# Lecture Classical Fourier series.

Let now $H = L^2([-\pi,\pi])$ - separable H-space with the inner product

$$\boxed{(f,g) := \int_{-\pi}^{\pi} f(x)g(x)dx}$$

H is a <u>completion</u> of $C[a,b]$ with respect to this inner product $\Rightarrow$ continuous functions are <u>dense</u> in H.

Consider the system of vectors $\{1, \sin nx, \cos nx\}$ in H.

<u>Lemma</u>: $\{1, \sin nx, \cos nx\}$ - orthogonal system in H.

<u>Proof</u>: Need to check

$$(1, \sin nx) = \int_{-\pi}^{\pi} \sin nx = 0 \; ; \quad (1, \cos nx) = \int_{-\pi}^{\pi} \cos nx = 0$$

$$(\sin nx, \sin mx) = \int_{-\pi}^{\pi} \sin nx \cdot \sin mx \, dx = \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos[(n-m)x] - \cos[(n+m)x]\right] dx = 0 \quad \text{if } \boxed{m \neq n}$$

$$(\cos nx, \cos mx) = \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos[(n-m)x] + \cos[(n+m)x]\right] dx = 0 \quad \boxed{m \neq n}$$

$$(\cos nx, \sin mx) = \int_{-\pi}^{\pi} \frac{1}{2}\left[\sin[(n+m)x] + \sin[(n-m)x]\right] dx = 0 \quad \forall m,n$$

$\square$

However, this system is not <u>orthonormal</u>:

$$\|\sin nx\|^2 = \int \sin^2 nx \, dx = \int \frac{1-\cos 2nx}{2} dx = \pi$$

$$\|\cos nx\|^2 = \int \cos^2 nx \, dx = \int \frac{1+\cos 2nx}{2} dx = \pi$$

$$\|1\|^2 = \int_{-\pi}^{\pi} 1 \, dx = 2\pi$$

$$\boxed{\text{Classical Fourier series}} = \boxed{\text{Fourier series w.r.t. } \{1, \sin nx, \cos nx\}}$$

II

$f \in L^2(-\pi,\pi)$

$$\boxed{f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx}$$

Classical Fourier series of $f$.

+ Bessel inequality

$$\boxed{2\pi a_0^2 + \pi \sum_{n=1}^{\infty} a_n^2 + b_n^2 \le \|f\|_{L^2}^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx}$$

in a fact there is an equality here (Parseval equality), but to know that we need to prove that $\{1, \sin nx, \cos nx\}$ is complete.

Natural questions about the convergence:

1. Completness and the convergence in mean (in the integral $L^2$-metric)?

$$\boxed{f_N(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx}$$

$$\boxed{\int_{-\pi}^{\pi} |f(x) - f_N(x)|^2 dx \to 0} \quad \boxed{N \to \infty}$$

From the abstract theory this convergence hold $\forall \boxed{f \in H}$ if $\{1, \sin nx, \cos nx\}$ is complete.

2. Point-wise convergence?

$f_N(x) \to f(x)$ or to something else $\forall$ fixed $x \in [-\pi,\pi]$

3. Uniform convergence = convergence in $C[-\pi,\pi]$?

$$\boxed{\max_{x \in [-\pi,\pi]} |f(x) - f_N(x)| \to 0} \quad \boxed{N \to \infty}$$

4. Smoothness and rate of convergence?

Simple, but important observations:

A) Uniform convergence is the strongest one. It implies point-wise and mean convergence.

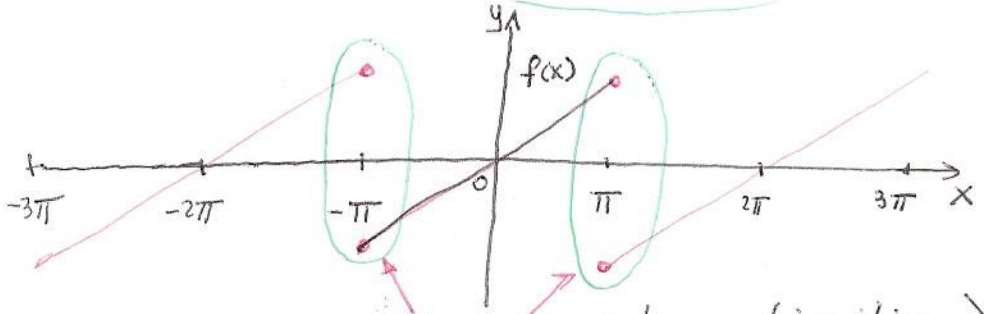
B) All $f_N(X)$ are continuous. Thus if $f_N \to f$ uniformly $\Rightarrow$ $f$ must be continuous.

