Introduction to Function spaces (MAT3004)

Unassessed Coursework I. Deadline: Thursday, Week 3.

Problem 1

a) Let (X, d) be a metric space and let

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Prove that $\tilde{d}$ is a metric on X. Show that $x_n \to x$ in (X, d) iff $x_n \to x$ in $(X, \tilde{d})$.

b) Let $\mathbb{R}^\infty$ be a space of all sequences (no matter bounded or unbounded) $x = (x_1, x_2, \cdots)$ and let

$$d(x, y) := \sum_{i=1}^{\infty} 2^{-i} \frac{|x_i - y_i|}{1 + |x_i - y_i|}.$$

Prove that d is a metric on $\mathbb{R}^\infty$. Prove that the convergence in $(\mathbb{R}^\infty, d)$ is a coordinate-wise convergence.

c) Is $(\mathbb{R}^\infty, d)$ a normed space? Justify your answer.

d) Show that the square sumable sequences l_2 are *dense* in $(\mathbb{R}^\infty, d)$.

Problem 2

a) Let $X = C[-1, 1]$ (space of continuous functions $f : [-1, 1] \to \mathbb{R}$). Is the following function a *norm* on X:

$$\|f\| := |f(0)| + \sup_{x \in [-1,1]} \left\{ \frac{|f(x) - f(0)|}{|x|} \right\}?$$

Justify your answer.

b) Is the same function a *norm* on the space $C^1[-1, 1]$ of continuously *differentiable* functions ($f \in C^1[-1, 1]$ iff $f, f' \in C[-1, 1]$)? Justify your answer.

c) Are the following two norms

$$\|f\|_1 := \sup_{x \in [-1,1]} \{x^2 |f(x)|\}, \quad \|f\|_2 := \sup_{x \in [-1,1]} \{x^4 |f(x)|\}$$

equivalent on $C[-1, 1]$? Justify your answer.

Problem 3

Let (X, d) and (Y, d) be two metric spaces and $f : X \to Y$ be a *continuous* function and $U \subset X$ be a set.

a) Prove that $f(\bar{U}) \subset \overline{f(U)}$.

b) Give an example where $f(\bar{U}) \neq \overline{f(U)}$.

Solutions

Problem 1.

a) Positivity and symmetry are evident and we only need to check the triangle inequality. Let $x, y, z \in X$ be arbitrary. Then, since the function $f(x) = \frac{x}{1+x}$ is monotone increasing ($f'(x) = 1/(1+x)^2 > 0$) and $d(x,z) \le d(x,y) + d(y,z)$,

$$\tilde{d}(x,z) = \frac{d(x,z)}{1+d(x,z)} \le \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)} = \\ = \frac{d(x,y)}{1+d(x,y)+d(y,z)} + \frac{d(y,z)}{1+d(x,y)+d(y,z)} \le \\ \le \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} = \tilde{d}(x,y) + \tilde{d}(y,z).$$

Thus, the triangle inequality holds and $\tilde{d}$ is a metric. The convergence in d implies the convergence in $\tilde{d}$ and vice versa since

$$\frac{1}{2} d(x,y) \le \tilde{d}(x,y) \le d(x,y)$$

if $d(x,y) \le 1$.

b) For all i, $\tilde{d}_i(x,y) = 2^{-i}\frac{|x-y|}{1+|x-y|}$ is a metric on $\mathbb{R}$ according to **a)**. Thus, the triangle inequality for $d(x,y) = \sum_{i=1}^{\infty} \tilde{d}_i(x_i, y_i)$ follows from the triangle inequalities for every d_i. Positivity and symmetry are evident as well as the fact that $d(x,y)$ is well-defined on $\mathbb{R}^\infty \times \mathbb{R}^\infty$. Thus, $d(x,y)$ is a metric.

Let us now $x^k \to x$ in d. Then, $d(x^k, x) \to 0$ as $k \to \infty$ and, in particular, for every $i \in \mathbb{N}$, $2^{-i}\frac{|x_i^k - x_i|}{1+|x_i^k - x_i|} \to 0$. Thus, $x_i^k \to x_i$ as $k \to \infty$ for every fixed i and $x^k \to x$ coordinate-wise.

