More precisely:

Definition 8 *A sequence (x_n) in a normed space $(E, \|\ \|)$ is* Cauchy *if*

$$\forall \varepsilon > 0\ \exists N\ \forall m, n \geq N\ \|x_m - x_n\| < \varepsilon.$$

In other words, $\|x_n - x_m\| \to 0$ as $m, n \to \infty$. The space E is complete *if every Cauchy sequence converges to a limit.*

Apart from $\mathbb{R}$, also $\mathbb{R}^n$ and $\mathbb{C}^n$ are complete for all finite n. For infinite dimensional normed spaces, completeness is more subtle.

Theorem 9 *The space $(\ell^2, \|\ \|_2)$ is complete.*

Proof. Let (x^n) be a Cauchy sequence in ℓ^2. We write the index n as a superscript, because these x^n are sequences in themselves, and we want to denote the coordinates of x^n by x_k^n, $k = 1, 2, 3, \ldots$. The proof consists of three steps:
1) find a candidate limit a.
2) show that $a \in \ell^2$, and
3) show that indeed $x^n \to a$ in $\| \, \|_2$.
To prove 1), observe that since x^n is Cauchy, also each of the coordinate sequences x_k^n (for fixed k) is a Cauchy sequence in $\mathbb{K}$. But $\mathbb{K}$ is complete, so x_k^n converges to some a_k as $n \to \infty$. Let $a = (a_k)_{k=1}^{\infty}$ be the candidate limit.
2) Given $\varepsilon > 0$, there exists N such that for all $m, n \geq N$, and all $K \geq 1$,

$$\sum_{k=1}^{K} |x_k^n - x_k^m|^2 \leq \sum_{k=1}^{\infty} |x_k^n - x_k^m|^2 < \varepsilon^2.$$

First let $m \to \infty$ to obtain $\sum_{k=1}^{K} |x_k^n - a_k|^2 \leq \varepsilon^2$, and then let $K \to \infty$ to obtain

$$\sum_{k=1}^{\infty} |x_k^n - a_k|^2 \leq \varepsilon^2 \tag{2}$$

This means that $x^n - a \in \ell^2$. But then also $a = x^k - (x^k - a) \in \ell^2$.
3) From (2) we obtain that for all $n \geq N$:

$$\|x^n - a\|_2 = \sqrt{\sum_{k=1}^{\infty} |x_k^n - a_k|^2} \leq \varepsilon.$$

So indeed $\lim_n x^n = a$ in $\| \, \|_2$. □

Definition 10 *A* Hilbert space *is a complete inner product space. A* Banach space *is a complete normed space.*

Examples: • $(\ell^2, \| \, \|_2)$ and $(L^2, \| \, \|_2)$ are Hilbert spaces.
• For $p \neq 2$, $(\ell^p, \| \, \|_p)$ and $(L^p, \| \, \|_p)$ are Banach spaces but not Hilbert spaces.
• $(C([a, b]), \| \, \|_2)$ is not a Hilbert space, since limits of continuous functions could be discontinuous, see the example earlier in the notes. However, $L^2([a, b])$ is the smallest Hilbert space containing $C([a, b])$. It is called the *completion* of $C([a, b])$.
• $(C([a, b]), \| \, \|_\infty)$ is a Banach space.

3 Orthonormal Bases in Hilbert Space and Fourier series

Fourier analysis was named after Joseph Fourier (1768-1830) who published a work on heat transport in which he described the technique of Fourier series [1]. In fact, Euler had

[1] Before writing this work, Fourier had already made a career as scientific adviser of Napoleon, and followed him on his campaign to Egypt.

the idea, and more elegant proofs, before Fourier, but the main subject of debate was that Fourier claimed that "any" function can be expressed as sum of sin and cos-functions. Fourier's contemporaries found this hard to swallow, not so surprisingly if you see, for example, an expression like:

$$t = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin nt \quad \text{for all } t \in (-\pi, \pi).$$

Over the years, Fourier analysis was put in the framework of linear algebra of infinite dimensional function spaces, but rigorous proofs of the questions unearthed by Fourier keep mathematicians busy until today. Let us just give an example of the usefulness of Fourier series.

A string of length L is attached on either end, pulled (or plucked) and then released. How does it move and what sound does it produce? Let $f(x,t)$ denote the displacement of the string from the rest-position for position $x \in [0, L]$ and time $t \geq 0$. The physics tell us that f should satisfy:

$$\begin{cases} c^2 \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial t} & \text{c is the speed of sound in the string,} \\ f(0,t) = f(L,t) = 0 & \text{this boundary condition expresses, that the string is attached on either end.} \\ f(x,0) = g_0(x) & \text{the initial condition. } g_0 \text{ is the shape of the plucked string at } t = 0. \end{cases}$$

Among the solutions of this partial differential equations are

$$f(x,t) = a \sin \frac{\pi}{L} nx \ \cos \frac{\pi c}{L} nt \quad \text{for any } a \in \mathbb{R} \text{ and } n \geq 1.$$

This solution vibrates with frequency $n\frac{\pi c}{L}$. The lowest pitch (the *fundamental*) that the string can produce is when $n = 1$. The *overtones* or *harmonics* have frequencies $2, 3, \ldots$ times as high, so they are $1, 2, \ldots$ octaves above the fundamental. These solutions tell you lot about what sounds the string can produce, but they don't, in general, satisfy the initial condition $f(x,t) = g_0(x)$. To make this happen, we need to take linear combinations

$$g_0(x) = \sum_{n \geq 1} a_n \sin \frac{\pi}{L} nx$$

and the trick is to find the numbers a_n. Fourier analysis is concerned with finding these a_n. Yet having found the a_n, we can tell how the string sounds, as they give the amount of the fundamental and each overtone present in the movement of the string.

Now let us start with the mathematical side of the subject.
Example in $\mathbb{R}^3$. Let $x = (1\ 2\ 3)^t$ and V be the plane spanned by $f_1 = (1\ 1\ 0)^t$ and $f_2 = (-1\ 1\ 1)^t$. What is the point $y \in V$ closest to x?
Answer: $y = Px$, the orthogonal projection of x onto V of course, but how to compute it

easily?
Write $y = \lambda_1 f_1 + \lambda_2 f_2$, use the inner product and fact that $x - y \perp f_1$ and $x - y \perp f_2$:

$$0 = \langle x - y, f_1 \rangle = \langle x, f_1 \rangle - \langle \lambda_1 f_1, f_1 \rangle - \langle \lambda_2 f_2, f_1 \rangle = \langle x, f_1 \rangle - \lambda_1 \langle f_1, f_1 \rangle,$$

where the last inequality follows because f_1 and f_2 happen to be perpendicular. Therefore:

$$\lambda_1 = \frac{\langle x, f_1 \rangle}{\langle f_1, f_1 \rangle} = \frac{3}{2} \text{ and similarly } \lambda_2 = \frac{\langle x, f_2 \rangle}{\langle f_2, f_2 \rangle} = \frac{4}{3}.$$

The calculation would have been even more simple if $\langle f_1, f_1 \rangle = \langle f_2, f_2 \rangle = 1$.

Definition 11 *A system of vectors* $\{e_i\}$ *is called* orthogonal *if* $\langle e_i, e_j \rangle = 0$ *for all* $i \neq j$. *If in addition,* $\langle e_i, e_i \rangle = 1$ *for all* i*, then the system is called* orthonormal.

Note: For orthogonal systems, Pythagoras theorem holds: $\| \sum_{i=1}^n e_i \|^2 = \sum_{i=1}^n \|e_i\|^2$.

Example (continued). We can make $\{f_1, f_2\}$ orthonormal by scaling:

$$e_1 := \frac{f_1}{\|f_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } e_2 := \frac{f_2}{\|f_2\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

Next we can extend $\{e_1, e_2\}$ to a orthonormal basis by either the Gram-Schmidt orthogonalisation process, or, in $\mathbb{R}^3$, by the exterior product:

$$e_3 = e_1 \times e_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Using the inner product, it is then easy to express x as linear combination of $\{e_1, e_2, e_3\}$:

$$x = \sum_{i=1}^{3} \langle x, e_i \rangle e_i = \frac{3}{\sqrt{2}} e_1 + \frac{4}{\sqrt{3}} e_2 + \frac{5}{\sqrt{6}} e_3.$$

We would like to apply this technique to arbitrary (infinite dimensional) Hilbert spaces.

Definition 12 *If* $\{e_i\}_{i=1}^n$ *or* $\{e_i\}_{i=1}^\infty$ *is a orthonormal system in a Hilbert space* H*, then the numbers* $\langle x, e_i \rangle$ *are called the* Fourier coefficients *of* x.

Theorem 13 *If* $\{e_i\}_{i=1}^n$ *is a orthonormal system in* H*, and* $x \in H$*, then the point* y *in the span of* $\{e_i\}_{i=1}^n$ *which is closest to* n *is*

$$y = \sum_{i=1}^{n} \langle x, e_i \rangle e_i,$$

and the distance $d = \|x - y\|$ *satisfies* $d^2 = \|x\|^2 - \sum_{i=1}^n |\langle x, e_i \rangle|^2$.

Proof. Write $c_i = \langle x, e_i \rangle$. We expand norms:

$$\begin{aligned} 0 &\leq \|x - \sum_{i=1}^n \lambda_i e_i\|^2 = \langle x - \sum_{i=1}^n \lambda_i e_i, x - \sum_{i=1}^n \lambda_i e_i \rangle \\ &= \langle x, x \rangle - \sum_{i=1}^n \lambda_i \langle e_i, x \rangle - \sum_{i=1}^n \overline{\lambda_i} \langle x, e_i \rangle + \sum_{i=1}^n \lambda_i \overline{\lambda_i} \\ &= \|x\|^2 + \sum_{i=1}^n |\lambda_i - c_i|^2 - \sum_{i=1}^n |c_i|^2. \end{aligned}$$

This expression is minimal if $\lambda_i = c_i$, so the closest y to x is indeed $y = \sum_{i=1}^n \langle x, e_i \rangle e_i$ and the distance satisfies $d^2 = \|x - y\|^2 = \|x\|^2 - \sum_{i=1}^n |c_i|^2$. $\square$

Example: The classical Fourier series are based on sin and cos functions: Let $H = L^2([-\pi, \pi])$ and the system $\{e_n\}_{n \in \mathbb{Z}}$ be defined by

$$e_n(t) = \begin{cases} \frac{1}{\sqrt{\pi}} \sin nt & \text{if } n \geq 1, \\ \frac{1}{\sqrt{2\pi}} & \text{if } n = 0, \\ \frac{1}{\sqrt{\pi}} \cos nt & \text{if } n \leq -1. \end{cases} \qquad \text{(Note that } \cos(-nt) = \cos(nt).\text{)}$$

Check your integration skills on showing that $\{e_n\}_{n \in \mathbb{Z}}$ is orthonormal. Let $f(t) = t$. Then the Fourier coefficients of f are

$$\langle f, e_n \rangle = \int_{-\pi}^{\pi} t e_n(t) dt = \begin{cases} (-1)^{n+1} \frac{2\sqrt{\pi}}{n} & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \\ 0 & \text{if } n \leq -1. \end{cases}$$

For $n \leq 0$, this answer is easy to guess, because you integrate an odd function over an interval symmetric with respect to 0. The case $n \geq 1$ is based on an integration by parts:

$$\begin{aligned} \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} t \sin nt dt &= \frac{1}{\sqrt{\pi}} \left\{ [t \cdot -\frac{1}{n} \cos nt]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} -\frac{1}{n} \cos nt dt \right\} \\ &= -\frac{2\sqrt{\pi}}{n} \cos n\pi + 0 \\ &= \frac{2\sqrt{\pi}}{n} (-1)^{n+1}. \end{aligned}$$

Therefore the best approximation of $f(t) = t$ by a combination of sin and cos functions is $\sum_{n \geq 1} (-1)^{n+1} \frac{2}{n} \sin nt$. Note that $\sum_{n \geq 1} (-1)^{n+1} \frac{2}{n} \sin nt$ is a 2π-periodic function, so equality to $f(t) = t$ can only hold for at most $t \in [\pi, \pi]$. In fact, $t = \sum_{n \geq 1} (-1)^{n+1} \frac{2}{n} \sin nt$ only for $t \in (-\pi, \pi)$, as we shall see later.