C) $f \in C[a,b]$ is not sufficient for uniform convergence.

Indeed, all $f_N(X)$ are $2\pi$-periodic $\Rightarrow$ the limit $f(X)$ must be $2\pi$-periodic. Let us extend the function $f \in C[-\pi,\pi]$ to $\mathbb{R}$ $2\pi$-periodically

$\boxed{y = f_{per}(X)}$

periodic extension of $f$.




jumps (discontinuities) if $\boxed{f(-\pi) \neq f(\pi)}$ !!

$\Rightarrow$

If $f_N \to f$ uniformly in $C[-\pi,\pi]$ $\Rightarrow$ $\boxed{f_N \to f_{per}}$ uniformly in $C(\mathbb{R})$ $\Rightarrow$ $f_{per}(X)$ must be continuous!

$\Rightarrow$ Necessary conditions for uniform convergence

a) $\boxed{f \in C(-\pi,\pi)}$ is continuous.

b) $\boxed{f(-\pi) = f(\pi)}$ – do not forget this!

Surprising fact: it is not sufficient for uniform (and even for the point-wise) convergence!!!

$\Rightarrow$ There exist CONTINUOUS periodic functions such that $f_N(X)$ do not converge to $f(X)$ point-wise!

Very difficult to construct an explicit example!

Necessary and sufficient conditions are not known!

# Sufficient conditions for point-wise convergence

Recall that $f$ is piece-wise continuously differentiable on $[a,b]$ if there are finitely many jump points $\{x_1, \dots x_n\}$ such that $\boxed{f \in C^1([a,b] \setminus \{x_1, \dots x_n\})}$

and

$$\boxed{f(x_i-) = \lim_{\substack{x \to x_i \\ x < x_i}} f(x)}, \quad \boxed{f(x_i+) := \lim_{\substack{x \to x_i \\ x > x_i}} f(x)}$$

$$\boxed{f'(x_i-) = \lim_{\substack{x \to x_i \\ x < x_i}} f'(x)}, \quad \boxed{f'(x_i+) := \lim_{\substack{x \to x_i \\ x > x_i}} f'(x)}$$

exist $\forall i$.

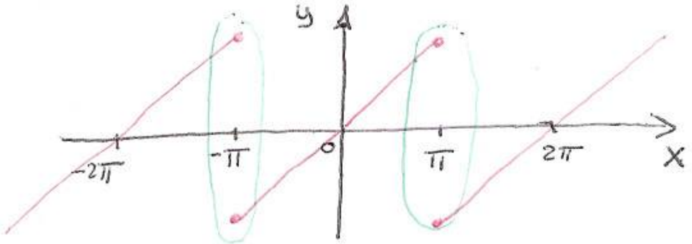
**Theorem** (Dirichlet) Let $f \in L^2([-\pi, \pi])$ and let $f_{per}(x)$ be a $2\pi$-periodic extension of $f$. Let $f_{per}(x)$ be piece-wise $C^1$. Then

$$\boxed{f_N(x) \to f_{per}(x)}$$ at every $x$ where $f_{per}(x)$ is continuous.

and

$$\boxed{f_n(x_i) \longrightarrow \frac{f_{per}(x_i+) + f_{per}(x_i-)}{2}}$$ at every jump point

Example: Please compute the Fourier expansions for $f(x) = x$ on $[-\pi, \pi]$




and verify that

$$\boxed{f_N(-\pi) = f_N(\pi) \equiv 0}$$

One more simple observation

$\sin nx$ is odd and $\cos nx$ is even

$\Rightarrow$ if $\boxed{f(x) \text{ is odd}} \Rightarrow \boxed{a_n = 0 \ \forall n}$ we have only sin in the expansion

if $\boxed{f(x) \text{ is even}} \Rightarrow \boxed{b_n = 0 \ \forall n}$ only cos in the expansions.

