MAT3004. UNASSESSED COURSEWORK II. DEADLINE: MAY 3, 2016.

Problem 1.** Consider the space $C_b(0,1)$ of bounded and continuous functions $f:(0,1)\to\mathbb{R}$ defined on the *open* interval $x\in(0,1)$ endowed by the sup-norm:

$$\|f\|_{C_b} := \sup_{x\in(0,1)} |f(x)|.$$

Let also $x_n := 2^{-n-1}$ and the spike-like function $\varphi_n(x)$ be such that $\varphi_n(x)=0$ for $x\notin[x_n,x_{n-1}]$, $\varphi_n(\frac{x_{n-1}+x_n}{2})=1$ and $\varphi_n(x)$ is *linear* on the segments $[x_n,\frac{x_{n-1}+x_n}{2}]$ and $[\frac{x_{n-1}+x_n}{2},x_{n-1}]$.

Finally, for any bounded sequence $a:=\{a_n\}_{n=1}^\infty\in l_\infty$, define a function

$$f_a(x) := \sum_{n=1}^\infty a_n\varphi_n(x).$$

a) Prove that the function $f_a\in C_b(0,1)$ for any $a\in l_\infty$.

b) Prove that $\|f_a\|_{C_b(0,1)} = \|a\|_{l_\infty}$.

c) Deduce from the previous result that the space $C_b(0,1)$ is *not* separable. You may use that l_∞ is not separable without proving this fact. *Hint:* prove that any subset of a separable metric space is separable.

Problem 2. Let $X:=\mathbb{R}^2$ with the Euclidean norm and let the map $F:X\to X$ be defined via

$$F(x) = Ax+b,\quad A=\begin{pmatrix}\frac12 & 1\\ 0 & \frac12\end{pmatrix},\quad x=(x_1,x_2)\in\mathbb{R}^2$$

and $b\in\mathbb{R}^2$ is some fixed (independent of x) vector.

a) Check that $F(x)$ and $F^{(2)}(x):=F(F(x))$ both are not *contractions* on X.

b) Prove that the third iteration $F^{(3)}(x)$ is a contraction on X.

Problem 3. Let $X:=[-10,10]\subset\mathbb{R}$ with the standard norm and $f:X\to\mathbb{R}$ is defined by $f(x):=\sqrt{x^2+1}$.

a) Prove that, for any $x,y\in X$, $|f(x)-f(y)|\le\frac{10}{\sqrt{101}}|x-y|$.

b) Prove that there are no fixed points of f.

c) Does this result contradict the Banach contraction theorem? Justify your answer.

Problem 4. Let V be an inner product space and $\|x\|:=\sqrt{(x,x)}$. Prove that, for any three vectors $x,y,z\in V$, the following identity holds:

$$\|x+y+z\|^2+\|x\|^2+\|y\|^2+\|z\|^2 = \|x+y\|^2+\|y+z\|^2+\|x+z\|^2.$$

Solutions

Problem 1. a) We need to check that $f_a(x)$ is continuous and bounded. The continuity of f_a at any $x_0 \in (0,1)$ follows from the fact that in the series for $f_a(x)$ for a fixed x, no more than one term is non-zero and, for x belonging to the sufficiently small neighbourhood of x_0, no more than two subsequent terms in the series are non-zero. So, the continuity of every spike $\varphi_n(x)$ implies the continuity of f_a. Let us check boundedness. Indeed, by the triangle inequality,

$$\|f_a\|_{C_b} \le \sum_{n=1}^{\infty} |a_n|\varphi_n(x) \le \|a\|_{l_\infty} \sum_{n=1}^{\infty} \varphi_n(x) \le \|a\|_{l_\infty}.$$

Thus, boundedness is also proved and we have verified that $f_a \in C_b(0,1)$.

b) We have already proved that $\|f_a\|_{C_b} \le \|a\|_{l_\infty}$. Let us prove the opposite inequality. Indeed,

$$\|f_a\|_{C_b} \ge \sup_{n\in\mathbb{N}} |f_a(\frac{1}{2}(x_{n-1}+x_n))| = \sup_{n\in\mathbb{N}} |a_n| = \|a\|_{l_\infty}$$

and the opposite inequality is also proved.

c) Since $f_{\alpha a+\beta b}(x) = \alpha f_a(x) + \beta f_b(x)$ for any $\alpha, \beta \in \mathbb{R}$, $a, b \in l_\infty$ and $x \in (0,1)$, the space l_∞ is isometrically embedded in $C_b(0,1)$. The desired statement follows now from a general fact that any subspace of a separable metric space is separable.