As a corollary to Theorem 13, we find for any x belonging to the span of $\{e_i\}_{i=1}^n$ that $x = y = \sum_{i=1}^n \langle x, e_i \rangle e_i$. We can extend these result to infinite orthonormal systems:

Theorem 14 *For any (infinite) orthonormal system $\{e_i\}_{i=1}^\infty$ the* Bessel inequality *holds:*

$$\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \leq \|x\|^2.$$

Proof. Start with a finite subsystem $\{e_i\}_{i=1}^n$ and rewrite to computation of the previous proof to $\|x\|^2 - \sum_{i=1}^n |\langle x, e_i\rangle|^2 \geq \|y\|^2 \geq 0$. Then let $n \to \infty$. $\square$

Example: In the space ℓ^2 with standard inner product, the system $\{f_i\}_{i=1}^\infty$ with

$$f_i = (0, 0, \dots, 0, 1, 0, \dots) \text{ with 1 on place } i+1,$$

is orthonormal. If $x \in \ell^2$, then the y in the span of $\{f_i\}_{i=1}^\infty$ closest to x is

$$y = \sum_{i=1}^\infty \langle x, f_i\rangle f_i = (0, x_2, x_3, x_4, x_5, \dots),$$

so we obviously miss the first coordinate. Note also, that the error vector $x - y$ is perpendicular to each f_i. We say that $x - y$ belongs to the orthogonal complement of $\{f_i\}$.

Definition 15 *Let $\{e_i\}$ be a collection of vectors in a Hilbert space H. The subspace X of H consisting of those vectors orthogonal to each e_i is called the* orthogonal complement *of $\{e_i\}$. The notation is $X = \{e_i\}^\perp$ or $X = H \ominus \{e_i\}$. (Note that X is closed!) The system $\{e_i\}$ is called* complete *if the only vector x orthogonal to all e_i is the zero vector: $x = 0$. A complete orthonormal system is called an* orthonormal basis *of H.*

Examples: • ℓ^2 has standard orthonormal basis $\{e_i\}_{i=1}^\infty$, where $e_1 = (1, 0, 0, \dots)$, $e_2 = (0, 1, 0, 0, \dots)$ etc.
• $\mathbb{P}([-1, 1]) = \{\text{all polynomials } p : [0, 1] \to \mathbb{K}\}$ has standard basis $\{e_i\}_{i=0}^\infty$, where $e_i(t) = t^i$. This basis is not orthonormal with respect to $\langle f, g\rangle = \int_{-1}^1 f(t)g(t)dt$, but it can be made orthogonal by means of the Gram-Schmidt orthogonalisation process. Then we get $q_0(t) = 1$, $q_1(t) = t$, $q_2(t) = \frac{1}{2}(3t^2 - 1)$, $q_3(t) = \frac{1}{2}(5t^3 - 3t) \dots$. The general formula is:

$$q_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n}[(t^2 - 1)^n].$$

These polynomials are called the *Legendre polynomials*. To make the system orthonormal, we need to scale: $\tilde{q}_n(t) = \sqrt{\frac{2n+1}{2}} q_n(t)$.
• For $C([-1, 1])$ the same basis $\{\tilde{q}_n(t)\}$ works. Note that neither $\mathbb{P}([-1, 1])$ nor $C([0, 1])$ are Hilbert spaces: they are not closed.

Theorem 16 *Let $\{e_n\}$ be an orthonormal systems in a Hilbert space H. The following statements are equivalent.*

1. *$\{e_i\}$ is complete.*
2. *$clin\{e_i\} = H$, where clin stands for the closure of the linear span,*
3. *$\|x\|^2 = \sum_i |\langle x, e_i\rangle|^2$, that is: the Bessel inequality is an equality.*

Proof. (1) $\Rightarrow$ (3): $x - \sum_i \langle x, e_i\rangle e_i \perp e_k$ for all k, so by assumption, $x - \sum_i \langle x, e_i\rangle e_i = 0$. By Pythagoras theorem:

$$\|x\|^2 = \|\sum_{i=1}^{\infty}\langle x, e_i\rangle e_i\|^2 = \sum_{i=1}^{\infty}|\langle x, e_i\rangle|^2\|e_i\|^2 = \sum_{i=1}^{\infty}|\langle x, e_i\rangle|^2.$$

(3) $\Rightarrow$ (2): Take $x \in (\text{clin}\{e_i\})^{\perp}$, so $\langle x, e_i\rangle = 0$ for each i. But $\|x\|^2 = \sum_i |\langle x, e_i\rangle|^2 = 0$, so $x = 0$. Therefore $(\text{clin}\{e_i\})^{\perp} = \{0\}$ and $\text{clin}\{e_i\} = H$.
(2) $\Rightarrow$ (1): Take $x \in H$ such that $x \perp e_i$ for all i. Let $E = \{x\}^{\perp}$. Then E contains every e_i, and hence every vector in the span of $\{e_i\}$. Also E is the kernel of the map $g : H \to \mathbb{K}$ defined by $g(y) = \langle x, y\rangle$. This map is continuous, so $E = g^{-1}(\{0\})$ is closed. In particular, E contains $\text{clin}\{e_i\} = H$. Thus $x = 0$ and $\{e_i\}$ is complete. $\square$
Example: As we will see later on, the orthonormal system of sin and cos functions in the earlier example is indeed complete. Therefore item 3 gives

$$\sum_{n\geq 1}\frac{4\pi}{n^2} = \sum_{n\in\mathbb{Z}}|\langle f, e_n\rangle|^2 = \|f\|^2 = \int_{-\pi}^{\pi}|t|^2 dt = \frac{2}{3}\pi^3.$$

Rearranging gives: $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

Definition 17 *A linear mapping $U : H \to K$, where H and K are Hilbert spaces, is a* unitary operator *if it preserves the inner product:*

$$\langle Ux, Uy\rangle_K = \langle x, y\rangle_H \text{ for all } x, y \in H.$$

If there exists such a unitary operator, then H and K are called isomorphic*.*

Remarks: From this definition, it follows that U is invertible. Using the polarisation formula, it is also easy to deduct that U is unitary if and only if $\|Ux\|_K = \|x\|_H$ for all $x \in H$.

Definition 18 *A Hilbert space is called* separable*, if there exists a countable orthonormal basis.*

Theorem 19 *Any separable Hilbert space is isomorphic to $\mathbb{K}^n$ for some $n \geq 1$ or to ℓ^2.*

Proof. We do the proof only for the infinite dimensional case. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis, so for each $x \in H$,

$$x = \sum_{i=1}^{\infty}\langle x, e_i\rangle e_i = \sum_{i=1}^{\infty}\xi_i e_i \text{ for } \xi_i = \langle x, e_i\rangle.$$

Define $Ux = \xi = (\xi_1, \xi_2, \dots)$. Obviously, U is linear. Since $\{e_i\}$ is complete, the Bessel inequality turns into an equality (see item 3 of Theorem 16). Therefore

$$\|\xi\|_2^2 = \sum_{i=1}^{\infty}|\langle x, e_i\rangle|^2 = \|x\|^2 < \infty.$$

This shows that U preserves the norm, and at the same time that $\xi = Ux \in \ell^2$. Since the inner product can be expressed in term of the norm (using the polarisation identity), U preserves the inner product as well. Check yourself that U is one-to-one and onto. □

Definition 20 *Let M be a closed subspace of Hilbert space H. The* orthogonal complement *of M is $M^\perp = \{x \in H \mid x \perp m \text{ for all } m \in M\}$.*

It is easy to see that $M^\perp$ is also a closed subspace. The space $M^\perp$ consists of vectors x who are closer to 0 than to any other $y \in M$.

Lemma 21 *$x \in M^\perp$ if and only if $\|x - y\| \geq \|x\|$ for all $y \in M$.*

Proof. ($\Rightarrow$) By Pythagoras theorem, $\|x\|^2 \leq \|x\|^2 + \|y\|^2 = \|x - y\|^2$ for all $y \in M$.
($\Leftarrow$) Take $y \in M$ arbitrary, thus $\lambda y \in M$ for all $\lambda \in \mathbb{K}$. Now

$$\|x\|^2 \leq \|x - \lambda y\|^2 = \langle x - \lambda y, x - \lambda y\rangle = \|x\|^2 - 2\mathrm{Re}\lambda\langle x, y\rangle + |\lambda|^2\|y\|^2,$$

and hence $2\mathrm{Re}\lambda\langle x, y\rangle \leq |\lambda|^2\|y\|^2$. Choose $\lambda = t\frac{\langle y,x\rangle}{|\langle x,y\rangle|}$ for some $t > 0$. Divide by $2t$, then we get

$$|\langle x, y\rangle| \leq \frac{t}{2}\|y\|^2 \to 0 \text{ as } t \to 0.$$

Therefore $\langle x, y\rangle = 0$. Because $y \in M$ was arbitrary, $x \in M^\perp$. □

Theorem 22 *Given a closed subspace $M \subset H$ and $x \in M$, there exist $y \in M$ and $z \in M^\perp$ such that $x = y + z$. Moreover, y and z are unique.*

Because of this unique decomposition of vectors $x \in H$, we say that H is the *orthogonal direct sum* of M and $M^\perp$: $H = M \oplus M^\perp$.

Proof. Let $y \in M$ be closest to x, so $\|x - y\| \leq \|x - m\|$ for all $m \in M$. The tricky part of this proof is to show that such closest y exists, and we reserve it for the end. Write $z = x - y$. Then

$$\|z\| = \|x - y\| \leq \|x - (y + m)\| = \|z - m\|$$

for all $m \in M$ (and hence $y + m \in M$). By the previous lemma, $z \in M^\perp$.

Now for the existence (and uniqueness) of y, let

$$\delta = \inf\{\|x - m\| \mid m \in M\} \geq 0.$$

Take $\{y_i\}$ a sequence in M such that

$$\|x - y_i\|^2 < \delta^2 + \frac{1}{i}. \tag{3}$$

We will show that $\{y_i\}$ is Cauchy. Apply the parallelogram law to get

$$\begin{aligned} \|(x-y_i)-(x-y_j)\|^2 + \|(x-y_i)+(x-y_j)\|^2 &= 2\|x-y_i\|^2 + 2\|x-y_j\|^2 \\ &< 4\delta^2 + \frac{2}{i} + \frac{2}{j}. \end{aligned}$$

Therefore

$$\|y_i - y_j\|^2 = \|(x-y_i)-(x-y_j)\|^2 < 4\delta^2 + \frac{2}{i} + \frac{2}{j} - 4\underbrace{\|x - \frac{y_i+y_j}{2}\|^2}_{\geq \delta^2} \leq \frac{2}{i} + \frac{2}{j} \tag{4}$$

which tends to 0 as $i, j \to \infty$. Hence $\{y_i\}$ is indeed Cauchy, and converging to some y in the Hilbert space H. Because M is closed, actually $y \in M$. Therefore $\|x - y\| \geq \delta$, but letting $i \to \infty$ in (3), we also get $\|x-y\|^2 \leq \delta^2$. Therefore $\|x-y\| = \delta = \inf\{\|x-m\| \mid m \in M\} \geq 0$. Now for uniqueness, suppose that $y = \lim_i y_i$ and $\tilde{y} = \lim_j \tilde{y}_j$, where $\{\tilde{y}_j\}$ is another sequence satisfying (3), were two points closest to x, then the calculation of (4) shows that $\|y_i - \tilde{y}_j\|^2 \leq \frac{2}{i} + \frac{2}{j}$. Now take the limit $i, j \to \infty$ to see that $y = \tilde{y}$. □

4 Classical Fourier Series

In the previous chapter of these notes, we used sin and cos functions as an orthonormal systems in the Hilbert space $L^2([-\pi, \pi])$. This led to the Fourier series

$$F(x) = \frac{a_0}{2} + \sum_{n \geq 1} (a_n \cos nx + b_n \sin nx)$$

of the function $f \in L^2([-\pi, \pi])$. The coefficients are computed as (check yourself, because we are not using an ortho**normal** system here)

$$\begin{cases} a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)dt, \quad \text{and for } n \geq 1 \\ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt dt, \\ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt dt. \end{cases}$$

This formula works in the real and complex space $L^2([-\pi, \pi])$. Due to the relations

$$\cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}, \quad \sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i},$$

we might as well, and it is much easier to, work with the orthonormal system[2]

$$\{e_n\}_{n \in \mathbb{Z}} \text{ defined as } e_n(z) = \frac{1}{\sqrt{2\pi}} e^{inz}$$

[2]Since $i = \sqrt{-1}$ is needed, we will no longer use i as an index in this chapter.

