Coursework II. Function Spaces. Solutions

Question 1.

a) A set X with a function $d : X \times X \to \mathbb{R}$ is a metric space if the function d satisfies the following properties:
1) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$.
2) $d(x, y) = d(y, x)$ for all $x, y \in X$.
3) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

b) Let $x, y, u, z \in X$. Then, by the triangle inequality

$$d(x, y) \leq d(x, u) + d(u, y) \leq d(x, u) + d(u, v) + d(v, y)$$
$$\Rightarrow \; d(x, y) - d(u, v) \leq d(x, u) + d(v, y)$$

and, analogously

$$d(u, v) \leq d(u, x) + d(x, v) \leq d(u, x) + d(x, y) + d(y, v)$$
$$\Rightarrow \; d(u, v) - d(x, y) \leq d(x, u) + d(v, y)$$

and the desired inequality is proved.

c) A point $x_0 \in \bar{X}$ iff there is a sequence $x_n \in X$ such that $x_0 = \lim_{n\to\infty} x_n$.

A point $x_0 \in \operatorname{int} X$ iff for some $\varepsilon > 0$, $B_\varepsilon(x_0) \subset X$.

A boundary ∂X is defined as $\bar{X} \setminus \operatorname{int} X$.

Let a sequence $x_n \in \bar{V}$ be such that $x_n \to x_0$ in X. By the definition of closure, for every n, there exists a sequence $x_n^k \in V$ such that $x_n^k \to x_n$ as $k \to \infty$. In particular, for every n, there exists $y_n \in V$ such that $d(x_n, y_n) \leq 1/n$. We claim that $y_n \to x_0$ as $n \to \infty$ (which proves that $x_0 \in \bar{V}_0$ and $\bar{V}_0$ is closed). Indeed,

$$d(y_n, x_0) \leq d(x_n, x_0) + d(x_n, y_n) \leq d(x_n, x_0) + 1/n \to 0$$

as $n \to \infty$ (since $x_n \to x_0$).

The boundary ∂V can be open if, for instance, it is empty. As an example, one may consider an open unit ball in the space of totally disconnected points.

d) Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent on V if there exist positive constants l, L such that

$$l\|x\|_1 \leq \|x\|_2 \leq L\|x\|_1$$

for all $x \in V$.

e) These two norms are not equivalent. Indeed, let

$$f_n(x) := \begin{cases} 1 - nx, & x \le 1/n, \\ 0, & x > 1/n. \end{cases}$$

Then f_n are continuous and $\|f_n\|_1 = 1$ for all n. On the other hand

$$\|f_n\|_2 = \max_{x \in [0,1/n]} \{x(1 - nx)\} = \frac{1}{4n} \to 0$$

as $n \to \infty$ and the norms cannot be equivalent.

Question 2.

a) A sequence of vectors $e_n \in H$ is an orthonormal system if $(e_n, e_m) = \delta_{n,m}$ (Kronecker delta).

b) $f_n := (f, e_n)$. Bessel inequality: $\sum_{n=1}^{\infty} f_n^2 \le \|f\|^2$. This inequality is an equality for all $x \in H$ if and only if the orthonormal system $\{e_n\}$ is complete (the equality $(f, e_n) = 0$ for all n implies that $f = 0$).

c) Note that the function f is even, therefore, $b_n \equiv 0$ and we only need to find a_n:

$$a_n := \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx)\, dx = 4\frac{(-1)^n}{n^2}, \ a_0 := \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2\, dx = \frac{\pi^2}{3}$$

(the first integral can be easily computed integrating by parts twice). Thus,

$$f(x) \sim \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$$

d) Extend the function $f(x) = x^2$ *periodically* from $x \in [-\pi, \pi]$ to the whole real line. Then the obtained function $f_{per}(x)$ will be *continuous* and piece-wise smooth. By the Dirichlet theorem, the Fourier sums converge point-wise to the limit function

$$f_{lim}(x) = f_{per}(x) \tag{1}$$

for every $x \in \mathbb{R}$. This convergence is uniform since the non-zero Fourier coefficients satisfy $\sum_{n=0}^{\infty} |a_n| < \infty$.

e) Take $x = \pi$ in the last formula. Then,

$$f_{lim}(\pi) = \pi^2 = \frac{\pi}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and

$$\sum_{n=0}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

f) The Parseval equality for the classical Fourier series reads:

$$\|f\|_{L^2}^2 := \int_{-\pi}^{\pi} f(x)^2\,dx = 2\pi a_0^2 + \pi \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

In our case, $\|f\|_{L^2}^2 = 2/5\pi^5$ and the Parseval equality gives

$$\frac{2\pi^5}{5} = \frac{2\pi^5}{9} + 16\pi \sum_{n=1}^{\infty} \frac{1}{n^4}$$

and $\sum \frac{1}{n^4} = \frac{\pi^4}{90}$.