MAT3004: Introduction to Function Spaces. ClassTest I.

March 15th 2016.

Problem I.

a) (1 point) Let X and Y be metric spaces. Define what it means for a function $f : X \to Y$ to be *continuous* at $x_0 \in X$.

b) (3 points) Let $X = Y = l_\infty$ be the space of all bounded sequences $x = \{x_n\}_{n=1}^\infty$ with the usual sup-norm and the function $f : X \to Y$ be defined via

$$f(x)_n := \sin(nx_n).$$

Is this function *continuous* at $x = 0$? Justify your answer.

Problem II.

a) (1 point) Let X be a metric space. Define the *closure* $\bar{V}$ of a set $V \subset X$.

b) (2 points) Let $A, B \subset X$ be two subsets of a metric space X. Prove that

$$\overline{A \cup B} = \bar{A} \cup \bar{B}.$$

Problem III.

a) (1 point) Define the Lebesgue space $L_1(0, 1)$.

b) (2 points) Let

$$f(x) := \frac{1}{\sqrt{\log(1 + x)}}.$$

Does this function belong to $L_1(0, 1)$? Justify your answer.

Solutions

Problem I.

a) A function $f : X \to Y$ is continuous at $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $d(f(x), f(x_0)) < \varepsilon$ if $d(x, x_0) < \delta$.

b) The function is *not* continuous at $x = 0$. Indeed, consider a sequence $x^k := \frac{\pi}{2k} e^k$, where $e^k = (0, \cdots, 1, \cdots)$ is the kth coordinate vector (one is on the kth position and all other elements are zeros). Then, clearly, $x^k \to 0$ in l_∞, but $f(x^k) = e^k$ does not converge to $f(0) = 0$.

Problem II.

a) A point x_0 belongs to the closure $\bar{V}$ if and only if there exists a sequence $x_n \in V$ such that $x_n \to x_0$ in X.

b) Let $x_0 \in \overline{A \cup B}$. Then, there exists a sequence $x_n \in A \cup B$ such that $x_n \to x_0$ in X. By the definition of a union, either $x_n \in A$ or $x_n \in B$ and at least in one them, say, A contains infinitely many terms of the sequence, we denoted them by x_{n_k}. Since $x_{n_k} \to x_0$, we conclude that $x_0 \in \bar{A}$ and therefore $x_0 \in \bar{A} \cup \bar{B}$.

Assume now that $x_0 \in \bar{A} \cup \bar{B}$. Then x_0 belongs at least to one of them, say, $x_0 \in \bar{A}$. By definition, there exists a sequence $x_n \in A$ such that $x_n \to x_0$. Since $x_n \in A \cup B$, we conclude that $x_0 \in \overline{A \cup B}$.

Problem III.

a) By definition, the Lebesgue space $L_1(0, 1)$ is a *completion* of the space $C[0, 1]$ of continuous functions on a segment $[0, 1]$ with respect to the norm

$$\|f\|_{L_1} := \int_0^1 |f(x)|\, dx.$$

b) The function f is continuous everywhere except of $x = 0$, so we only need to study the neighbourhood of zero. By the criterion, we need to check whether or not the limit

$$L := \lim_{\varepsilon \to 0} \int_\varepsilon^1 \frac{1}{\sqrt{\log(1 + x)}}\, dx$$

is infinite or finite. We recall that $\log(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + \cdots \geq x - x^2/2 \geq x/2$ and, consequently $f(x) \geq x/2$ on $[0, 1]$. Thus

$$L \leq \lim_{\varepsilon \to 0} \int_\varepsilon^1 \frac{\sqrt{2}}{\sqrt{x}}\, dx = 2\sqrt{2} \lim_{\varepsilon \to 0} \sqrt{x}\Big|_{x=\varepsilon}^{x=1} = 2\sqrt{2} \lim_{\varepsilon \to 0}(1 - \sqrt{\varepsilon}) = 2\sqrt{2} < \infty$$

and $f \in L_1(0, 1)$.