Check that this is indeed an orthonormal system. The formula for the Fourier series simplifies to

$$F(z) = \sum_{k\in\mathbb{Z}} c_k e^{ikz} \quad \text{with } c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}dt.$$

In this chapter we want to show that $\{e_n\}_{n\in\mathbb{Z}}$ is a complete system in $L^2([-\pi,\pi])$, and then the completeness of the system $\{1, \cos x, \sin x, \cos 2x, \sin 2x, \dots\}$ follows too.
In the following theorem we will use a condition for a real functions f:

$$\left.\begin{array}{ll} f(x^+) = \lim_{y\searrow x} f(y), & f'(x^+) = \lim_{y\searrow x} \frac{f(y)-f(y^+)}{y-x}, \\ f(x^-) = \lim_{y\nearrow x} f(y), & f'(x^-) = \lim_{y\nearrow x} \frac{f(y)-f(y^-)}{y-x}, \end{array}\right\} \text{all exist} \qquad (5)$$

This is true for differentiable functions of course, but in (5) we are allowing discontinuous functions, as long as the left and right limits of f and left and right derivatives at the discontinuities exist.

Theorem 23 (Dirichlet) *Let f be a 2π-periodic function such that $\int_{-\pi}^{\pi}|f(t)|dt < \infty$ and (5) holds for x. Then the Fourier series*

$$F(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \quad \textit{converges to} \quad \frac{f(x^+)+f(x^-)}{2},$$

that is, the average value of $f(x^+)$ and $f(x^-)$.

In particular, if f is a 2π-periodic C^1-function[3], then $F(x) = f(x)$.

◇ **Proof.** Write $F_n(z) = \sum_{k=-n}^{n} c_k e^{-kiz}$. Since all functions involved are 2π-periodic, we can translate them to shift z to 0. Therefore it suffices to prove the result for $z = 0$. Geometric sums $\sum a^n$ can be simplified by multiplying and dividing by $1-a$. This is what we do for the following sum:

$$\begin{aligned} \frac{1}{2\pi}\sum_{k=-n}^{n} e^{-ikz} &= \frac{1}{2\pi}\frac{1-e^{iz}}{1-e^{iz}}\left(e^{-inz} + e^{-i(n-1)z} + \cdots + e^{inz}\right) \\ &= \frac{1}{2\pi}\frac{1}{1-e^{iz}}\left([e^{-inz} - e^{-i(n-1)z}] + [e^{-i(n-1)z} - e^{-i(n-2)z}] + \cdots + \right. \\ &\qquad \left. + \cdots + [e^{inz} - e^{i(n+1)z}]\right) \\ &= \frac{1}{2\pi}\frac{1}{1-e^{iz}}(e^{-inz} - e^{i(n+1)z}) \\ &= \frac{1}{2\pi}\frac{-e^{iz/2}}{1-e^{iz}}(e^{i(n+\frac{1}{2})z} - e^{-i(n+\frac{1}{2})z}) \\ &= \frac{1}{2\pi}\frac{2i}{e^{iz/2}-e^{-iz/2}}\frac{e^{i(n+\frac{1}{2})z} - e^{-i(n+\frac{1}{2})z}}{2i} \\ &= \frac{1}{2\pi}\frac{\sin(n+\frac{1}{2})z}{\sin\frac{1}{2}z} =: D_n(z). \end{aligned}$$

[3] C^n stands for the functions that are n times continuously differentiable

The quantity D_n is called the n-th *Dirichlet kernel.*[4] This kernel is an even function (because it is the quotient of two odd functions $\sin(n+\frac{1}{2})z$ and $\sin\frac{1}{2}z$). When integrating over $(-\pi,\pi)$, only the term with $k=0$ in the sum (left-hand side of the above displayed formula) gives a contribution. In other words: $\int_{-\pi}^{\pi}\frac{1}{2\pi}\sum_{k=-n}^{n}e^{-ikz}dz = \frac{1}{2\pi}\int_{-\pi}^{\pi}1dz=1$. Therefore

$$\int_{-\pi}^{\pi}D_n(t)dt=1 \text{ and } \int_{0}^{\pi}D_n(t)dt=\int_{-\pi}^{0}D_n(t)dt=\frac{1}{2}.$$

Moreover, D_n is 2π-periodic. We have

$$\begin{aligned}F_n(0)=\sum_{k=-n}^{n}c_k &= \sum_{k=-n}^{n}\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)e^{-ikt}dt\\ &= \int_{-\pi}^{\pi}\frac{1}{2\pi}\sum_{k=-n}^{n}f(t)e^{-ikt}dt=\int_{-\pi}^{\pi}D_n(t)f(t)dt.\end{aligned}$$

We split the integral into integrations over $(-\pi,0)$ and $(0,\pi)$. The integral over $(0,\pi)$ is

$$\int_{0}^{\pi}D_n(t)f(t)dt=\frac{f(0^+)}{2}+\int_{0}^{\pi}D_n(t)(f(t)-f(0^+))dt. \tag{6}$$

and split the integrand

$$\begin{aligned}D_n(t)[f(t)-f^+(0)] &= \frac{1}{2\pi}\frac{f(t)-f^+(0)}{t}\frac{t}{\sin t/2}\sin(n+\frac{1}{2})t\\ &= \frac{1}{2\pi}\frac{f(t)-f^+(0)}{t}\frac{t}{\sin t/2}(\cos\frac{t}{2}\sin nt+\sin\frac{t}{2}\cos nt).\end{aligned}$$

By assumption, $\lim_{t\searrow 0}\frac{f(t)-f^+(0)}{t}$ exists, and so does $\lim_{t\searrow 0}\frac{t}{\sin t/2}$. Therefore the integral

$$\int_{0}^{\pi}D_n(t)[f(t)-f(0^+)]dt=\int_{0}^{\pi}\frac{2p(t)}{\sqrt{\pi}}\sin ntdt+\int_{0}^{\pi}\frac{2q(t)}{\sqrt{\pi}}\cos ntdt,$$

where p and q are functions in L^2. We can extend p and q to an even respectively odd L^2 function on $(-\pi,\pi)$ and hence $p(t)\sin nt$ and $q(t)\cos nt$ both become even. Then the integral can we written as

$$\int_{-\pi}^{\pi}p(t)\frac{\sin nt}{\sqrt{\pi}}dt+\int_{-\pi}^{\pi}q(t)\frac{\cos nt}{\sqrt{\pi}}dt.$$

The trick is now to recognise these integrals as Fourier coefficients $p_n=\langle p,\frac{1}{\sqrt{\pi}}\sin nx\rangle$ and $q_n=\langle q,\frac{1}{\sqrt{\pi}}\cos nx\rangle$. By the Bessel inequality,

$$\sum_{n\geq 1}|p_n|^2\leq\|p\|_2^2<\infty \text{ and } \sum_{n\geq 1}|q_n|^2\leq\|q\|_2^2<\infty.$$

[4]This use of the word “kernel”. is entirely different from a kernel of a (linear) transformation.

Therefore $\lim_{n\to\infty} p_n = 0$ and $\lim_{n\to\infty} q_n = 0$. Hence, by (6)

$$\int_0^\pi D_n(t)f(t) \to \frac{1}{2}f(0^+) \quad \text{as} \quad n \to \infty.$$

The same argument for the integral over $(-\pi, 0)$ gives $\int_{-\pi}^0 D_n(t)f(t) \to \frac{1}{2}f(0^-)$. Taking the sums of both integrals again, we get

$$F_n(0) = \int_0^\pi D_n(t)f(t)dt + \int_{-\pi}^0 D_n(t)f(t)dt \to \frac{f(0^+) + f(0^-)}{2},$$

as asserted. □

Example: If $f(x) = x$ on $\mathbb{R}$, then we can make it into a 2π-periodic function by cutting at $-\pi$ and π. So let $g(x) = f(x)$ for $x \in [-\pi, \pi)$ and continue periodically: $g(x + 2k\pi) = g(x)$ for $k \in \mathbb{Z}$. The previous theorem says that the Fourier series G converges to g for all x except the discontinuity points. At $x = \pi + 2k\pi$, the Fourier series $G(x) = \frac{1}{2}(\lim_{y\nearrow\pi} g(y) + \lim_{y\searrow\pi} g(y)) = 0$.

If f is sufficiently smooth, the convergence of the Fourier series is uniform (i.e. in $\| \ \|_\infty$).

Theorem 24 *Let f be a continuously differentiable (i.e. $f \in C^1$) 2π-periodic function. Then its Fourier series F converges uniformly to f.*

To explain notation: $C^k([a, b])$ is the space of all functions $f : [a, b] \to \mathbb{K}$ that are k times *continuously differentiable*: they are k times differentiable and the k-th derivative is still continuous. In this terminology, $C^0([a, b]) = C([a, b])$.

Proof. Because f' is continuous on the compact interval $[-\pi, \pi]$, it is bounded, and therefore $\|f'\|_2 = \sqrt{\int_{-\pi}^\pi |f'(t)|^2 dt} < \infty$. Let $c_k = \frac{1}{2\pi}\int_{-\pi}^\pi f(t)e^{-ikt}dt$ and $d_k = \frac{1}{2\pi}\int_{-\pi}^\pi f'(t)e^{-ikt}dt$ be the Fourier coefficients of f resp. f'. Integration by parts gives (for $k \neq 0$)

$$d_k = \frac{1}{2\pi}\int_{-\pi}^\pi f'(t)e^{-ikt}dt = \frac{1}{2\pi}\left([f(t)e^{-ikt}]_{-\pi}^\pi + \int_{-\pi}^\pi ikf(t)e^{-ikt}dt\right) = ikc_k,$$

By the Cauchy-Schwarz inequality and Bessel's inequality

$$\sum_k |c_k| = |c_0| + \sum_{k\neq 0}\frac{|d_k|}{k} \leq |c_0| + \sqrt{\sum_{k\neq 0}\frac{1}{k^2}}\sqrt{\sum_{k\neq 0}|d_k|^2} \leq |c_0| + \frac{\pi}{\sqrt{3}}\|f'\| < \infty.$$

Let $\varepsilon > 0$ be given. Because $\sum_k |c_k| < \infty$, there exists k_0 such that $\sum_{|k|>k_0} |c_k| < \varepsilon$. Then also

$$|F(x) - F_{k_0}(x)| = |\sum_{k\in\mathbb{Z}} c_k e^{ikx} - \sum_{|k|\leq k_0} c_k e^{ikx}| \leq \sum_{|k|>k_0} |c_k||e^{ikx}| < \varepsilon,$$

for all values of x. Hence the convergence is uniform. $\square$

We have seen now conditions under which Fourier sequences converge. If f is C^1, then $F(x) = f(x)$, so this is an example where the Fourier series converges to a continuous function. If f is not continuous, neither will be the Fourier series. But there are also examples, where f is continuous (but not C^1), where the Fourier series is discontinuous. For many years, one of the main open questions in the field has been to show that Fourier series cannot be too wildly discontinuous: the set of discontinuities or a Fourier series has Lebesgue measure 0. This is *Lusin's conjecture*, and it has been solved in 1966 by the Swedish mathematician Lennart Carleson.

