Unassessed CourseWork II: (Introduction to) Function Spaces.

Deadline: Monday, week 12 (January 9th, 2012)

Question 1

(a) Give the definition of a *metric space* (X, d). [3]

(b) Let (X, d) be a metric space. Prove that the following inequality

$$|d(x, y) - d(u, v)| \leq d(x, u) + d(y, v)$$

holds for every $x, y, u, v \in X$. Hint: use the triangle inequality. [4]

(c) Let (X, d) be a metric space and let $V \subset X$ be a subset of it.

- Give the definitions of the *closure* $\bar{V}$, *interior* $\operatorname{int} V$ and *boundary* ∂V. [2]
- Prove by first principles that the *closure* $\bar{V}$ is a *closed* set in X. [5]
- Can the boundary ∂V of the set V be an *open* set? Justify your answer. [4]

(d) Let V be a vector space. Define what does it mean to say that two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ on V are *equivalent*. [2]

(e) Let $V := C[0, 1]$ be the space of continuous functions $f : [0, 1] \to \mathbb{R}$ and let

$$\|f\|_1 := \max_{x \in [0,1]} |f(x)|, \quad \|f\|_2 := \max_{x \in [0,1]} \{x|f(x)|\}.$$

Are these norms *equivalent*? Justify your answer. [5]

Question 2

(a) Define what is an *orthonormal* system in an inner product space H. [2]

(b) Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal system in a *Hilbert* space H and let

$$f \sim \sum_{n=1}^{\infty} f_n e_n$$

be the abstract Fourier expansion of the element f.

- Write down the formula for the Fourier coefficients f_n. [1]
- State the Bessel inequality. [2]
- State the condition on $\{e_n\}$ which guarantees that this inequality is an *equality* for all $f \in H$. [2]

(c) Let $f(x) = x^2$, $x \in [-\pi, \pi]$. Find the Fourier coefficients a_n and b_n for the Fourier sums

$$f_N(x) := a_0 + \sum_{i=1}^{N} a_n \cos(nx) + b_n \sin(nx)$$

of the function $f(x)$. [4]

(d) What is the point-wise limit of $f_N(x)$ as $N \to \infty$? Is the convergence *uniform* with respect to $x \in [-\pi, \pi]$? Explain your answer. [5]

(e) Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

[5]

(f) Using the Parseval equality, prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

[4]