INTRODUCTION TO FUNCTION SPACES (MAT3004)

UNASSESSED COURSEWORK II. DEADLINE: THURSDAY, WEEK 9.

Problem 1 Let $f(x) = \frac{e^x}{\sqrt{x(1-x)}}$.

a) Is it true that $f \in L^1(0,1)$? Justify your answer.

b) Does $f \in L^2(0,1)$. Justify your answer.

c) Let $g(x) := \frac{\sin(\pi x)}{\sqrt{x(1-x)}}$. Prove that $g \in L^p(0,1)$ for all $1 \le p < \infty$.

Problem 2

a) Let $X := \mathbb{R}$ with the usual norm and $f(x) = \frac{\pi}{2} + x - \arctan x$.

(1) Prove that $|f(x) - f(y)| < |x - y|$ for all x, y, $x \neq y$.

(2) Find all fixed points of this map.

(3) Do the obtained results contradict the Banach contraction theorem? Explain your answer.

b) Let $X := C[0,1]$ with the standard sup-norm and let

$$(Ff)(x) := 1 + x^2 + \int_0^1 sf(s)\, ds.$$

(1) Prove that F is a contraction on X.

(2) Find all fixed points of F in X.

Problem 3 Let $H := L^2(-1,1)$ with the standard inner product.

a) Find the angle between functions $f(x) = x + 1$ and $g(x) = x^2 - x + 1$ in H.

b) Orthogonalize $\{1, x, x^2, x^3\}$ using the Gram orthogonalization.

Solutions

Problem 1.

a) By the criterion, we need to check whether or not

$$\int_0^1 \frac{e^x}{\sqrt{x(1-x)}}\,dx < \infty,$$

where the integral is understood as improper Riemann integral. The function $f(x) = \frac{e^x}{\sqrt{x(1-x)}}$ has two singular points on the interval of integration: $x = 0$ and $x = 1$, therefore

$$\int_0^1 \frac{e^x}{\sqrt{x(1-x)}}\,dx := \lim_{\varepsilon\to 0}\int_\varepsilon^{1-\varepsilon} \frac{e^x}{\sqrt{x(1-x)}}\,dx < \infty.$$

Furthermore, since $1 < e^x < e$ when $x \in (0,1)$, it is sufficient to check whether or not the following (simpler) integral is finite:

$$\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,dx := \lim_{\varepsilon\to 0}\int_\varepsilon^{1-\varepsilon} \frac{1}{\sqrt{x(1-x)}}\,dx < \infty.$$

The last integral can be computed explicitly: $\int \frac{1}{\sqrt{x(1-x)}}\,dx = \arcsin(2x-1)$ and

$$\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,dx :=$$
$$= \lim_{\varepsilon\to 0}\int_\varepsilon^{1-\varepsilon} \frac{1}{\sqrt{x(1-x)}}\,dx = \lim_{\varepsilon\to 0} 2\arcsin(1-2\varepsilon) = 2\arcsin 1 = \pi < \infty.$$

Thus $f \in L^1(0,1)$.

Remark: NB! You need not to compute explicitly the anti-derivative of $f(x)$ in order to solve the problem. This anti-derivative cannot be found in elementary functions and you just waste time! You need only to check whether or not this integral is finite by replacing f by *simpler* functions for which the integral can be found explicitly. Actually, you even need not to know that $\int \frac{1}{\sqrt{x(1-x)}}\,dx = \arcsin(2x-1)$ and may argue as follows:

$$\int_\varepsilon^{1-\varepsilon} \frac{1}{\sqrt{x(1-x)}}\,dx = \int_\varepsilon^{1/2} \frac{1}{\sqrt{x(1-x)}}\,dx + \int_{1/2}^{1-\varepsilon} \frac{1}{\sqrt{x(1-x)}}\,dx =$$
$$= 2\int_\varepsilon^{1/2} \frac{1}{\sqrt{x(1-x)}}\,dx < 2\int_\varepsilon^{1/2} \frac{1}{\sqrt{x(1-1/2)}}\,dx =$$
$$= 2\sqrt{2}\int_\varepsilon^{1/2} \frac{1}{\sqrt{x}}\,dx = 4(1-\sqrt{2\varepsilon})$$

and the integral is finite (we have implicitly used that the function $\sqrt{x(1-x)}$ is symmetric with respect to $x = 1/2$).

b) We now need to check whether or not the integral

$$\int_0^1 f^2(x)\,dx = \int_0^1 \frac{e^{2x}}{x(1-x)}\,dx < \infty.$$

As before, using that $1 < e^{2x} < e^2$ for $x \in (0,1)$, we only need to answer the same question about the integral

$$\int_0^1 \frac{1}{x(1-x)}\,dx = \lim_{\varepsilon\to 0} \int_\varepsilon^{1-\varepsilon} \frac{1}{x(1-x)}\,dx$$