Theorem 25 *The system* $\{\frac{1}{\sqrt{2\pi}}e^{inx}\}_{n\in\mathbb{Z}}$ *is complete in* $L^2([-\pi,\pi])$.

Proof. Recall from the definition of Theorem 16, that to prove completeness, it suffices to show that $\operatorname{clin}\{e^{-inx}\} = L^2([-\pi,\pi])$, in the $\| \|_2$ norm. In other words, for every $f \in L^2([-\pi,\pi])$, and $\varepsilon > 0$, there is a linear combination F of functions e^{-ikx} such that $\|F - f\|_2 < \varepsilon$.

Choose $\varepsilon > 0$. We use a result from measure theory that says that the closure (in $L^2([-\pi,\pi])$ with norm $\| \|_2$) of the space $C([-\pi,\pi])$ is $L^2([-\pi,\pi])$: given f, there exists $\tilde{f} \in C([-\pi,\pi])$ such that $\|f - \tilde{f}\|_2 < \frac{\varepsilon}{10}$.

Secondly, every function $\tilde{f} \in C([-\pi,\pi])$ can be approximated (in norm $\| \|_\infty$) by a function $\hat{f} \in C^1([-\pi,\pi])$, i.e. there is $\hat{f} \in C^1([-\pi,\pi])$ such that $\|\tilde{f} - \hat{f}\|_\infty < \frac{\varepsilon}{10}$.

In Theorem 24, we saw that every C^1 function can be approximated (in norm $\| \|_\infty$), by linear combinations of $\{e^{-inx}\}$, hence there is a finite Fourier sum F such that $\|F - \hat{f}\|_\infty < \frac{\varepsilon}{10}$.

To compare the two norms that we are using, check that

$$\|h\|_2 = \sqrt{\int_{-\pi}^{\pi} |h(t)|^2 dt} \leq \sqrt{\int_{-\pi}^{\pi} \sup_x |h(x)|^2 dt} = \sqrt{2\pi}\|h\|_\infty.$$

Putting this together, we get

$$\begin{aligned} \|F - f\|_2 &\leq \|F - \hat{f}\|_2 + \|\hat{f} - \tilde{f}\|_2 + \|\tilde{f} - f\|_2 \\ &\leq \sqrt{2\pi}\|F - \hat{f}\|_\infty + \sqrt{2\pi}\|\hat{f} - \tilde{f}\|_\infty + \|\tilde{f} - f\|_2 \\ &\leq \sqrt{2\pi}\frac{\varepsilon}{10} + \sqrt{2\pi}\frac{\varepsilon}{10} + \frac{\varepsilon}{10} < \varepsilon. \end{aligned}$$

Hence, $\operatorname{clin}\{e^{-inx}\} = L^2([-\pi,\pi])$ as asserted. $\square$

Theorem 26 (Parseval) *Let* $f, g \in L^2([-\pi,\pi])$ *have Fourier series* $\sum_k c_k e^{ikx}$ *respectively* $\sum_k d_k e^{ikx}$. *Then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}dt = \sum_{k\in\mathbb{Z}} c_k\overline{d_k}.$$

In particular, $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2 dt = \sum_{k\in\mathbb{Z}} |c_k|^2$.

Proof. $L^2([-\pi,\pi])$ is a separable Hilbert space with its countable orthogonal basis $\{e^{-int}\}_{n\in\mathbb{Z}}$. Therefore, as we have seen before, $Uf = (\xi_n)_{n\in\mathbb{Z}}$ with $\xi_n = \langle f, e^{-int}\rangle$ is an isomorphism between $L^2([-\pi,\pi])$ and $\ell^2_{\mathbb{Z}} = \{(x_n)_{n\in\mathbb{Z}} \mid \sum_{n\in\mathbb{Z}} |x_n|^2 < \infty\}$. Write $c = (c_n)_{n\in\mathbb{Z}}$ and $d = (d_n)_{n\in\mathbb{Z}}$. Then we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt = \frac{1}{2\pi}\langle f,g\rangle = \frac{1}{2\pi}\langle Uf,Ug\rangle = \frac{1}{2\pi}\langle\sqrt{2\pi}c,\sqrt{2\pi}d\rangle = \sum_{n\in\mathbb{Z}} c_n\overline{d_n},$$

as required. □

Remark: Note that the factor $\frac{1}{2\pi}$ comes from the fact $\{e^{inz}\}_{n\in\mathbb{Z}}$ is not ortho**normal**. In the orthonormalcase, Parseval's equality reads: $\langle f,g\rangle = \sum_k c_k\overline{d_k}$.

Theorem 27 (Weierstrass or Stone-Weierstrass) *The set of polynomials $\mathbb{P}([a,b])$ is dense in $C([a,b])$ in $\|\ \|_\infty$. In other words, given a compact interval $[a,b]$ and a continuous function $f : [a,b] \to \mathbb{K}$, there is a sequence of polynomials $p_n : [a,b] \to \mathbb{K}$ that converges uniformly to f.*

It is important that $[a,b]$ is indeed compact, otherwise the theorem is false. There is a well-known constructive proof of this theorem by Bernstein. "Constructive" here means that the proof uses explicit (now called *Bernstein*) polynomials

$$B_n(x,f) = \sum_{i=0}^{n} \binom{i}{n} x^i(1-x)^{i-n} f(\frac{i}{n}) \quad \text{for } x \in [0,1].$$

and shows that $\|f - B_n\|_\infty \to 0$ with explicit bounds. We will use a proof based on Theorem 24.

Proof. The proof, in telegram style, reads: $C^1([a,b])$ lies dense in $C^1([a,b])$ in the $\|\ \|_\infty$ norm. Fourier series converge uniformly to C^1 functions. Taylor polynomials converge uniformly on compact intervals to exponential functions comprising the Fourier series. Hence polynomials converge uniformly to continuous functions.

Now the details: start by scaling $\tilde{f}(x) = f(\frac{b-a}{\pi}x + \frac{b+a}{2})$, which is a continuous function $[-\frac{\pi}{2},\frac{\pi}{2}]$. Find $g \in C^1([-\frac{\pi}{2},\frac{\pi}{2}])$ whose graph lies between $\tilde{f} - \frac{1}{n}$ and $\tilde{f} + \frac{1}{n}$. Extend g to a C^1 2π-periodic function. Find, by Theorem 24, a finite Fourier series $G = \sum_{k=-l}^{l} c_k e^{-ikx}$ whose graph lies between $g - \frac{1}{n}$ and $g + \frac{1}{n}$. Each function $c_k e^{-ikx}$ used in this Fourier series can be approximated uniformly on $[-\pi,\pi]$ by its Taylor polynomials $T_m(x) = c_k(1 + (-ikx) + \frac{1}{2}(-ikx)^2 + \cdots + \frac{1}{m!}(-ikx)^m)$. Find a linear combination $\tilde{p}_n$ of such Taylor polynomials whose graph lies between $G - \frac{1}{n}$ and $G + \frac{1}{n}$. This shows that on $[-\frac{\pi}{2},\frac{\pi}{2}]$, the graph of $\tilde{p}_n$ lies between $\tilde{f} - \frac{3}{n}$ and $\tilde{f} + \frac{3}{n}$, so, scaling back to polynomials $p_n : [a,b] \to \mathbb{K}$, $\|f - p_n\|_\infty < \frac{3}{n}$. Since this can be done for all $n \geq 1$, uniform converges $p_n \to f$ follows. □

5 Functionals and Dual Spaces

Definition 28 *Given a vector space E over field $\mathbb{K}$, a linear* functional *is a linear map $f : E \to \mathbb{C}$. (Most of the time, we just say functional, implicitly assuming that the functional is indeed linear.)*

Examples: • If $E = L^1([0,1])$, then $Fg = \int_0^1 g(t)dt$ is a functional.
• If $E = C^1([0,1])$ with norm $\| \ \|_\infty$, then $Fg = g'(0)$ is a functional.
• If $E = \ell^1$ and y some bounded sequence. Then $F_y(x) = \sum_{n=1}^\infty y_n x_n$ is a functional.
• If E is some Hilbert space, and $x \in E$, then $F(y) = \langle y, x\rangle$ is a functional.

We tend to think of linear maps as continuous maps, but in infinite dimensional spaces this is not always the case! In the second example above, let $g_n(x) = \frac{1}{\sqrt{n}}(1-x)^n$. Then g_n is a Cauchy sequence in $(C^1([0,1]), \| \ \|_\infty)$ with limit $g(x) \equiv 0$, but still $Fg_n = g_n'(0) = \sqrt{n} \to \infty$.

Continuity of functions is related to the notion of *boundedness* of functionals, see item 3. of the below theorem.

Theorem 29 *Let F be a linear functional on a normed space $(E, \| \ \|)$, then the following three statements are equivalent:*

1. *F is continuous;*
2. *F is continuous at 0;*
3. $\sup\{|F(y)| \mid \|y\| \le 1\} < \infty$, *that is: F is* bounded.

Proof. 1. $\Rightarrow$ 2. This is obvious.
2. $\Rightarrow$ 3. Assume that F is continuous at 0, then there exists $\delta > 0$ such that for any $z \in Y$ with $\|z - 0\| < \delta$, $|F(z)| < 1$. But F is linear, so for any y with $\|y\| \le 1$, $\|\frac{\delta}{2}y\| < \delta$, and $|F(y)| = |\frac{2}{\delta}|F(\frac{\delta}{2}y))| < \frac{2}{\delta} < \infty$. Therefore F is bounded.
3. $\Rightarrow$ 1. Let $x \in E$ arbitrary. Take $\varepsilon > 0$ and $\delta = \varepsilon(\sup_{\|z\|\le 1}|F(z)|)^{-1}$. If $y \in E$ is such that $\|y - x\| < \varepsilon$, then

$$|F(y) - F(x)| \le \|y - x\| \; |F(\frac{y-x}{\|y-x\|})| \le \delta \sup_{\|z\|\le 1} |F(z)| = \varepsilon.$$

Since x was arbitrary, F is continuous everywhere. □

Definition 30 *The space of all bounded (and hence continuous) linear functionals $F : E \to \mathbb{K}$ is called the* dual space *of a E, and it is denoted as E^*. The quantity*

$$\|F\| = \sup\{|F(y)| \mid \|y\| \le 1\}$$

is the norm of the dual space.

Note that by linearity

$$|F(x)| = |F(\frac{x}{\|x\|})|\ \|x\| \leq \|F\|\ \|x\| \tag{7}$$

for all $x \in E$.

Theorem 31 *If $(E, \|\ \|_E)$ is a Banach space, then the dual space $(E^*, \|\ \|)$ is also a Banach space.*

Proof. Let us first check that $\|F\| = \sup\{|F(y)| \mid \|y\| \leq 1\}$ is indeed a norm:
- $\|F\|$ is finite, because E^* only contains bounded functionals.
- $\|\lambda F\| = \sup_{\|y\|\leq 1} |\lambda F(y)| = |\lambda| \sup_{\|y\|\leq 1} |F(y)| = |\lambda| \|F\|$
- $\|F + G\| = \sup_{\|y\|\leq 1} |F(y) + G(y)| \leq \sup_{\|y\|\leq 1} |F(y)| + \sup_{\|y\|\leq 1} |G(y)| = \|F\| + \|G\|$
- $\|F\| \geq 0$ is obvious.

