Introduction To Function Spaces (Autumn 2012). Coursework I

Deadline: Thursday, November 1st

Problem 1. Let $X = \mathbb{R}^2$ and

$$d(x, y) = \sqrt{|x_1 - y_1|} + \sqrt{|x_2 - y_2|}, \quad x, y \in \mathbb{R}^2.$$

Prove that (X, d) is a metric space. Is $d(x, 0)$ a norm? Justify your answer.

Problem 2. Consider the following two norms on the space $X = C[0, \pi/2]$:

$$\|f\|_1 = \sup_{x\in[0,\pi/2]} \{\sin(x)|f(x)|\}, \quad \|f\|_2 := \sup_{x\in[0,\pi/2]} \{(x + x^2)|f(x)|\}.$$

Are these two norms equivalent? Justify your answer.

Problem 3. a) Let (X, d) be a metric space, $U, V \subset X$. Prove that

$$\partial(U \cup V) \subset \partial U \cup \partial V.$$

b) Give an example when the inclusion is strict.

Problem 4. Let $l_{00} \subset l_\infty$ be the space of sequences which have only *finitely many* non-zero elements

$$x = (x_1, x_2, \cdots, x_N, 0, 0, \cdots)$$

with the usual l_∞-norm (N may be different for different sequences x).

a) Prove that the space l_{00} is not complete.

b)* Prove that the completion of l_{00} is the space $c_0 \subset l_\infty$ of all sequences tending to zero as $n \to \infty$.

Solutions

Problem 1. The positivity and symmetry are obvious, so we only need to check the triangle inequality. To do that we use the elementary inequality

$$\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$$

which is true for all nonnegative $a, b \in \mathbb{R}$:

$$\sqrt{|x_i - y_i|} \le \sqrt{|x_i - z_i| + |z_i - y_i|} \le \sqrt{|x_i - y_i|} + \sqrt{|y_i - z_i|}, \quad i = 1, 2.$$

Summing these inequalities, we get the desired triangle inequality.

$d(x, 0)$ is not a norm since $d(\lambda x, 0) = \sqrt{|\lambda|} d(x, 0) \neq |\lambda| d(x, 0)$.

Problem 2. These norms are equivalent. Indeed, for $x \in [0, \pi/2]$, the following inequality holds:

$$\frac{2}{\pi} x \le \sin(x) \le x$$

and, therefore, $\|f\|_1 \le \|f\|_2$ and

$$\|f\|_2 \le \sup\{(1 + \frac{\pi}{2}) x |f(x|\} \le \sup\{\frac{\pi}{2}(1 + \frac{\pi}{2}) \sin(x) |f(x)|\} = \frac{\pi}{2}(1 + \frac{\pi}{2}) \|f\|_1.$$

Problem 3. Let $x \in \partial(U \cup V)$. Then, by definition, $x \in \overline{U \cup V}$ and $x \notin$ U ∪ V. Since

$$\overline{U \cup V} = \bar{U} \cap \bar{V} \quad \text{and} \quad \operatorname{int}(U) \cup \operatorname{int}(V) \subset \operatorname{int}(U \cup V),$$

see lectures, we conclude that $x \in \overline{U}$ OR $x \in \overline{V}$ AND x is not in the interiors neither of U nor of V. In the first case, $x \in \partial U$ and in he second case $x \in \partial V$ and the inclusion is proved.

Let $X = \mathbb{R}$, $U = [0, 1]$, $V = [1, 2]$. Then, $\partial U \cup V = \{0, 2\}$ and $\partial U \cup \partial V = \{0, 1, 2\}$.

Problem 4. The space l_{00} is not complete since it is not closed in l_∞. Indeed, the sequence $x^k := (1, 1/2, \cdots, 1/k, 0, \cdots)$ belongs to l_{00} and converges in l_∞ to the vector $x = (1, 1/2, 1/3, \cdots)$ which is not in l_{00}.

We need to prove that closure of l_{00} in l_∞ is exactly c_0. Let $x = (x_1, x_2, \cdots) \in c_0$ and let $x^k := (x_1, x_2, \cdots, x_k, 0, 0, \cdots)$. Then the sequence $x^k \in l_{00}$ and

$$\|x - x^k\|_{l_\infty} = \sup_{n \ge k} |x_n| \to 0$$

as $k \to \infty$ since $\lim_{n\to\infty} x_n = 0$. Thus, $x^k \to x$ in l_∞ and $x \in \overline{l_{00}}$.

Let now $x \notin c_0$. Then,

$$\limsup_{n\to\infty} |x_n| \ge \alpha > 0$$

and, for any $y \in l_{00}$, we have

$$\|x - y\|_{l_\infty} \ge \alpha > 0,$$

so $x \notin \overline{l_{00}}$. Thus, we have proved that $\overline{l_{00}} = c_0$.