Indeed, let (X,d) be a separable metric space, $\{z_n\}_{n=1}^{\infty}$ be a countable dense set in X and $Y \subset X$. Let $d_n = d_n(z_n, Y) := \inf_{y\in Y} d(z_n, Y)$ be a distance from z_n to the set Y. By definition, it means that there exists $y_n \in Y$ such that $d(y_n, z_n) \le 2d_n$. We claim that $\{y_n\}_{n=1}^{\infty} \subset Y$ is a countable dense set in Y. Let $y_0 \in Y$ be arbitrary. Then, by assumptions, there exists a subsequence $z_{n_k} \in X$ such that $z_{n_k} \to y_0$ in X. In particular, this means that $d_{n_k} \to 0$ as $k \to \infty$. Then, by triangle inequality

$$d(y_{n_k}, y_0) \le d(z_{n_k}, y_0) + d(z_{n_k}, y_{n_k}) \le d(z_{n_k}, y_0) + 2d_{n_k} \to 0$$

Thus, $y_{n_k} \to y_0$ in X and $\{y_n\}_{n=1}^{\infty}$ is a countable dense set in Y.

Problem 2. Obviously $F^{(2)}(x) = A(Ax+b)+b = A^2x+b_1$, $F^{(3)}(x) = A(A^2x+b_1)+b = A^3x+b_2$, where the constant vectors $b_1 := Ab+b$ and $b_2 = A^2b+Ab+b$. Computing the powers of the matrix A, we see

$$A^2 = \begin{pmatrix} \frac{1}{4} & 1 \\ 0 & \frac{1}{4} \end{pmatrix}, \quad A^3 = \begin{pmatrix} \frac{1}{8} & \frac{3}{4} \\ 0 & \frac{1}{8} \end{pmatrix}.$$

a) To prove that both $F(x)$ and $F^{(2)}(x)$ are not contractions, we take $x = (0,1)$ and $y = (0,0)$. Then, $\|x-y\| = \|x\| = 1$ and

$$\|F(x)-F(y)\| = \|Ax\| = \|(1, \frac{1}{2})\| = \frac{\sqrt{5}}{2} > 1$$

and F is not a contraction. Analogously,

$$\|F^{(2)}(x) - F^{(2)}(y)\| = \|A^2x\| = \|(1, \frac{1}{4})\| = \frac{\sqrt{17}}{4} > 1$$

and $F^{(2)}$ is also not a contraction.

b) Let us prove that $F^{(3)}(x)$ is a contraction. We use that $F^{(3)}(x) - F^{(3)}(y) = A^3(x-y)$ and denote $z := x - y = (z_1, z_2)$. Then

$$\|A^3 z\|^2 = (\frac{1}{8}z_1 + \frac{3}{4}z_2)^2 + \frac{1}{64}z_2^2 = \frac{1}{64}(z_1^2 + z_2^2) + \frac{3}{16}z_1 z_2 + \\ + \frac{9}{16}z_2^2 \le (\frac{1}{64} + \frac{9}{16}(z_1^2 + z_2^2) + \frac{3}{16}(\frac{1}{2}(z_1^2 + z_2^2)) = \frac{43}{64}(z_1^2 + z_2^2).$$

Thus, $F^{(3)}$ is a contraction with a contraction factor $\kappa = \frac{\sqrt{43}}{8}$. Actually, more accurate computations show that the sharp contraction factor is $\kappa = \frac{1}{8}\sqrt{10} + \frac{3}{8}$, but we need not this to prove that $F^{(3)}$ is a contraction.

Problem 3. a) Due to the mean value theorem, $|f(x) - f(y)| = |f'(\xi)||x-y|$. Since $f'(\xi) = \frac{\xi}{\sqrt{\xi^2+1}}$, we see that

$$|f'(\xi)| \le \|f'\|_{C[-10,10]} = \frac{10}{\sqrt{101}}$$

and $|f(x) - f(y)| \le \frac{10}{\sqrt{101}}|x-y|$ for all $x, y \in X$.

b) There are no fixed points for this map since any fixed point p must satisfy $p = \sqrt{p^2+1}$ which gives $p^2 = p^2 + 1$ and $0 = 1$.

c) This does not contradict the Banach contraction theorem since f maps X to $\mathbb{R}$, but not X to X. Indeed $f(10) = \sqrt{101} \notin X$.

Problem 4. This is just a routine calculation

$$\|x+y+z\|^2 = (x+y+z, x+y+z) = \|x\|^2 + \|y\|^2 + \|z\|^2 + 2(x,y) + 2(y,z) + 2(x,z)$$

and analogously

$$\|x+y\|^2 + \|y+z\|^2 + \|x+z\|^2 = 2\|x\|^2 + 2\|y\|^2 + 2\|z\|^2 + 2(x,y) + 2(y,z) + 2(x,z).$$