Finally, to show the completeness of E^*, consider a Cauchy sequence (F_n) in $\|\ \|$. Then $\|F_n - F_m\| \to 0$ as $m, n \to \infty$. In particular,

$$|F_n(y) - F_m(y)| = \|y\|_E\ |F_n(\frac{y}{\|y\|_E}) - F_m(\frac{y}{\|y\|_E})| \leq \|y\|_E \|F_n - F_m\| \to 0$$

pointwise, and since the field $\mathbb{K}$ is complete, $F_n(y)$ converges. Call the limit $F(y)$. This defining a new functional F. (Check that it is linear.) Since $|F(y)| \leq |F_n(y) - F(y)| + |F_n(y)| \leq 1 + \|F_n\|$ for all $\|y\| \leq 1$ and n sufficiently large, F is indeed bounded. This shows that $F \in E^*$. □

This theorem creates new Banach spaces from old ones, and we might go on, creating E^{**}, the dual of the dual space, etc. If we think of isomorphic spaces (defined in the previous chapter for Hilbert spaces, but equally applicable to Banach spaces), it turns out that few of these Banach spaces are actually new.

Theorem 32 *If $p > 1$ and $q > 1$ are conjugate exponents, then*

$$(\ell^p)^* \simeq \ell^q \quad \textit{and} \quad (\ell^q)^* \simeq \ell^p.$$

(Here $\simeq$ denotes: is isomorphic to.) Furthermore

$$(\ell^1)^* \simeq \ell^\infty, \quad \textit{but} \quad c_0^* \simeq \ell^1,$$

for $c_0 = \{x = (x_1, x_2, , \dots) \mid x_n \in \mathbb{K}$ and $\lim_{n\to\infty} x_n = 0\}$ equipped with the norm $\|\ \|_\infty$.

From this theorem we see that if $p > 1$, then $(\ell^p)^{**} = \ell^p$. Spaces E with the property that E^{**} are called *reflexive*. So ℓ^1 is an example of a non-reflexive Banach space. Note also that ℓ^2 is isomorphic to its own dual space. It is no coincidence here that among all space ℓ^p, only ℓ^2 is a Hilbert space.

◇ **Proof.** We will only do the proof that $(\ell^1)^*$ is isomorphic to ℓ^∞. The proof for the other isomorphisms is similar, but much more technical.

Assume that $\{e_n\}_{n\geq 1}^\infty$ is the standard basis of ℓ^1. Let us define a *dual basis* $\{e_n^*\}_{n\geq 1}^\infty$, by setting

$$e_n^*(e_k) = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k. \end{cases}$$

Then, if $x = (x_1, x_2, \dots) \in \ell^1$, we get $e_n^*(x) = e_n^*(\sum_{k=1}^\infty x_k e_k) = x_n$. Next define

$$T : \ell^\infty \to (\ell^1)^*, \qquad Ty = \sum_{n=1}^\infty y_n e_n^*.$$

The image Ty is a functional, and $(Ty)(x) = \sum_n y_n x_n$. The map T should be a linear isometry between ℓ^∞ and $(\ell^1)^*$, and for this we need to check:

- T is linear. This is easy; check it yourself.
- T preserves norms. For Ty we need the functional norm $\|\ \|$, which requires estimates over $\{x \in \ell^1 \mid \|x\|_1 \leq 1\}$:

$$\begin{aligned} \sup_{\|x\|_1\leq 1} |(Ty)(x)| &\leq \sup_{\|x\|_1\leq 1} |\sum_{n=1}^\infty y_n x_n| \\ &\leq \sup_{n\geq 1} |y_n| \cdot \sup_{\|x\|_1\leq 1} \sum_{n=1}^\infty |x_n| \\ &\leq \|y\|_\infty \sup_{\|x\|_1\leq 1} \|x\|_1 \\ &\leq \|y\|_\infty. \end{aligned}$$

  This shows that $\|Ty\| \leq \|y\|_\infty$. On the other hand,

$$\sup_{\|x\|_1\leq 1} |(Ty)(x)| \geq \sup_{n\geq 1} |(Ty)(e_n)| \geq \sup_{n\geq 1} |y_n| = \|y\|_\infty.$$

  Therefore also $\|Ty\| \geq \|y\|_\infty$, so $\|Ty\| = \|y\|_\infty$.
- T is onto. In other words, for every bounded linear functional $g \in (\ell^1)^*$, there is a $y \in \ell^\infty$ such that $Ty = g$. Since g is bounded, $\sup_n |g(e_n)| < \infty$. Define $y = (y_1, y_2, y_3, \dots)$ by $y_n = g(e_n)$. Then $y \in \ell^\infty$. Moreover

$$g(x) = g(\sum_{n=1}^\infty x_n e_n) = \sum_{n=1}^\infty x_n g(e_n) = \sum_{n=1}^\infty x_n y_n = (Ty)(x)$$

  for all x. Therefore $g = Ty$.

□

The main result about dual Hilbert spaces is called the Riesz-Fréchet Theorem. If we look back at the examples of functionals on a Hilbert space $(H, \langle\ ,\ \rangle)$, we could define functional $F(y) = \langle y, x\rangle$ for each fixed $x \in H$. This functional is bounded, because by the Cauchy-Schwarz inequality,

$$|F(y)| = |\langle y, x\rangle| \leq \|y\|\ \|x\|, \text{ so } \|F\| \leq \|x\|.$$

By substituting the unit vector $y = x/\|x\|$ be find that $\|F\| \geq \|x\|$, so in fact, $\|F\| = \|x\|$. The Riesz-Fréchet Theorem states that all continuous linear functionals on a Hilbert space are of this type.

Theorem 33 (Riesz-Fréchet) *If F is a continuous linear functional on a Hilbert space $(H, \langle\ ,\ \rangle)$, then there exists a unique $x \in H$ such that $F(y) = \langle y, x\rangle$ for all y. Moreover $\|F\| = \|x\|$.*

Proof. The equality$\|F\| = \|x\|$ was proven above. If there are two vectors x and $x' \in H$ such that $F(y) = \langle y, x\rangle = \langle y, x'\rangle$ for all y, then $\langle y, x - x'\rangle = 0$ for all y. Take $y = x - x'$, then we find $\langle x - x', x - x'\rangle = 0$, so $x - x' = 0$ and indeed the x is unique. Therefore it suffices to show that such vector x exists.

If $F(y) = 0$ for all y, then $x = 0$ solves the problem. So assume that the kernel $M = \ker(F) = \{y \in H \mid F(y) = 0\}$ is a proper subspace of H. Since F is continuous $M = F^{-1}(\{0\})$ is closed, and hence $H = M \oplus M^\perp$. Take $\xi \in M^\perp$, such that $F(\xi) = 1$. By scaling ξ, this can always be arranged. Then we can write

$$y = \underbrace{y - F(y)\xi}_{\in M} + \underbrace{F(y)\xi}_{\in M^\perp}.$$

Check that the first term indeed belongs to M by applying F to it! Now take the inner product

$$\langle y, \xi\rangle = \langle y - F(y)\xi, \xi\rangle + \langle F(y)\xi, \xi\rangle = \langle F(y)\xi, \xi\rangle = F(y)\|\xi\|^2.$$

But then, if we take $x = \xi/\|\xi\|^2$,

$$\langle y, x\rangle = \frac{1}{\|\xi\|^2}\langle y, \xi\rangle = F(y).$$

□

We said earlier that ℓ^2 was isomorphic to its own dual space was not surprising because ℓ^2 is a Hilbert space. This holds, namely, for all Hilbert spaces (although we will only prove it for real Hilbert spaces).

Theorem 34 *Let H be a real Hilbert space, then H^* is isomorphic to H.*

♢ **Proof.** For each $F \in H^*$, let $UF := \eta$ be the corresponding vector in H satisfying the Riesz-Fréchet Theorem: $F(x) = \langle x, \eta\rangle$ for all $x \in H$. We will show that

$U : H^* \to H$ is unitary.
• If $UF = \eta$ and $UG = \zeta$, then $(F+G)(y) = F(y)+G(y) = \langle y, \eta\rangle + \langle y, \zeta\rangle = \langle y, \eta+\zeta\rangle$ for all y, so U(F+G) = UF + UG.
• If $UF = \eta$ and $\lambda \in \mathbb{R}$, then $(\lambda F)(y) = \lambda F(y) = \lambda\langle y, \eta\rangle = \langle y, \lambda\eta\rangle$ for all y (note that we used here that H is a real Hilbert space), so $U(\lambda F) = \lambda UF$.
• For each $\eta \in H$, the functional F defined as $F(y) = \langle y, \eta\rangle$ is bounded and satisfies $UF = \eta$, so U is surjective.
We know already from the Riesz-Fréchet Theorem that U preserves the norm, hence it is unitary. We can define the inner product of H^* explicitly by means of the isomorphism and the polarisation formula. □

6 Linear Operators

Functionals were maps from a linear space into $\mathbb{R}$ or $\mathbb{C}$. Now we shift gear, and look at linear maps from one linear space E to another linear space F:

$$T : E \to F \qquad \text{with} \qquad T(\lambda x + \mu y) = \lambda Tx + \mu Ty.$$

These are called linear *operators*. (Note that E and F should be linear spaces over the same field $\mathbb{K}$.) If E and F are normed spaces, then we can again speak of *bounded operators* if there exists $M > 0$ such that

$$\|Tx\|_F \leq M\|x\|_E \quad \text{for all } x \in E,$$

and the *operator norm* is

$$\|T\| = \sup_{\|x\|_E \leq 1} \|Tx\|_F.$$

Theorem 35 *Let $T : E \to F$ be a linear operator between normed spaces $(E, \| \|_E)$, and $(F, \| \|_F)$, then the following three statements are equivalent:*

1. *T is continuous;*
2. *T is continuous at 0;*
3. *T is* bounded.

Proof. The proof is the same as for Theorem 29 □

Definition 36 *The* kernel *of an operator $T : E \to F$ is the set $\{x \in E \mid Tx = 0\}$ and denoted as ker(T). The* range *of T is the set $TE = \{y \in F \mid$ there is an $x \in E$ such that $Tx = y\}$. Notation: $R(T)$.*

Note that the kernel is a subspace of E; if T is continuous, then it is even a closed subspace. The range is a subspace of F, but it need not be space.

