# MAT3004. Unassessed Coursework I. Deadline: March 8, 2016.

**Problem 1.**

**a)** Let $(X, d)$ be a metric space. Prove that $\tilde{d}(x, y) := \min\{1, d(x, y)\}$ is a metric on $X$.

**b)** Prove that $x_n \to x$ in $(X, d)$ if and only if $x_n \to x$ in $(X, \tilde{d})$.

**c)** Let $\mathbb{R}^\infty$ be the space of all sequences $x = (x_1, x_2, \cdots)$ (not necessarily bounded). Check that the function

$$d(x, y) := \sum_{i=1}^{\infty} 2^{-i} \min\{1, |x_i - y_i|\}$$

is a metric on $\mathbb{R}^\infty$. Prove that $x^k = (x_1^k, x_2^k, \cdots)$ is convergent to a sequence $x = (x_1, \cdots)$ if and only if $x_i^k \to x_i$ for every $i \in \mathbb{N}$ (coordinate-wise convergence).

**Problem 2*.** Let $C^1[0,1]$ be the space of continuously differentiable functions $f : [0,1] \to \mathbb{R}$ $(f, f' \in C[0,1])$ endowed with the standard norm $\|f\|_{C^1} := \|f\|_{sup} + \|f'\|_{sup}$. Prove that

$$\|f\|_{lip} := |f(0)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}$$

is an *equivalent* norm on $C^1[0,1]$. *Hint:* use the mean value theorem.

**Problem 3.** Let $X$ and $Y$ be metric spaces and let $f : X \to Y$ be a map.

**a)** Prove that, for any $U \subset Y$,

$$f^{-1}(Y \backslash U) = X \backslash f^{-1}(U).$$

**b)** Based on this identity verify that $f$ is *continuous* if and only if the inverse image of every *closed* set is *closed*.

**Problem 4.** Let $X = C[0,1]$ be the space of continuous functions endowed with the norm

$$\|f\|_w := \sup_{x \in [0,1]} \{x|f(x)|\}.$$

Is this space *complete*? Justify you answer. *Hint:* look at the function $f(x) = \frac{1}{\sqrt{x}}$ and approximate it by continuous functions.

## Solutions

**Problem 1. a)** Positivity and symmetry are obvious, so we need to check the triangle inequality. Let $x, y, z \in X$. According to the triangle inequality for metric $d$, we have

$$d(x, y) \le d(x, z) + d(y, z).$$

If $d(x, z)$ and $d(y, z)$ both less than than one, we get

$$\tilde{d}(x, y) \le d(x, y) \le d(x, z) + d(y, z) = \tilde{d}(x, z) + \tilde{d}(y, z).$$

Assume now that at least one of them is greater than one. Then $\tilde{d}(x, z)+\tilde{d}(y, z) \ge 1$ and $\tilde{d}(x, y) \le 1$. Thus, in all cases the triangle inequality is satisfied.

**b)** Since $\tilde{d}(x, y) \le d(x, y)$, the convergence $x_n \to x_0$ in $(X, d)$ obviously implies that $x_n \to x_0$ in $(\tilde{X}, d)$. Vise versa, if $x_n \to x_0$ in $(\tilde{X}, d)$, there exists $N$ such that $\tilde{d}(x_n, x_0) < 1$ for $n > N$ and, therefore, $d(x_n, x_0) = \tilde{d}(x_n, x_0)$ for $n > N$. This implies that $x_n \to x_0$ in $(X, d)$.

**c)** Let $D(a, b) = |a-b|$ be a standard metric on $\mathbb{R}$ and $\tilde{D}(a, b) := \min\{1, |a-b|\}$. Then, $\tilde{D}$ is also a metric on $\mathbb{R}$ which gives the same convergence in $\mathbb{R}$ and

$$d(x, y) = \sum_{i=1}^{\infty} 2^{-i} \tilde{D}(x_i, y_i).$$

The series is convergent since $\tilde{D}(a, b) \le 1$. Positivity and symmetry are obvious and the triangle inequality follows from the triangle inequality for $\tilde{D}(a, b)$. Thus, $d(x, y)$ is a metric on $\mathbb{R}^\infty$. Let now a sequence $x^k \to x$ in $\mathbb{R}^\infty$. Then, since

$$2^{-i} \tilde{D}(x_i^k, x_i) \le d(x^k, x),$$

we have $x_i^k \to x_i$ for every fixed $i$ in $(\mathbb{R}, \tilde{D})$ and by the previous result $x_i^k \to x_i$ in $\mathbb{R}$. Let now $x^k \to x$ coordinate-wise. Then, for every $\varepsilon > 0$, we may find $N = N(\varepsilon)$ such that

$$\sum_{i=N+1}^{\infty} 2^{-i} \tilde{D}(a_i, b_i) \le \sum_{i=N+1}^{\infty} 2^{-i} = 2^{-N} < \varepsilon/2$$

