(Introduction to) Function spaces (MAT3004 and MAT3010)

Unassessed Coursework I. Deadline: Friday, Week 6

Problem 1.
a) Let (X, d) be a metric space and let

$$\tilde{d}(x, y) = \min\{d(x, y), 1\}.$$

Prove that $\tilde{d}$ is a metric on X. Show that $x_n \to x$ in (X, d) iff $x_n \to x$ in $(X, \tilde{d})$.
b) Let $\mathbb{R}^\infty$ be a space of all sequences (no matter bounded or unbounded) $x = (x_1, x_2, \cdots)$ and let

$$d(x, y) := \sum_{i=1}^{\infty} 2^{-i} \min\{|x_i - y_i|, 1\}.$$

Prove that d is a metric on $\mathbb{R}^\infty$. Prove that the convergence in $(\mathbb{R}^\infty, d)$ is a coordinate-wise convergence.
c) Is $(\mathbb{R}^\infty, d)$ complete? Justify your answer.
d) Show that the square sumable sequences l_2 are *dense* in $(\mathbb{R}^\infty, d)$.

Problem 2.
a) Let $X = C[-1, 1]$ (space of continuous functions $f : [-1, 1] \to \mathbb{R}$) and let

$$\|f\|_1 := \max_{x\in[-1,1]} \{(x^2 - x^3)|f(x)|\}, \quad \|f\|_2 := \max_{x\in[-1,1]} \{(x^2 + x^3)|f(x)|\}.$$

Check that both of them are norms on X. Are these norms equivalent? Justify your answer.
b) Consider a set

$$V := \{f \in X, \ \sup_{x\in[0,1]} \frac{|f(x)|}{|x|} < 1\}.$$

Find the $\operatorname{int} V$.

Problem 3. Let (X, d) be a metric space and let ∂V means "boundary of V".
a) Prove that $\partial(U \cup V) \subset \partial U \cup \partial V$.
b) Give an example such that $\partial(U \cup V) \neq \partial U \cup \partial V$.

Solutions

Problem 1.

a) Positivity and symmetry are evident and we only need to check the triangle inequality. Let $x, y, z \in X$ be arbitrary. Then, if $d(y,z)$ or $d(x,z)$ are greater than one, we have

$$\tilde{d}(x,y) \le 1 \le \tilde{d}(x,z) + \tilde{d}(y,z)$$

So, we only need to consider the case $d(x,z) \le 1$ and $d(y,z) \le 1$. Then,

$$\tilde{d}(x,z) = d(x,z), \quad \tilde{d}(y,z) = d(y,z)$$

and, since d is a metric,

$$\tilde{d}(x,y) \le d(x,y) \le d(x,z) + d(y,z) \le \tilde{d}(x,z) + \tilde{d}(y,z).$$

Thus, $\tilde{d}$ is also a metric. The convergence in d implies the convergence in $\tilde{d}$ since $\tilde{d}(x,y) = d(x,y)$ if $\tilde{d}(x,y) < 1$ or $d(x,y) < 1$.

b) For all i, $d_i(x,y) = 2^{-i}\min\{|x-y|, 1\}$ is a metric on $\mathbb{R}$ according to **a)**. Thus, the triangle inequality for $d(x,y) = \sum_{i=1}^{\infty} d_i(x_i, y_i)$ follows from the triangle inequalities for every d_i. Positivity and symmetry are evident as well as the fact that $d(x,y)$ is well-defined on $\mathbb{R}^\infty \times \mathbb{R}^\infty$. Thus, $d(x,y)$ is a metric.

Let us now $x^k \to x$ in d. Then, $d(x^k, x) \to 0$ as $k \to \infty$ and, in particular, for every $i \in \mathbb{N}$, $2^{-i}\min\{|x_i^k - x_i|, 1\} \to 0$. Thus, $x_i^k \to x_i$ as $k \to \infty$ for every fixed i and $x^k \to x$ coordinate-wise.

Vise versa, let $x^k \to x$ coordinate-wise. We need to prove that $d(x^k, x) \to 0$ as $k \to \infty$. Fix any $\varepsilon > 0$. Then, there exists $N = N(\varepsilon)$ such that $\sum_{i=N+1}^{\infty} 2^{-i} < \varepsilon/2$ and, therefore,

$$d(x^k, x) \le \varepsilon/2 + \sum_{i=1}^{N} |x_i^k - x_i|$$

Due to coordinate-wise convergence $x^k \to x$, we may fix $K = K(\varepsilon)$ such that the second term in the right-hand side is less than $\varepsilon/2$ for $k > K(\varepsilon)$. Thus, $d(x^k, x) < \varepsilon$ for $k > K(\varepsilon)$ and $x^k \to x$ in d. So, we have checked that convergence in d is a point-wise convergence.

c) This space is complete. Indeed, let x^k be a Cauchy sequence in $(\mathbb{R}^\infty, d)$. Then, for every $i \in \mathbb{N}$, $\{x_i^k\}_{i=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $x_i^k \to x_i$ for every i and $x^k \to x$ coordinate-wise. Finally, due to **b)**, $x^k \to x$ in d and $(\mathbb{R}^\infty, d)$ is complete.

d) Actually, even the finite sequences are dense in $\mathbb{R}^\infty$. Indeed, let e_i be ith coordinate vector and $x^k := \sum_{i=1}^{k} x_i e_i$. Then,

$$d(x, x^k) = \sum_{i=k+1}^{\infty} 2^{-i}\min\{1, |x_i|\} \le \sum_{i=k+1}^{\infty} 2^{-i} = 2^{-k} \to 0$$

as $k \to \infty$.

Problem 2.

a) The axioms of the norm are obviously satisfied for both $\|f\|_1$ and $\|f\|_2$ since $x^2 - x^3$ and $x^2 + x^3$ are strictly positive on $(-1, 1)$. However, the norms are *not equivalent* and, factually, none of them is bounded by another one. Indeed, the first is very small for functions with support close to $x = 1$ and the second one is small if this support is close to $x = -1$. For instance, let $f_n \in C[-1, 1]$ be a sequence such that $\|f_n\|_{sup} = f_n(1) = 1$ and $f_n(x) = 0$ for $x \leq 1 - 1/n$. Then,

$$\|f_n\|_2 = 2, \quad \|f_n\|_1 \leq 1/n$$

and the norms cannot be equivalent.

b) The interior of this set is empty. Indeed, for every $f \in V$, we must have $f(0) = 0$ (otherwise the supremum will be infinite) and the set of functions vanishing at $x = 0$, obviously, has an empty interior in C.

Problem 3.

a) Let $x \in \partial(U \cup V)$. Then, $x \in \bar{(U \cup V)} = \bar{U} \cup \bar{V}$ (see lectures) and $x \notin \text{int}(U \cup V)$. Since

$$\text{int}(U) \cup \text{int}(V) \subset \text{int}(U \cup V)$$

(check and give an example that the equality may fail here!), we conclude that $x \notin \text{int}(U)$ and $x \notin \text{int}(V)$. Then, from $x \in \bar{U} \cup \bar{V}$, we conclude that $x \in \bar{U}$ or $x \in \bar{V}$ and, therefore, $x \in \partial(U)$ or $x \in \partial(V)$.

b) You may take, for example, $x = \mathbb{R}$ with $U = \mathbb{Q}$ and $V = \mathbb{R} - \mathbb{Q}$.