Examples: • If $g : [0,1] \to \mathbb{K}$ is a bounded function, then $T : L^p([0,1]) \to L^p([0,1])$ defined by $(Tf)(t) = g(t) \cdot f(t)$ is a linear operator. It is also bounded, because

$$\|Tf\|_p = \sqrt[p]{\int_0^1 |g(t)f(t)|^p dt} \le \sqrt[p]{\sup_{t\in[0,1]} |g(t)|^p \int_0^1 |f(t)|^p dt} = \|g\|_\infty \|f\|_p.$$

• If $k : [a,b] \times [c,d] \to \mathbb{K}$ is a continuous function, then the integral operator $(Tf)(t) = \int_a^b k(s,t)f(s)ds$ is linear operator from $L^2([a,b])$ to $L^2([c,d])$. It is also bounded, because (using the Cauchy-Schwarz inequality)

$$|Tf(t)|^2 = \left|\int_a^b k(s,t)f(s)ds\right|^2 \le \int_a^b |k(s,t)|^2 ds \int_a^b |f(s)|^2 ds = \int_a^b |k(s,t)|^2 ds\ \|f\|_2^2,$$

and therefore

$$\|Tf\|_2^2 = \int_c^d \left|\int_a^b k(s,t)f(s)ds\right|^2 dt \le \int_c^d \int_a^b |k(s,t)|^2 dsdt\ \|f\|_2^2.$$

• If $E = C^\infty(\mathbb{R})$, then the differential operator $Df = f'$ is linear. In the $\| \|_\infty$-norm it is not a bounded.operator, as can be seen from the example $f_n(x) = \sin nx$. Composite differential operator are very common, for example: $L = D^2 - x^2 D - I$ defined as $Lf(x) = f''(x) + x^2 f'(x) - f(x)$.
• Partial differential operators, for example the Laplacian: $\Delta(f) = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$ for maps $f : \mathbb{R}^n \to \mathbb{R}$. • If $E = \ell^\infty$, then $S : E \to E$ defined as

$$S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$$

is a bounded linear operator. It is called the *right-shift* operator. The *left-shift* operator S^* shifts the string in the other direction:

$$S^*(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$$

• Different branches of mathematics have their own favourite operators. If $\tau : X \to X$ is some transformation of a space X, then you might be interested in the behaviour of orbits: $\{x, \tau(x), \tau \circ \tau(x), \dots\}$. Operators in use for this study are the *Koopman operator:* $K : L^\infty(X) \to L^\infty(X)$ defined by $Kf = f \circ \tau$. Because we used the $\| \|_\infty$-norm, K is bounded. For the space $(L^p(X), \| \|_p)$ this need not be the case anymore.
• The *transfer operator* is $(L_g f)(x) = \sum_{y,\ \tau(y)=x} g(y)f(y)$. The boundedness of the transfer operator depends on g and on the space on which L_g is defined.

Definition 37 *Let $\mathcal{L}(E,F)$ denote the space of continuous (and hence bounded) linear operators from E to F. If $E = F$ then we simply write $\mathcal{L}(E)$.*

Theorem 38 *If F is a Banach space, then $\mathcal{L}(E, F)$ is also a Banach space.*

Proof. This is proven in the same way as Theorem 31 $\square$

Lemma 39 *If $A \in \mathcal{L}(E, F)$ and $B \in \mathcal{L}(F, G)$, then the composition $BA \in \mathcal{L}(E, G)$ and its norm $\|BA\| \leq \|B\| \, \|A\|$.*

Proof. It is clear that $BA : E \to G$ and linearity is easy to check. Next, if $x \in E$, then (using formula (7) twice)

$$\|BAx\|_G \leq \|B\| \|Ax\|_F \leq \|B\| \|A\| \|x\|_E$$

Take the supremum over all $x \in E$ with $\|x\|_E \leq 1$, and derive $\|BA\| \leq \|B\| \, \|A\|$. $\square$

Note that the strict inequality $\|BA\| < \|B\| \, \|A\|$ is possible. By induction, it is easy to see that if $A \in \mathcal{L}(E)$, then the n-fold iterate $A^n = \underbrace{A \dots A}_{n \text{ times}}$ satisfies $\|A^n\| \leq \|A\|^n$.

When we want to solve the f in the equation

$$Af = g,$$

for some linear operator A and a given g, the easiest would be to have the inverse operation to A. In the rest of this section, we will discuss when operators are invertible.

Definition 40 *Let E and F be normed spaces. An operator $A \in \mathcal{L}(E, F)$ is called* invertible *if there exists an operator $B \in \mathcal{L}(F, E)$ such that*

$$BA = I_E \quad \text{and} \quad AB = I_F.$$

Here I_E (resp. I_F) denotes the identity on E (resp. F). If it exists, B is unique, and denoted as A^{-1}.

In spaces of finite dimension, invertibility of linear operators $A : E \to E$ is rather simple (see a course on linear algebra). You just need to check one of the following equivalent conditions:

1. A is invertible.

2. A is one-to-one.

3. A is onto.

4. There exists $B \in \mathcal{L}(E)$ such that $AB = I$.

5. There exists $B \in \mathcal{L}(E)$ such that $BA = I$.

6. The determinant of some (any) matrix representation of A is different from 0.

For infinite dimensional spaces, none of these conditions is necessarily equivalent to any other.

Examples: • The right and left-shift operators are not each other's inverse, because

$$SS^* = I \text{ but } S^*S \neq I.$$

• The multiplication operator $T : L^2([0,1]) \to L^2([0,1]$ defined by $(Tf)(t) = t^2 f(t)$ is not onto, because there is (for example) no f such that $Tf \equiv 1$.

Theorem 41 *Let E be a Banach space. If $A \in \mathcal{L}(E)$ and $\|A\| < 1$, then $I - A$ is invertible, and $(I-A)^{-1} = \sum_{n=0}^{\infty} A^n$. (Note: $A^0 = I$ by definition.)*

Proof. First we need say clearly what $\sum_{n=0}^{\infty} A^n$ means. It is a limit of a Cauchy sequence of operators B_k. Indeed, let $B_k = \sum_{n=0}^{k} A^n$, so $B_k x = A^0 x + A^1 x + A^2 x + \cdots + A^k x$. The sequence (B_k) is Cauchy in the operator norm $\| \ \|$, because

$$\|(B_k - B_l)x\|_E = \| \sum_{n=l+1}^{k} A^n x\|_E \leq \sum_{n=l+1}^{k} \|A\|^n \|x\|_E \leq \frac{\|A\|^l}{1 - \|A\|} \|x\|_E \to 0$$

as $l < k \to \infty$. Therefore (B_k) converges in the Banach space $\mathcal{L}(E)$. Let B be the limit. Multiply with $(I - A)$, then

$$(I - A)B_k x = Ix - Ax + Ax - A^2 x + \cdots - A^{k+1}x = x - A^{k+1}x \to x$$

for each x. In the limit $(I - A)B = I$, and a similar computation gives $B(I - A) = I$. □

7 Adjoint and Self-Adjoint Operators

Definition 42 *Given two Hilbert spaces $(E, \langle\ ,\ \rangle_E)$ and $(F, \langle\ ,\ \rangle_F)$, and a bounded linear operator $A : E \to F$, we say that an operator*[5] *$A^* : F \to E$ is the* adjoint operator *of A if*

$$\langle Ax, y\rangle_F = \langle x, A^* y\rangle_E \text{ for all } x \in E \text{ and } y \in F.$$

Examples: • You may have seen the notation A^* earlier in a linear algebra course, because if A is the matrix representing a linear transformation of $\mathbb{C}^n$, then $A^* = \overline{A^t}$, and $\langle Ax, y\rangle = \langle x, A^* y\rangle$ is true for the standard inner product on $\mathbb{C}^n$.

• If $A : L^2([0,1]) \to L^2([0,1])$ is the multiplication operator $Ax(t) = f(t)x(t)$ for some fixed function f, then $A^* y(t) = \overline{f(t)} y(t)$. Indeed,

$$\langle Ax, y\rangle = \int_0^1 f(t)x(t) \cdot \overline{y(t)} dt = \int_0^1 x(t) \cdot \overline{\overline{f(t)} y(t)} dt = \langle x, A^* y\rangle.$$

[5]Unfortunately, the superscript * is used both for adjoint operator and for dual space. If it is clear whether A is an operator or a space, no confusion will arise.

• If $A = L^2([a,b]) \to L^2([c,d])$ is the *integral operator* with *kernel* $k : [a,b] \times [c,d] \to \mathbb{K}$, i.e.

$$Af(t) = \int_a^b k(s,t)f(s)ds,$$

then

$$\begin{aligned}
\langle Af, g\rangle &= \int_c^d \left(\int_a^b k(s,t)f(s)ds\right)\overline{g(t)}\ dt \\
&= \int_c^d \int_a^b k(s,t)f(s)\overline{g(t)}\ ds\ dt \\
&= \int_a^b f(s)\left(\overline{\int_c^d \overline{k(s,t)}g(t)dt}\right)\ ds = \langle f, A^*g\rangle,
\end{aligned}$$

so from this computation, we can read off that $A^*g(t) = \int_c^d \overline{k(t,s)}g(t)dt$. Note the change in the order of the arguments of k!

• The adjoint of the right-shift operator on ℓ^2 is the left-shift operator, and vice versa.

Theorem 43 *For each $A \in \mathcal{L}(E,F)$ where $(E, \langle\ ,\ \rangle_E)$ and $(F, \langle\ ,\ \rangle_F)$ are Hilbert spaces, the adjoint operator A^* exists and belongs to $\mathcal{L}(F,E)$. Moreover, $A^{**} = A$ and $\|A^*\| = \|A\|$.*

Proof. The Riesz-Fréchet Theorem will be useful to find A^*. Given $y \in F$, the map

$$x \mapsto \langle Ax, y\rangle_F$$

is a linear functional on E. It is also bounded because $|\langle Ax, y\rangle| \le \|Ax\|_F\|y\|_F \le \|x\|_E\|A\|\|y\|_F$, so the norm of the functional is at most $\|A\|\|y\|_F$. By the Riesz-Fréchet Theorem, we can find $z \in E$ such that

$$\langle Ax, y\rangle_F = \langle x, z\rangle_E.$$

Define A^* by $A^*y = z$, so obviously $A^* : F \to E$. Now we need to check:

- A^* is linear. Take $z_1, z_2 \in F$ and $\lambda_1, \lambda_2 \in \mathbb{K}$. Then

$$\begin{aligned}
\langle x, A^*(\lambda_1 z_1 + \lambda_2 z_2)\rangle_E &= \langle Ax, (\lambda_1 z_1 + \lambda_2 z_2)\rangle_F \\
&= \overline{\lambda_1}\langle Ax, z_1\rangle_F + \overline{\lambda}_2\langle Ax, z_2\rangle_F \\
&= \overline{\lambda_1}\langle x, A^*z_1\rangle_E + \overline{\lambda}_2\langle x, A^*z_2\rangle_E \\
&= \langle x, \lambda_1 A^*z_1 + \lambda_2 A^*z_2\rangle_E.
\end{aligned}$$

Since this is true for all x, we have $A^*(\lambda_1 z_1 + \lambda_2 z_2) = \lambda_1 A^*z_1 + \lambda_2 A^*z_2$.

- A^* is bounded. For this, take any $y \in F$ with $\|y\|_F \leq 1$. To show that A^* is bounded, we need not worry about those y for which $\|A^*y\|_E = 0$, so let us assume that $\|A^*y\|_E > 0$. By the Cauchy-Schwarz inequality:

  $$\|A^*y\|_E^2 = \langle A^*y, A^*y\rangle_E = \langle AA^*y, y\rangle_F \leq \|AA^*y\|_F\|y\|_F \leq \|A\|\|A^*y\|_E\|y\|_F$$

  Divide out one factor of $\|A^*y\|_E$, and we find $\|A^*y\|_E \leq \|A\|\|y\|_F$, so

  $$\|A^*\| \leq \|A\| < \infty. \tag{8}$$

Now we show that $A^{**} = A$. Write $B = A^*$. Then

$$\langle x, B^*y\rangle_F = \langle Bx, y\rangle_E = \langle A^*x, y\rangle_E = \overline{\langle y, A^*x\rangle_E} = \overline{\langle Ay, x\rangle_F} = \langle x, Ay\rangle_F$$

for all $x \in F$ and $y \in E$. Therefore $A^{**} = B^* = A$. Finally, (8) showed that $\|A^*\| \leq \|A\|$, and applying this to A^*, we obtain $\|A\| = \|A^{**}\| \leq \|A^*\|$. Therefore $\|A^*\| = \|A\|$. □

Definition 44 *An operator $A \in \mathcal{L}(E)$ is called* self-adjoint *or* Hermitian, *if $A^* = A$. (Note that here the domain and range must be the same space.)*

Examples: • The multiplication operator $Ax(t) = f(t)x(t)$ is self-adjoint if and only if $f(t)$ is a real function.
• If $A = L^2([a,b]) \to L^2([a,b])$ is the *integral operator* with *kernel* $k : [a,b] \times [a,b] \to \mathbb{K}$, then it is self-adjoint if and only if $k(s,t) = \overline{k(t,s)}$ for all $s, t \in [a,b]$.
• If E is the space of real infinitely differentiable 2π-periodic functions with inner product $\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)dt$, then the differential operator $D^2 f = f''$ is self-adjoint. This follows from integration by parts:

$$\begin{aligned}\langle D^2 f, g\rangle &= \int_{-\pi}^{\pi} f''(t)g(t)\ dt \\ &= [f'(t)g(t)]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(t)g'(t)\ dt \\ &= -[f(t)g'(t)]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} f(t)g''(t)\ dt = \langle f, D^2 g\rangle.\end{aligned}$$

(Here we ignored the detail that E is not a Hilbert space: it is not complete.)