## Lecture  Classical Fourier series Part II (I)

Regularity and the rate of convergence — more observations

$$\boxed{f(x) \sim a_0 + \sum_{n=1}^{\infty}{}' a_n \cos nx + b_n \sin nx}$$

Assume that $f'(x)$ exists ($f \in C'[-\pi, \pi]$). Then formally

$$(*) \quad \boxed{f'(x) \sim \sum_{n=1}^{\infty}{}' (n b_n) \cos nx + (-n a_n) \sin nx}$$

Justification: Let us find the F-coefficients of $f' \in H$.

$$\boxed{a_0(f') = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)\,dx = \frac{1}{2\pi}[f(\pi) - f(-\pi)]}$$

$$a_n(f') = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\,dx = \frac{1}{\pi} f(x)\cos nx\Big|_{-\pi}^{\pi} + \frac{1}{\pi}\int f(x) n \sin nx = $$
$$= n\, b_n(f) + (-1)^n/\pi\, [f(+\pi) - f(-\pi)]$$

$$b_n(f') = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin nx = \frac{1}{\pi} f(x)\sin nx\Big|_{-\pi}^{\pi} - n\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$
$$= -n\, a_n(f).$$

Conclusion: if $\boxed{f \in C'([-\pi, \pi])}$ and $\boxed{f(-\pi) = f(\pi)}$

$\Rightarrow$ (*) is a Fourier series for the function $f'(x)$

no jump condition for $f_{per}(x)$

Attention!: if $\boxed{f(-\pi) \neq f(\pi)}$ we are unable to differentiate the Fourier series! (the result of formal differentiation is a divergent series)

Corollary: Let $\boxed{f \in C'(-\pi, \pi)}$ and $\boxed{f(-\pi) = f(\pi)}$.

Then

$$\boxed{\sum_{n=1}^{\infty}{}' n^2 (a_n^2 + b_n^2) \leq \| f' \|_{L^2}^2 < \infty}$$

just the Bessel inequality for the function $f'$.

Analogously, if $f_{per}(x) \in C^k(\mathbb{R})$ - k-times continuously differentiable, then

$$\sum_{n=1}^{\infty}{}' n^{2k}(a_n^2 + b_n^2) \le \|f^{(k)}\|^2_{L^2} < \infty$$

in particular, $a_n \le \dfrac{c}{n^k}$, $b_n \le \dfrac{c}{n^k}$

Rate of decay of $a_n$ and $b_n$ $\sim$ Rate of convergence of $f_N(x)$ $\sim$ Smoothness of $f_{per}(x)$

Corollary: Let $f \in C'(-\pi,\pi)$ and $f(-\pi) = f(\pi)$. Then $f_N(x) \rightrightarrows f(x)$ uniformly (in the sup-metric)

Proof: i) I claim that

$$\sum_{n=1}^{\infty}{}' |a_n| + |b_n| < \infty$$

Indeed, due to Cauchy-Schwartz inequality

$$\sum_{n=1}^{\infty}{}' |a_n| + |b_n| = \sum_{n=1}^{\infty}{}' \frac{1}{n} \cdot n a_n + \sum_{n=1}^{\infty}{}' \frac{1}{n} \cdot n b_n \le \left(\sum_{n=1}^{\infty}{}' \frac{1}{n^2}\right)^{1/2} \left[\left(\sum_{n=1}^{\infty}{}' n^2 a_n^2\right)^{1/2} + \left(\sum_{n=1}^{\infty}{}' n^2 b_n^2\right)^{1/2}\right] \le 2\left(\frac{\pi}{6}\right)^{1/2} \left(\sum_{n=1}^{\infty}{}' n^2(a_n^2 + b_n^2)\right)^{1/2} < \infty.$$

Bessel inequality for $f'$.

2) $f_N(x)$ is a Cauchy sequence in $C[-\pi,\pi]$:

Indeed,

$$\|f_N - f_{N+M}\|_{sup} = \left\|\sum_{n=N+1}^{N+M}{}' a_n \cos nx + b_n \sin nx\right\| \le \sum_{n=N+1}^{N+M}{}' |a_n| + |b_n| \le$$

$$\le \sum_{n=N+1}^{\infty}{}' |a_n| + |b_n| \to 0 \quad \text{as } N \to \infty$$

$\Rightarrow$ $f_N(x)$ is a Cauchy sequence in $C(-\pi,\pi)$ $\Rightarrow$ (the space C is complete) $f_N \to f_0 \in C(-\pi,\pi)$ in the sup-metric.

By Dirichlet theorem $f_N(x) \to f(x)$ point-wise

$\Rightarrow$ $f_0(x) = f(x)$

$\square$

Corollary: Let $\boxed{f \in C^1(-\pi,\pi)}$ and $\boxed{f(-\pi)=f(\pi)}$

Then,

$$\boxed{\| f - f_N \|_{L^2} \to 0} \quad \boxed{N \to \infty}$$

Proof: Indeed, since $f_N \rightrightarrows f$ (uniformly),

$$\| f - f_N \|^2_{L^2} := \int_{-\pi}^{\pi} | f_N(x) - f(x) |^2 dx \to 0 \quad \text{as } \boxed{N \to \infty}$$

$$\leq 2\pi \| f_N - f \|_{sup}$$

$\square$

**Theorem**: The orthogonal system $\{1, \sin nx, \cos nx\}$ is complete in $\boxed{H = L^2(-\pi,\pi)}$

Proof: Due to the completness criterium, we only need to check that $f_N \to f$ in the $L^2$-norm for all $f$ from the dense set $V \subset H$. Take

$$\boxed{V = \{ f \in C^1(-\pi,\pi),\ f(-\pi) = f(\pi) \}.}$$

Only need to prove that $V$ is dense in $H$ (in the $L^2$-norm).