This, gives

$$d(x^k, x) \le \sum_{i=1}^{N} \tilde{D}(x_i^k, x_i) + \varepsilon/2.$$

Since $x^k \to x$ cordinate-wise, we may find $K = K(N, \varepsilon)$ such that, for every $k > K$, $2^{-i} \tilde{D}(x_i^k, x_i) < \varepsilon/(2N)$ and, consequently,

$$d(x^k, x) \le \varepsilon/(2N) \sum_{i=1}^{N} 1 + \varepsilon/2 = \varepsilon/2 + \varepsilon/2 = \varepsilon$$

and the convergence is proved.

**Problem 2.** According to the mean value theorem

$$\sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|}\le \sup_{\xi\in[0,1]}|f'(\xi)|.$$

Vise versa

$$|f'(x)|=\lim_{\Delta x\to 0}\frac{|f(x+\Delta x)|}{|\Delta x|}\le \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|}.$$

Thus, we proved that

$$\sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|}=\|f'\|_{sup}.$$

This immediately gives that $\|f\|_{lip}\le\|f\|_{sup}+\|f'\|_{sup}=\|f\|_{C^1}$.

To verify the opposite inequality, it is sufficient to note that, due to the Newton-Leibnitz formula,

$$f(x)=f(0)+\int_0^x f'(y)\,dy$$

and, therefore,

$$\|f\|_{sup}\le|f(0)|+\|f'\|_{sup}=\|f\|_{lip}.$$

Thus, $\|f\|_{C^1}\le\|f\|_{lip}+\|f\|_{lip}=2\|f\|_{lip}$ and the norms are equivalent.

**Problem 3. a)** Let $x\in f^{-1}(Y\backslash U)$. This means, $y=f(x)\in Y\backslash U$ and $f(x)\notin U$. Then, by definition, $x\notin f^{-1}(U)$ which implies that $x\in X\backslash f^{-1}(U)$.

Assume now that $x\in X\backslash f^{-1}(U)$. Then $x\notin f^{-1}(U)$ and $y=f(x)\notin U$. Thus, $y=f(x)\in Y\backslash U$ and $x\in f^{-1}(Y\backslash U)$.

**b)** Due to the criterion via open sets, we only need to check that the property I="inverse image of any closed set is closed" is equivalent to the property II="inverse image of any open set is open". Indeed, let $U\subset Y$ be closed. Then $Y\backslash U$ is open and II implies $f^{-1}(Y\backslash U)=X\backslash f^{-1}(U)$ is open and then $f^{-1}(U)$ is closed and I is satisfied. Vise versa, if I is satisfied and $V\subset Y$ is open, then $Y\backslash V$ is closed and $f^{-1}(Y\backslash V)=X\backslash f^{-1}(V)$ is closed which implies that $f^{-1}(V)$ is open. Thus, I is equivalent to II.

**Problem 4.** The space is not complete. The problem here is that this space should "naturally" contain more functions, e.g., like $f(x)=\frac{1}{\sqrt{x}}$ which are not continuous and does not not belong to $X$. Indeed, let us consider the approximating sequence

$$f_n(x):=\begin{cases}\frac{1}{\sqrt{x}}, & x\ge\frac{1}{n}\\ \sqrt{n}, & x\le\frac{1}{n}\end{cases}$$

These functions are continuous $f_n\in C[0,1]$. Moreover, since $f_n(x)\le f(x)$ for all $x$

$$\|f_n-f\|_w=\sup_{x\in(0,\frac{1}{n}]}\{x|f_n(x)-f(x)|\}\le 2\sup_{x\in(0,1/n]}\{xf(x)\}=\frac{2}{\sqrt{n}}.$$

This shows the convergence $f_n\to f$ in the $\|\cdot\|_w$-norm. In particular, this means that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence. We claim that this sequence has no limit in

$X$. Indeed, if $g \in C[0,1]$ is a limit, then arguing as in the proof of the uniqueness of the limit, we conclude that

$$\|f - g\|_w = \sup_{x \in (0,1]} \{x|f(x) - g(x)|\} = 0.$$

In particular, this means that $g(x) = f(x) = \frac{1}{\sqrt{x}}$ for all $x \in (0,1)$ and this function cannot be continuous at $x = 0$. Thus, $X$ is not complete.