8 Compact Operators

◇ **Definition 45** *Let E and F be Banach spaces. An operator $A \in \mathcal{L}(E, F)$ is called* compact *is for every bounded sequence $(x_n)_{n=1}^{\infty} \subset E$, the sequence $(Ax_n)_{n=1}^{\infty}$ has a convergent subsequence.*

Examples: • An operator A is of *finite rank* if the *rank*, i.e. the dimension of the range $R(A)$ is finite. For example, the orthogonal projection on a finite dimensional subspace has finite rank. Every bounded finite rank operator A is compact. Indeed, if $\{x_n\}_n$ is bounded, and A is bounded, then $\{Ax_n\}_n$ is a bounded sequence in a finite dimensional space. We know that such sequences have convergent subsequences (Heine-Borel). The boundedness of A is important. A counter-example would be:

$$A : \ell^1 \to \ell^1, \quad Ae_n = ne_1.$$

This operator has rank 1, but is not compact.

- Let $A : \ell^1 \to \ell^1$ be defined by $Ae_n = \frac{1}{n}e_n$. Then A is bounded, of infinite rank, but still compact. The reason for this is that A is the limit of finite rank operators

$$A_k : \ell^1 \to \ell^1, \quad A_k e_n = \begin{cases} \frac{1}{n}e_n & \text{if } n \leq k, \\ 0 & \text{if } n > k. \end{cases}$$

It is easy to see that the rank of A_k is k. And $\lim_k A_k = A$ in the operator norm, because for each $x \in \ell^1$ with $\|x\|_1 \leq 1$ we have

$$\|(A - A_k)x\|_1 = \sum_{n>k} |\frac{1}{n}x_n| \leq \frac{1}{k+1} \sum_{n>k} |x_n| \leq \frac{1}{k}\|x\|_1,$$

so $\sup_{\|x\|_1 \leq 1} \|(A - A_k)x\|_1 \leq \frac{1}{k+1} \to 0$. To conclude this example, we need an important theorem about compact operators.

Theorem 46 *Let E and F be Banach spaces. The set of compact operators in $\mathcal{L}(E, F)$ is a closed subset with respect to the operator norm.*

Proof. Let $\{A_k\}_k \subset \mathcal{L}(E, F)$ be a sequence of compact operators converging in the operator norm to A. Let $\{x_n\}_n$ be any bounded sequence in E, say $\|x_n\|_E \leq M$ for all $n \geq 1$. We need to show that $\{Ax_n\}_n$ contains a converging subsequence. To do this, we use a kind of diagonal argument.

- A_1 is compact, so there exists a subsequence, say $\{x_{1,n}\}_n$ of $\{x_n\}_n$, such that $\{A_1 x_{1,n}\}_n$ is convergent.
- A_2 is compact, so there exists a subsequence, say $\{x_{2,n}\}_n$ of $\{x_{1,n}\}_n$, such that $\{A_2 x_{2,n}\}_n$ is convergent.

In general:

- A_k is compact, so there exists a subsequence, say $\{x_{k,n}\}_n$ of $\{x_{k-1,n}\}_n$, such that $\{A_k x_{k,n}\}_n$ is convergent.

All the above convergent sequences are of course also Cauchy sequences. Now for the diagonal construction, for each k, take $n(k)$ such that

$$\|A_k x_{k,m} - A_k x_{k,m'}\|_F < \frac{1}{k} \text{ for all } m, m' \geq n(k). \tag{9}$$

The vectors $y_k := x_{k,n(k)}$ form a subsequence of $\{x_n\}_n$. We show that $\{Ay_k\}_k$ is Cauchy sequence in F. Indeed, for $l \geq k$ we have

$$\begin{aligned}
\|Ay_k - Ay_l\|_F &\leq \|Ay_k - A_k y_k\|_F + \|A_k y_k - A_k y_l\|_F + \|A_k y_l - Ay_l\|_F \\
&\leq \|A - A_k\| \; \|y_k\|_E + \|A_k x_{k,n(k)} - A_k x_{k,m'}\|_F + \|A_k - A\| \; \|y_l\|_E \\
&\leq \|A - A_k\| M + \frac{1}{k} + \|A_k - A\| M \\
&\leq 2M\|A - A_k\| + \frac{1}{k} \to 0 \text{ as } k \to \infty.
\end{aligned}$$

Here we used in the second line that $y_l = x_{k,m'}$ for some $m' \geq n(k)$ and in the third line that (9) holds and that $\{y_k\}_k$ is a bounded (by M) sequence. Cauchy sequences are convergent in the Banach space F. This shows that all convergent sequences of compact operators have a compact limit operator. Therefore the set of compact operators is closed. □

Definition 47 *Let E and F be Hilbert spaces. An operator $A \in \mathcal{L}(E, F)$ is called a* Hilbert-Schmidt operator *if there exists an orthonormal basis $\{e_n\}_{n\geq 1}$ of E such that $\sum_{n\geq 1} \|Ae_n\|^2 < \infty$. (A priori, the finiteness of the sum $\sum_{n\geq 1} \|Ae_n\|^2$ depends on the choice of orthonormal basis. A nice thing about Hilbert-Schmidt operators is that the choice does not matter! But we will not prove this.)*

Examples: • The *Volterra operator* $V : L^2([0,1]) \to L^2([0,1])$, defined as

$$Vf(t) = \int_0^t f(s)ds,$$

is Hilbert-Schmidt. Indeed, take the orthonormal basis $e_n(t) = e^{-2\pi i n t}$, then

$$\|Ae_n\|_2^2 = \int_0^1 \left| \int_0^t e^{-2\pi i n x} dx \right|^2 dt = \int_0^1 \left| [\frac{1}{2\pi i n} e^{-2\pi i n x}]_0^t \right|^2 dt \leq \int_0^1 (\frac{2}{2\pi n})^2 dt = \frac{1}{\pi^2 n^2}.$$

Therefore $\sum_{n\geq 1} \|Ae_n\|_2^2 \leq \sum_{n\geq 1} \frac{1}{\pi^2 n^2} = \frac{1}{6}$.

Theorem 48 *Every Hilbert-Schmidt operator is compact.*

Proof. The proof of this theorem uses the same idea as the above example, namely, we will write the Hilbert-Schmidt operator A as limit of finite rank operators. Let $\{e_n\}_{n\geq 1}$ be an orthonormal basis of E, so each $x \in E$ can be written as $x = \sum_{n=1}^{\infty} x_n e_n$. By the Cauchy-Schwarz inequality and Pythagoras Theorem, it follows that

$$\begin{aligned}
\|Ax\|_F &= \|A(\sum_{n=1}^{\infty} x_n e_n)\|_F \\
&\leq \sum_{n=1}^{\infty} |x_n| \|Ae_n\|_F \\
&\leq \sqrt{\sum_{n=1}^{\infty} |x_n|^2 \sum_{n=1}^{\infty} \|Ae_n\|_F^2} \\
&\leq \|x\|_E \sqrt{\sum_{n=1}^{\infty} \|Ae_n\|_F^2},
\end{aligned}$$

so A is a bounded operator. Define

$$A_k : E \to F, \quad A_k(x) = A(\sum_{n=1}^{k} x_n e_n),$$

then the rank of A_k is at most k, and $\|A_k x\|_F \leq \|Ax\|_F$, so A_k is a bounded operator. Therefore, the operators A_k are all compact. Moreover, $\lim_k A_k = A$ in the operator norm because (as above)

$$\|Ax - A_k x\|_F = \|A(\sum_{n=k+1}^{\infty} x_n e_n)\|_F \leq \|x\|_E \sqrt{\sum_{n=k+1}^{\infty} \|Ae_n\|_F^2},$$

for all $x \in E$. Because $\sum_{n=1}^{\infty} \|Ae_n\|_F^2 < \infty$, we have $\sum_{n=k+1}^{\infty} \|Ae_n\|_F^2 \to 0$ as $k \to \infty$. So if we take the supremum over all $x \in E$ with $\|x\|_E \leq 1$, we obtain

$$\|A - A_k\| \leq \sqrt{\sum_{n=k+1}^{\infty} \|Ae_n\|_F^2} \to 0 \quad \text{as } k \to \infty.$$

The statement follows now from Theorem 46. □

9 Spectral Properties

Apart from the equation $Af = g$, quite often the equation

$$Af - \lambda f = g$$

comes up in applications. Here $\lambda \in \mathbb{C}$ is some number, and depending on the value of λ solutions may or may not exist.

Definition 49 *Let A be a bounded operator on a Banach space E. For $\lambda \in \mathbb{C}$, we call*

$$R_\lambda(A) = (\lambda I - A)^{-1}$$

the resolvent operator *of A. The* resolvent set *of A is the set*

$$\rho(A) = \{\lambda \in \mathbb{C} \mid R_\lambda(A) \text{ exist and is bounded}\}.$$

The spectrum *of A is the complement of $\rho(A)$, so*

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid R_\lambda(A) \text{ does not exists or is unbounded}\}.$$

We call λ an *eigenvalue* if there exists and $x \neq 0$ such that $Ax = \lambda x$. Such x is called an *eigenvector*. For eigenvalues λ, $x \in \ker(\lambda I - A)$, so R_λ does not exist. Eigenvalues, therefore, belong to the spectrum. If E is a finite dimensional space, then $\sigma(A)$ is precisely the set of eigenvalues of A, but for infinite dimensional spaces, the spectrum can be bigger. For example, if

$$A : \ell^2 \to \ell^2, \quad Ae_n = \frac{1}{n} e_n,$$

then the eigenvalues of A are $\{\frac{1}{n} \mid n \geq 1\}$, but also the value $\lambda = 0$ belongs to the spectrum, because the inverse of A satisfies $A^{-1}e_n = ne_n$, so it is not a bounded operator.

Theorem 50 *The spectrum of a bounded operator is a compact set.*

Proof. By the Heine-Borel theorem, we need to check that

- $\sigma(A)$ is bounded: Take $|\lambda| > \|A\|$. Then $\|\frac{1}{\lambda}A\| = \frac{1}{|\lambda|}\|A\| < 1$, so $B := (I - \frac{1}{\lambda}A)^{-1}$ exists and is bounded. But then also

  $$(\lambda I - A)^{-1} = [\lambda(I - \frac{1}{\lambda}A)]^{-1} = \frac{1}{\lambda}(I - \frac{1}{\lambda}A)^{-1} = \frac{1}{\lambda}B$$

  exists and is bounded. Hence $\sigma(A)$ is contained in the disk $\{\lambda \in \mathbb{C} \mid |\lambda| \leq \|A\|\}$.