Using that $\frac{1}{x(1-x)} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{1-x}\right)$, we compute the last integral:

$$\int_\varepsilon^{1-\varepsilon} \frac{1}{x(1-x)}\,dx = \int_\varepsilon^{1-\varepsilon} \frac{1}{x}\,dx = \log\frac{1-\varepsilon}{\varepsilon}.$$

Thus, the limit as $\varepsilon \to 0$ is infinite and $f \notin L^2(0,1)$.

c) Since $\sin\pi x = 0$ for $x = 0$ and $x = 1$, the function $g(x) := \frac{\sin(\pi x)}{\sqrt{x(1-x)}}$ is *continuous* at the segment $x \in [0,1]$ ($g \in C[0,1]$). By this reason $g \in L^p(0,1)$ for all p.

Problem 2.

a) (1) By the mean value theorem

$$f(x) - f(y) = f'(\xi)(x - y)$$

and $f'(\xi) = 1 - \frac{1}{1+\xi^2} = \frac{\xi^2}{1+\xi^2}$. Thus $|f'(\xi)| < 1$ for all ξ.

(2) To find the fixed point ξ, we need to solve $f(\xi) = \xi$ which is equivalent to $\arctan \xi = \pi/2$. Thus, there are no fixed points for this map.

(3) No contradiction with the Banach contraction theorem since f is *not* a contraction. Indeed, since $f'(\xi) \to 1$ as $\xi \to 1$, there are no $\kappa < 1$ such that

$$|f(x) - f(y)| \le \kappa |x - y|.$$

b) Actually, there was a misprint in the statement of the problem: the upper limit of integration should be x, not 1. But this only makes the problem a bit less standard and more interesting!

(1) Contraction: there is no difference here between x and 1 upper limits. Let f and g be two functions $f, g \in C[0,1]$. Then

$$|(Ff)(x) - (Fg)(x)| \le |\int_0^x s(f(s) - g(s))\, ds| \le \int_0^1 s|f(s) - g(s)|\, ds \le$$
$$\le \|f - g\|_{sup} \int_0^1 s\, ds = \frac{1}{2}\|f - g\|_{sup}.$$

Thus, $\|Ff - Fg\|_{sup} \le \frac{1}{2}\|f - g\|_{sup}$ and F is a contraction.

(2) Here there is a big difference between the upper limit x and upper limit 1. Let us start with the first one. We need to solve the equation

$$y(x) = 1 + x^2 + \int_0^x sy(s)\, ds$$

Differentiating this equation in x, we get $y'(x) = 2x + xy(x)$ and taking $x = 0$, we see that the initial condition $y(0) = 1$. Solving the ODE, we get

$$y(x) = 3e^{x^2/2} - 2.$$

Consider now the case of 1 in the upper limit. Then, we need to solve

$$y(x) = 1 + x^2 + \int_0^1 sy(s)\, ds.$$

Since the integral is independent of x, $y(x) = x^2 + C$ and we only need to find C:

$$x^2 + C = 1 + x^2 + \int_0^1 s(s^2 + C)\, ds = 1 + x^2 + C/2 + 1/4$$

Thus, $C/2 = 5/4$, $C = 5/2$ and $y(x) = x^2 + 5/2$.

Problem 3.

a) The angle between f and g is found by $\cos\alpha = \frac{(f,g)}{\|f\|\|g\|}$. Thus

$$(f,g) = \int_1^1 (x+1)(x^2-x+1)\,dx = 2,\ \|f\|^2 = \int_{-1}^1 (x+1)^2\,dx = 8/3,$$

$$\|g\|^2 = \int_{-1}^1 (x^2-x+1)^2\,dx = 22/5$$

and $\cos\alpha = \frac{1}{44}\sqrt{660}$.

b) (Legendre polynomials). Note that $\{1, x^2\}$ are even and $\{x, x^3\}$ are odd, so any function from the first group is already orthogonal to any function from the second group and we only need to orthogonalize the functions inside of every group. Since $\|1\|^2 = 2$, we take $e_1 = \frac{1}{\sqrt{2}}$. Then,

$$\tilde{e}_3 = x^2 - (x^2, e_1)e_1 = x^2 - \frac{1}{2}\int_{-1}^1 x^2\,dx = x^2 - \frac{1}{3},\ \|\tilde{e}_3\|^2 = \frac{8}{45}$$

and $e_3 = \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3})$. Furthermore, $\|x\|^2 = 2/3$ and $e_2 = \sqrt{\frac{3}{2}}x$. Finally

$$\tilde{e}_4 = x^3 - (x^3, e_2)e_2 = x^3 - \frac{3}{2}x\int_{-1}^1 x^4\,dx = x^3 - \frac{3}{5}x,\ \|\tilde{e}_4\|^2 = \frac{8}{175}$$

and $e_4 = \sqrt{\frac{175}{8}}(x^3 - \frac{3}{5}x)$.