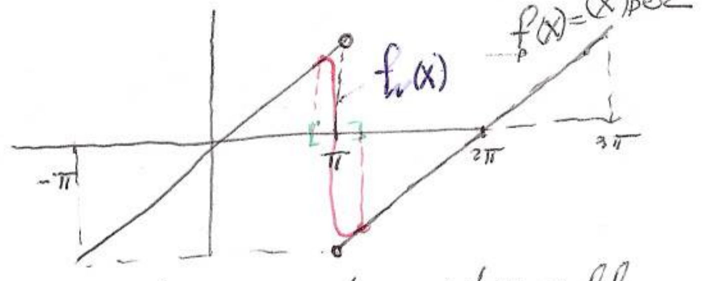
a) $C[-\pi,\pi]$ is dense in $H$ (by definition)

b) ~~$\{X^n\}_{n=0}^{\infty}$~~ algebraic polynoms are dense in $C(-\pi,\pi]$ (in sup-norm $\Rightarrow$ in $L^2$-norm as well) (Weierstrass theorem)

$\Rightarrow$ Algebraic polynoms are dense in $H$.

$\Rightarrow$ it is sufficient to approximate any function $\boxed{f(x) = x^n}$ by functions from $V$.

For even $n$, $x^n \in V$. We only need to consider $\boxed{n = 2k+1}$




$f_N \in V$

correct $f(x)$ in a small interval near $\boxed{x = \pi}$ and $\boxed{x = -\pi}$ in such way that $\boxed{f_N(-\pi) = f_N(\pi) = 0}$ (see picture)

since the correction intervall is small (say of length $2/N$) $\boxed{f_N \to f}$ in the $L^2$-metric. $\Rightarrow$ $V$ is dense in $H$ and $\{1, \sin nx, \cos nx\}$ is complete $\square$

# Lecture (last) Proof of the Dirichlet theorem

We have proved before that $\{1, \cos nx, \sin nx\}$ is complete in $H = L^2(-\pi,\pi)$, i.e., that for every $f \in L^2$

$$\boxed{\|f - f_N\|_{L^2}^2 = \int_{-\pi}^{\pi} |f(x) - f_N(x)|^2 dx \to 0}$$

Based on the Dirichlet theorem

Theorem (Dirichlet) Let $f \in L^2(-\pi,\pi)$ and let the periodic extension $f_{per}(x)$ of the function is piece-wise continuously differentiable. Then

$$\boxed{f_N(x_0) := a_0 + \sum_{n=1}^{N} a_n \cos nx + b_n \sin nx \to \frac{f_{per}(x_0+) + f_{per}(x_0-)}{2}}$$

for all $x_0 \in \mathbb{R}$.

Proof: Let us first obtain the integral formula for $f_N(x)$:

$$a_n \cos nx + b_n \sin nx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \cos ny \, dy \cdot \cos nx + \frac{1}{\pi} \int f(y) \sin ny \, dy \, \sin nx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) [\cos nx \cdot \cos ny + \sin nx \cdot \sin ny] dy$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \cos n(x-y) \, dy.$$

$$f_N(x) = \int_{-\pi}^{\pi} f(y) \left[\frac{1}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{N} \cos n(x-y)\right)\right] dy =$$

$$= \int_{-\pi}^{\pi} f(y) D_N(x-y) \, dy$$

Where the Diriclet kernel $D_N$ is

check!

$$\boxed{D_N(z) := \frac{1}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{N} \cos Nz\right]} = \frac{1}{\pi} \frac{\sin(N+\frac{1}{2})z}{\sin \frac{1}{2} z}$$

Hint: use that $\boxed{\cos nx = \frac{e^{inx} + e^{-inx}}{2}}$ + the summation rule for the geometric progression.