- $\sigma(A)$ is closed, or in other words: its complement is open. Take $\lambda \notin \sigma(A)$, so $R_\lambda = (\lambda I - A)^{-1}$ exists and is bounded. Let μ be such that $|\lambda - \mu| < \|R_\lambda\|^{-1}$ and therefore $\|(\lambda - \mu)R_\lambda\| < 1$. This means that

  $$\begin{aligned} I - (\lambda - \mu)R_\lambda &= I + [(\mu I - A) - (\lambda I - A)]R_\lambda \\ &= I + (\mu I - A)R_\lambda - I \\ &= (\mu I - A)R_\lambda \end{aligned}$$

  has a bounded inverse; call it S. But then $R_\lambda S$ is the inverse of $(\mu I - A)$ because $(\mu I - A)R_\lambda S = I$ and also $R_\lambda S(\mu I - A) = R_\lambda S(\mu I - A)R_\lambda R_\lambda^{-1} = R_\lambda R_\lambda^{-1} = I$. The norm $\|R_\lambda S\| \leq \|R_\lambda\| \, \|S\| < \infty$ as well. This shows that the $\|R_\lambda\|^{-1}$-neighbourhood of λ is disjoint from $\sigma(A)$, hence the complement of $\sigma(A)$ is open.

□

Theorem 51 *If A is a bounded self-adjoint operator on a Hilbert space E, then the eigenvalues are real, and the eigenvectors of different eigenvalues are perpendicular. Also the entire spectrum $\sigma(A)$ is real.*

Proof. If λ is an eigenvalue of A, belonging to a unit eigenvector v, then

$$\lambda = \lambda\langle v, v\rangle = \langle \lambda v, v\rangle = \langle Av, v\rangle = \langle v, Av\rangle = \langle v, \lambda v\rangle = \overline{\lambda}\langle v, v\rangle = \overline{\lambda}.$$

Therefore λ is real. If $\lambda \neq \mu$ are two different eigenvalues, belonging to eigenvectors v and w, then

$$\lambda\langle v, w\rangle = \langle \lambda v, w\rangle = \langle Av, w\rangle = \langle v, Aw\rangle = \langle v, \mu w\rangle = \overline{\mu}\langle v, w\rangle,$$

and because $\lambda \neq \mu = \overline{\mu}$, the only possibility is $\langle v, w\rangle = 0$.

The proof that $\sigma(A)$ is real is a bit more involved. Take $\lambda \in \mathbb{C} \setminus \mathbb{R}$, so $\text{Im } \lambda \neq 0$. To show that $\lambda I - A$ has a bounded inverse, we need to check several things:

- $\lambda I - A$ is one-to-one: We have

$$\begin{aligned} \text{Im } \langle(\lambda I - A)u, u\rangle &= \frac{1}{2}(\langle(\lambda I - A)u, u\rangle - \overline{\langle(\lambda I - A)u, u\rangle}) \\ &= \frac{1}{2}(\lambda\|u\|^2 - \overline{\lambda}\|u\|^2 + \langle Au, u\rangle - \overline{\langle Au, u\rangle}) \\ &= \text{Im } \lambda\|u\|^2, \end{aligned}$$

  because $\langle Au, u\rangle = \langle u, Au\rangle = \overline{\langle Au, u\rangle}$. Therefore, by the Cauchy-Schwarz inequality,

$$|\text{Im } \lambda| \; \|u\|^2 = |\text{Im } \langle(\lambda I - A)u, u\rangle| \leq |\langle(\lambda I - A)u, u\rangle| \leq \|\lambda I - A)u\| \; \|u\|.$$

  If $u \neq 0$, then we can divide out a factor $\|u\|$, so

$$|\text{Im } \lambda| \; \|u\| \leq \|(\lambda I - A)u\|. \tag{10}$$

  Because $\text{Im } \lambda \neq 0$, we obtain $\ker(\lambda I - A) = \{0\}$, or in other words, $\lambda I - A$ is one-to-one.

- The inverse R_λ is bounded: If v belongs to the range $R(\lambda I - A)$, then (10) shows that $\|R_\lambda v\| \leq |\text{Im } \lambda|^{-1} \; \|v\|$, so $\|R_\lambda\| \leq |\text{Im } \lambda|^{-1} < \infty$.

- The range $R(\lambda I - A)$ lies dense in E: First let D be the closure of the range $R(\lambda I - A)$. Since E is a Hilbert space, $E = D \oplus D^\perp$, and if $v \in D^\perp$, then

$$0 = \langle(\lambda I - A)u, v\rangle = \langle u, (\overline{\lambda} I - A)v\rangle \text{ for all } u \in E.$$

  But this means that $(\overline{\lambda} I - A)v = 0$, and hence either $\overline{\lambda}$ is an eigenvalue of A (which is impossible, because $\overline{\lambda}$ is not real) or $v = 0$. Therefore $D^\perp = \{0\}$ and $D = E$, so the range of $\lambda I - A$ lies dense in E.

- $R(\lambda I - A)$ is closed: Take any $y \in E$. There exists a sequence $\{y_n\}_n \subset R(\lambda I - A)$ that converges to y. Let $x_n = R_\lambda(y_n)$. Because $\{y_n\}_n$ is Cauchy, and R_λ is bounded, also $\{x_n\}_n$ is Cauchy, and therefore convergent in the Hilbert space E. Call the limit x. Then by continuity of $\lambda I - A$,

  $$(\lambda I - A)x = \lim_{n\to\infty} (\lambda I - A)x_n = \lim_{n\to\infty} y_n = y.$$

  Therefore $y \in R(\lambda I - A)$. Because $y \in D$ was arbitrary, $R(\lambda I - A) = D$.

Together, this shows that $R(\lambda I - A) = E$. □

We know that the spectrum of an operator A is contained in the disk of radius $\|A\|$. In many cases, we can actually find eigenvalues on the boundary of this disk.

♢ **Theorem 52** *If A is a compact self-adjoint operator on a Hilbert space, then at least one of the numbers $\|A\|$ and $-\|A\|$ is an eigenvalue of A.*

Proof. See Kreyszig (Theorems 9.2-2 and 9.2-3) or Young (Theorems 7.18 and 8.10). □

The main theorem of this chapter is called the Spectral Theorem of compact self-adjoint operators.

Theorem 53 *If A is a compact self-adjoint operator on a Hilbert space H, then there exists a finite or infinite sequence of eigenvector $\{v_n\}_n$ corresponding to real eigenvalues $\{\lambda_n\}_n$ such that*

$$Ax = \sum_n \lambda_n \langle x, v_n\rangle v_n \text{ for all } x \in H.$$

Moreover, if $\{\lambda_n\}_n$ is infinite, then $\lambda_n \to 0$ as $n \to \infty$.

This theorem states that A has a basis of eigenvectors. For each $\lambda_n \neq 0$, the eigenspace is finite dimensional. Even if $\dim(H) = \infty$ and there are only finitely many eigenvalues, then 0 is also an eigenvalue, and the corresponding eigenspace is an infinite dimensional Hilbert space, so any orthonormal basis of it is automatically an orthonormal basis of eigenvectors (with eigenvalue 0).

Proof. The theorem is obviously true if $Ax \equiv 0$. From previous results, we already know that all eigenvalues are real, and that for each $\varepsilon > 0$, there are only finitely many eigenvalues with $|\lambda_n| > \varepsilon$. So let us start finding eigenvectors.

By Theorem 52, there exists at least one eigenvector v_1 with eigenvalue $\lambda_1 = \pm\|A\| \neq 0$. Assume that $\|v_1\| = 1$. Clearly A leaves $\text{span}(v_1)$ invariant, but also $\{v_1\}^\perp$ is invariant, because

$$0 = \langle v_1, u\rangle = \frac{1}{\lambda_1}\langle \lambda_1 v_1, u\rangle = \frac{1}{\lambda_1}\langle Av_1, u\rangle = \frac{1}{\lambda_1}\langle v_1, Au\rangle,$$

for all $u \in \{v_1\}^\perp$.

Write $A_1 = A$ and $H_2 = \{v_1\}^\perp$, then the restriction $A_2 := A_1|H_2$ is again a compact self-adjoint operator, and $\|A_2\| \leq \|A_1\|$. Therefore we can repeat the above argument to find the next unit eigenvector v_2, corresponding to the next eigenvalue λ_2, with $|\lambda_2| \leq |\lambda_1|$.

We continue by induction: $H_n = \{v_1, v_2, \ldots, v_{n-1}\}^\perp$ and the restriction $A_n = A_{n-1}|H_n$ is again a compact self-adjoint operator, having unit eigenvector v_n with eigenvalue $\lambda_n = \pm\|A_n\|$. Note that v_n is perpendicular to all previous eigenvectors, so $\{v_k\}_k$ becomes automatically orthonormal.

The induction stops if $\|A_N\| = 0$ for some N. But then

$$Ax = \sum_{n=1}^{N-1} \lambda_n \langle x, v_n\rangle v_n + A_N x = \sum_{n=1}^{N-1} \lambda_n \langle x, v_n\rangle v_n.$$

Otherwise, i.e. if $\|A_n\| > 0$ for all n, the inductions gives an infinite system of orthonormal eigenvectors. Observe that

$$y_k = x - \sum_{n=1}^{k-1} \langle x, v_n\rangle v_n \in H_k.$$

Hence $x = y_k + \sum_{n=1}^{k-1}\langle x, v_n\rangle v_n$, and by Pythagoras Theorem

$$\|x\|_H^2 = \|y_k\|_H^2 + \sum_{n=1}^{k-1} |\langle x, v_n\rangle|^2.$$

This shows that the sequence $\{y_k\}_k$ is bounded by $\|x\|_H$. In the limit, we find

$$\begin{aligned} \|Ax - \sum_{n=1}^{\infty} \lambda_n \langle x, v_n\rangle v_n\|_H &= \lim_{k\to\infty} \|Ax - \sum_{n=1}^{k-1} \lambda_n \langle x, v_n\rangle v_n\|_H \\ &= \lim_{k\to\infty} \|A(x - \sum_{n=1}^{k-1} \langle x, v_n\rangle v_n)\|_H \\ &\leq \lim_{k\to\infty} \|A_k\| \; \|x - \sum_{n=1}^{k-1} \langle x, v_n\rangle v_n\|_H \\ &= \lim_{k\to\infty} \|A_k\| \; \|y_k\|_H \leq \lim_{k\to\infty} \|A_k\| \|x\|_H = 0 \end{aligned}$$

This proves the theorem. $\square$

Notation used for several linear spaces:

$\mathbb{R}^n, \mathbb{C}^n, \mathbb{K}^n,\ \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

$M_{m\times n}(\mathbb{K}) = \{A : A$ is an $m \times n$ matrix with entries in $\mathbb{K}\}$.

$P^d([a,b]) = \{p : [a,b] \to \mathbb{K} : p$ polynomial of degree $\leq d\}$.

$P^d([a,b]) = \{p : [a,b] \to \mathbb{K} : p$ polynomial of any degree $\}$.

$C([a,b],\mathbb{K}) = \{f : [a,b] \to \mathbb{K} : f$ is continuous $\}$.

$C^k([a,b],\mathbb{K}) = \{f : [a,b] \to \mathbb{K} : f$ is k times continuously differentiable $\}$.

$\ell^p = \{x = (x_n)_{n=1}^{\infty} \mid x_n \in \mathbb{K},\ \sum_n |x_n|^p < \infty\}$.

$\ell^\infty = \{x = (x_n)_{n=1}^{\infty} \mid x_n \in \mathbb{K},\ \sup_n |x_n| < \infty\}$.

$L^p([a,b]) = \{f : [a,b] \to \mathbb{K} \mid \int_a^b |f(t)|^p dt < \infty\}$

$L^\infty([a,b]) = \{f : [a,b] \to \mathbb{K} \mid \sup\{|f(t)| : t \in [a,b]\} < \infty\}$.