Vise versa, let $x^k \to x$ coordinate-wise. We need to prove that $d(x^k, x) \to 0$ as $k \to \infty$. Fix any $\varepsilon > 0$. Then, there exists $N = N(\varepsilon)$ such that $\sum_{i=N+1}^{\infty} 2^{-i} < \varepsilon/2$ and, therefore,

$$d(x^k, x) \le \varepsilon/2 + \sum_{i=1}^{N} \frac{|x_i^k - x_i|}{1+|x_i^k - x_i|} \le \varepsilon/2 + \sum_{i=1}^{N} |x_i^k - x_i|.$$

Due to coordinate-wise convergence $x^k \to x$, we may fix $K = K(\varepsilon)$ such that the second term in the right-hand side is less than $\varepsilon/2$ for $k > K(\varepsilon)$. Thus, $d(x^k, x) < \varepsilon$ for $k > K(\varepsilon)$ and $x^k \to x$ in d. So, we have checked that convergence in d is a point-wise convergence.

c) No, it is not normed, since $d(\lambda x, \lambda y) \ne |\lambda| d(x,y)$ in general.

d) Actually, even the finite sequences are dense in $\mathbb{R}^\infty$. Indeed, let e_i be ith coordinate vector and $x^k := \sum_{i=1}^{k} x_i e_i$. Then,

$$d(x, x^k) = \sum_{i=k+1}^{\infty} 2^{-i}\frac{|x_i|}{1+|x_i|} \le \sum_{i=k+1}^{\infty} 2^{-i} = 2^{-k} \to 0$$

as $k \to \infty$.

Problem 2.

a) No, it is *not* a norm on $X = C[-1,1]$ since it is not defined on the whole X. Indeed, $\|\sqrt{x}\| = \infty$.

b) It is a norm on $C^1[-1,1]$. Indeed, checking all axioms of the norm is straightforward and, in contrast to **a)**, it is well-defined on $C^1[-1,1]$ since, due to the mean value theorem,

$$|f(x) - f(0)| \le |f'(\xi)||x| \le |x| \max_{\xi\in[-1,1]}\{|f(\xi)|\} = \|f'\|_{sup}|x|$$

and $\|f\| \le |f(0)| + \|f'\|_{sup} < \infty$.

c) These norms are not equivalent. Indeed, let

$$f_n(x) := \begin{cases} 1 - n|x|, & |x| \le 1/n, \\ 0, & |x| > 1/n. \end{cases}$$

Then

$$\|f_n\|_1 = \sup_{|x|\le 1/n}\{x^2(1 - n|x|)\} = 1/n^2 \max_{y\in[0,1]}\{y^2(1-y)\} = an^{-2},$$

where $a = \frac{4}{27}$, but the concrete value is not important, we only need that $a > 0$. Analogously

$$\|f_n\|_2 = \sup_{|x|\le 1/n}\{x^4(1 - n|x|)\} = 1/n^4 \max_{y\in[0,1]}\{y^4(1-y)\} = bn^{-4},$$

with $b = \frac{4^4}{5^5}$ and again we only need that $b > 0$. Thus, $\|f_n\|_1$ decays as n^{-2} as $n \to \infty$, but $\|f_n\|_2$ decays as n^{-4} and the norms cannot be equivalent.

Problem 3.

a) Let $y \in f(\bar{U})$. Then, there exists a sequence $x_n \in U$, such that $x_n \to x_0$ in X and $f(x_0) = y$. Since f is continuous, $f(x_n) \to f(x_0) = y$ and, since $f(x_n) \in f(U)$, $y = \overline{f(U)}$.

b) You may take, for example, $X = Y = U = \mathbb{R}$ and $f(x) := e^x$. Then, $U = \bar{U}$ and $f(U) = \mathbb{R} \cap \{x > 0\}$ - is not closed.