Properties of the Diriclet kernel:

a) $D_N(z)$ is <u>even</u> and $2\pi$-periodic

b) $\boxed{\int_{-\pi}^{\pi} D_N(z)\,dz = 1}$ and

$$\boxed{\int_{0}^{\pi} D_N(z)\,dz = \int_{-\pi}^{0} D_N(z)\,dz = \frac{1}{2}}$$

c) replacing $f$ by $f_{per}$, we may write

$$f_N(x) = \int_{-\pi}^{\pi} f_{per}(y)\, D_N(y-x)\,dy = \int_{-\pi-x}^{\pi-x} f_{per}(z+x)\, D_N(z)\,dz =$$

$$= \int_{-\pi}^{\pi} f_{per}(z+x)\, D_N(z)\,dz$$

integrals over the period of a periodic function

It is sufficient to prove the convergence at $x=0$ (other $x$'s are analogous).

$$f_N(0) - \frac{f(0+)-f(0-)}{2} = \left[\int_{-\pi}^{0} f(z) D_N(z) - \frac{f(0-)}{2}\right] +$$

$$+\left[\int_{0}^{\pi} f(z) D_N(z) - \frac{f(0+)}{2}\right] = \int_{-\pi}^{0} [f(z)-f(0-)]\, D_N(z)\,dz$$

$$+ \int_{0}^{\pi} [f(z)-f(0+)]\, D_N(z)\,dz = \int_{-\pi}^{\pi} \varphi(z)\, D_N(z)\,dz$$

where

$$\boxed{\varphi(z) := \begin{cases} f_{per}(z) - f_{per}(0-), & z \le 0 \\ f_{per}(z) - f_{per}(0+), & z \ge 0 \end{cases}}$$

Need to prove that

$$\boxed{\int_{-\pi}^{\pi} \varphi(z)\, D_N(z)\,dz \to 0} \quad \boxed{N \to \infty}$$

Since $f_{per}$ is piece-wise smooth, we know that

$$\boxed{\varphi(0) = 0} \qquad \boxed{|\varphi(z)| \le C|z|} \text{ near zero}$$

important!

(since $D_N$ is concentrated near zero)!

$$\int_{-\pi}^{\pi}{}' \varphi(z) D_N(z)\,dz = \frac{1}{\pi} \int_{-\pi}^{\pi}{}' \varphi(z) \frac{\sin(N+\frac{1}{2})z}{\sin\frac{1}{2}z}\,dz =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}{}' \varphi(z)\left[\cos Nz + \sin Nz \cdot \mathrm{ctg}\,\tfrac{1}{2}z\right] dz =$$

$$= \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi}{}' \varphi(z)\cos Nz\,dz}_{\text{Fourier coefficient of } \varphi(z)} + \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi}{}' \varphi(z)\,\mathrm{ctg}\,\tfrac{1}{2}z \cdot \sin Nz\,dz}_{\text{Fourier coefficient for } \varphi(z)\mathrm{ctg}\frac{1}{2}z}$$

The function $\varphi$ is from $L^2 \Rightarrow$ Fourier coefficients must tend to zero $\Rightarrow$ first term tends to zero
(Bessel inequality) as $\boxed{N \to \infty}$

(First term is trivial and do not require any assumptions on $f$).

The second one:

$$\varphi(z)\,\mathrm{ctg}\,\tfrac{1}{2}z = \underbrace{\frac{\varphi(z)}{z}}_{\text{Bounded}} \cdot \underbrace{\cos\tfrac{1}{2}z \cdot \frac{z}{\sin\frac{1}{2}z}}_{\text{Bounded}}$$

$\Rightarrow \varphi(z)\,\mathrm{ctg}\,\frac{1}{2}z \in L^2 \Rightarrow$ the second term also tend to 0. $\square$

Observations AFTER the proof:

1) Localization: the convergence at point $x = x_0$ depends only on the behavior of $f$ near $x = x_0$.
($\varphi(z)\,\mathrm{ctg}\,z/2$ can have only one pole at $z=0$)
the behavior outside of zero neighborhood is not essential.

2) The assumptions on $f$ can be weakened
For convergence at $x = 0$, it is sufficient that

$$\boxed{\int_{-\pi}^{0} \frac{|f(x)-f(0-)|}{|x|}\,dx + \int_{0}^{\pi}{}' \frac{|f(x)-f(0+)|}{|x|}\,dx < \infty}$$

Dini condition

(Used the fact that $\boxed{\int' f(x)\sin Nx\,dx \to 0}$ $\forall\ \boxed{f \in L^1}$

3) It is not difficult to prove that
$\boxed{\int_{-\pi}^{\pi}{}' |D_N(z)|\,dz \sim \log N}$ unbounded, but $\boxed{\int_{-\pi}^{\pi}{}' D_N(z)\,dz = 1}$
main obstacle for the validity of the Dirichlet theorem for any continuous function!