MAT3004: Introduction to Function Spaces. ClassTest II.

May 10th 2016.

Problem I. Let (X, d) be a metric space.

a) (1 point) Define what it means for a function $f : X \to X$ to be the *contraction* on X.

b) (1 point) State the Banach Contraction Theorem.

c) (3 points) Is the function

$$f(x) := \frac{\pi}{2} + x - \arctan(x)$$

a contraction on the space $\mathbb{R}$ endowed with the standard norm? Justify your answer.

Problem II. Let $X := C[0, T]$, $T > 0$, be the space of continuous functions $f : [0, T] \to \mathbb{R}$ with the standard sup-norm and let $F : X \to X$ be defined by

$$F(f)(t) := 1 + 2\int_0^t f(s)\, ds.$$

a) (3 points) Prove that F is a contraction on X *if and only if* $T < \frac{1}{2}$.

b) (2 points) Find all fixed points of this map.

Problem III.

a) (1 point) Define the inner product (x, y) on a vector space H.

b) (1 point) State the Cauchy-Schwarz inequality.

c) (1 point) Define the angle between two vectors x and y in the inner product space H.

d) (2 points) Let $x := \{2^{-n}\}_{n=1}^{\infty}$ and $y := \{3^{-n}\}_{n=1}^{\infty}$. Compute the angle between the sequences x in y in the space $H = l_2$ with the standard Euclidean inner product.

Solutions

Problem I.

a) A function $f : X \to X$ is a contraction on X if there exists $\kappa < 1$ such that

$$d(f(x), f(y)) \le \kappa d(x, y)$$

for all $x, y \in X$.

b) BCT: If $f : X \to X$ is a *contraction* on a *complete* metric space X then f has a *unique* fixed point $p \in X$ (i.e., $p = f(p)$).

c) This function is NOT a contraction on $X = \mathbb{R}$. Indeed, according to the mean value theorem,

$$|f(x) - f(y)| = |f'(\xi)||x - y|$$

for some ξ inbetween x and y. But $f'(\xi) = 1 - \frac{1}{1+\xi^2} = \frac{\xi^2}{1+\xi^2}$. Although $f'(\xi) < 1$ for every finite ξ, we see that $\lim_{\xi\to\infty} f'(\xi) = 1$ and by this reasonm for every $\kappa < 1$, the inequality

$$|f(x) - f(y)| \le \kappa|x - y|$$

will be violated if x and y are both large enough.

Problem II.

a) Let $T < 1/2$ and let us check that F is a contraction. Take $f_1, f_2 \in X$, then

$$|F(f_1)(t) - F(f_2)(t)| \le 2 \int_0^t |f_1(s) - f_2(s)|\, ds \le$$

$$\le 2\|f_1 - f_2\|_{sup} \int_0^T ds = 2T\|f_1 - f_2\|_{sup}.$$

Taking the supremum with respect to x, we see that F is a contraction with the contraction factor $\kappa = 2T < 1$.

Let now $T \ge 1/2$. Consider two functions $f_1(t) = 0$ and $f_1(t) = 1$. Then, obviously, $\|f_1 - f_2\|_{sup} = 1$ and $F(f_1)(t) = 1$, $F(f_2)(t) = 1 + 2t$. Thus,

$$\|F(f_1) - F(f_2)\|_{sup} = \|2t\|_{sup} = 2T \ge 1$$

and F is not a contraction.

b) To find a fixed point $p(t)$, we need to solve the integral equation

$$p(t) = 1 + 2 \int_0^t p(s)\, ds$$

in the space of continuous functions. Since $p(t)$ is continuous, the integral is continuously differentiable and from the equality, we see that $p \in C^1[0, T]$. Moreover, from the equation, we see that $p(0) = 1$. Differentiating it by t, we end up with the ODE $p'(t) = 2p(t)$. Thus $p(x) = p(0)e^{2t}$ and finally $p(t) = e^{2t}$.

Problem III.

a) The function $(x, y) : H \times H \to \mathbb{R}$ is an inner product on H if it is

1) Linear with respect to every argument: $(\alpha x_1 + \beta x_2, y) = \alpha(x_1, y) + \beta(x_2, y)$ for every $x_1, x_2, y \in H$, $\alpha, \beta \in \mathbb{R}$ and the analogous identity holds with respect to the variable y (in other words, (x, y) is *bi-linear*);

2) Symmetric: $(x, y) = (y, x)$ for all $x, y \in H$;

3) Positive definite: $(x, x) \geq 0$ and $(x, x) = 0$ if and only if $x = 0$.

b) Cauchy-Schwartz inequality: $|(x, y)| \leq \|x\|\|y\|$ for any $x, y \in H$.

c) The angle between two vectors $x, y \in H$ is defined by

$$\varphi := \arccos\left(\frac{(x, y)}{\|x\|\|y\|}\right).$$

d) We first need to compute (x, y), $\|x\|$ and $\|y\|$. Indeed, using the summation rule for the geometric power series, we have

$$(x, y) = \sum_{n=1}^{\infty} 2^{-n}3^{-n} = \frac{1}{5}, \|x\|^2 = \sum_{n=1}^{\infty} 4^{-n} = \frac{1}{3}, \quad \|y\|^2 = \sum_{n=1}^{\infty} 9^{-n} = \frac{1}{8}.$$

Thus, $\varphi = \arccos(\frac{2\sqrt{6}}{5}) \sim 11.5°$.