INFORMAL solutions for MAT3004 Exam (2016)

Question 1.

a) (i) A set $\bar{U}$ is a closure of U if it coincides with the set of all limit points of U in X. Namely, $x_0 \in \bar{U}$ if there exists a sequence $x_n \in U$ such that $x_n \to x_0$ in X. An interior $\operatorname{int} U$ consists of all points $x_0 \in U$ such that there exists $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subset U$.

(ii) Let $x_n \in \bar{U}$ and $x_n \to x_0$ in X. We need to prove that $x_0 \in \bar{U}$. By the definition of the closure, for any $x_n \in \bar{U}$, there exists a sequence of elements belonging to U converging to x_n. In particular, there exists $y_n \in U$ such that $d(x_n, y_n) < 1/n$. Then, by the triangle inequality,

$$d(x_0, y_n) \le d(x_n, y_n) + d(x_n, x_0) < 1/n + d(x_n, x_0).$$

Thus, the convergence $x_n \to x_0$ implies that $y_n \to x_0$. So, $x_0 \in \bar{U}$ and $\bar{U}$ is closed.

(iii)) Let $x_0 \in \overline{U \cap V}$. Then, there exists a sequence $x_n \in U \cap V$ such that $x_n \to x_0$. Then, $x_n \in U$ and $x_n \in V$ and, therefore, $x_0 \in \bar{U}$ and $x_0 \in \bar{V}$ which means that $x_0 \in \bar{U} \cap \bar{V}$.

(iv) Let $X = \mathbb{R}$ with the usual norm, $U := \mathbb{Q}$ (rational numbers) and $V = \mathbb{R}\backslash\mathbb{Q}$.

(v) Yes. Let (X, d) be the totally disconnected metric space. Then, *any* set is open and closed simultaneously.

b) (i) Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent on V if there exist positive constants l, L such that

$$l\|x\|_1 \le \|x\|_2 \le L\|x\|_1$$

for all $x \in V$.

(ii) These two norms are equivalent. Indeed, let us consider a function $\varphi(x) := \frac{x - \sin x}{x^3}$. This function is continuous at zero since $\sin x = x - x^3/3! + \cdots$. Therefore, $\varphi \in C[0,1]$ and its max and min exist and strictly positive (since $\sin x < x$ for $x > 0$). Thus, there exist positive constants l, L such that

$$l \le \varphi(x) \le L \quad \text{or} \quad lx^3 \le (x - \sin x) \le Lx^3$$

for all $x \in [0,1]$. Multiplying this inequality by $|f(x)|$ and integrating, we end up with

$$l\|f\|_2 \le \|f\|_1 \le L\|f\|_2$$

and the norms are equivalent.

Question 2.

a) (i) The space (X, d) is *complete* if *every* Cauchy sequence is convergent. It is totally bounded if for every $\varepsilon > 0$, there exists a covering of X by *finitely-many* ε-balls. It is separable if there exists a *countable* and *dense* set in it.

(ii) A metric space (X, d) is compact if *every* sequence in it contains a convergent subsequence.

(iii) Let $A := \sup_{x\in X} f(x)$. Then, by the definition of the supremum, there exists a sequence $x_n \in X$ such that $A = \lim_{n\to\infty} f(x_n)$. Since X is compact, there exists a convergent subsequence $x_{n_k} \to x_0$ in X. Since f is continuous,

$$A = \lim_{k\to\infty} f(x_{n_k}) = f(\lim_{k\to\infty} x_{n_k}) = f(x_0)$$

and the maximum is achieved at x_0.

(iv) Let, say, $X = \mathbb{R}$ with the usual metric and $f(x) = -e^x$.

(v) Let us argue by contradiction. Assume that f^{-1} is not continuous at $y_0 \in Y$. Then, there exists $\varepsilon_0 > 0$ and a sequence $x_n = f^{-1}(y_n) \in X$ such that $x_n \notin B_\varepsilon(x_0)$ where $x_0 = f^{-1}(y_0)$ and $d(y_n, y_0) \le 1/n$. Therefore, $y_n \to y_0$ in Y. Using the compactness of X, we conclude that there exist a subsequence $x_{n_k} \to \bar{x} \notin B_\varepsilon(x_0)$ ($X \backslash B_\varepsilon(x_0)$ is closed). Thus, $\bar{x} \neq x_0$, but $f(\bar{x}) = y_0 = f(x_0)$ by continuity. This contradicts the fact that f is one-to-one.

b) The Lebesgue space $L^p(0,1)$ is a *completion* of the space $C[0,1]$ with respect to the norm

$$\|f\|_{L^p} := \left(\int_0^1 |f(x)|^p\, dx\right)^{1/p}.$$

c) According to the criterion stated in lectures, we need to check whether or not

$$\int_0^1 |f(x)|\, dx = \int_0^1 \frac{1}{(1-x)\sqrt{x}}\, dx < \infty$$

in the sense of improper Riemann integration. Since the function f has *two* singularities at $x = 0$ and $x = 1$ and is continuous everywhere outside, we need to take care about every of them separately. Let us start with $x = 1$ (where the singularity is stronger):

$$\int_{1/2}^1 \frac{1}{(1-x)\sqrt{x}}\, dx \ge \int_{1/2}^1 \frac{dx}{1-x} = \lim_{\varepsilon\to 0} \int_{1/2}^{1-\varepsilon} \frac{dx}{1-x} =$$

$$= -\lim_{\varepsilon\to 0} \log(1-x)\Big|_{x=1/2}^{x=1-\varepsilon} = \lim_{\varepsilon\to 0}(\log\frac{1}{\varepsilon} - \log 2) = \infty$$

and, therefore, $f \notin L^1(0,1)$. We need not to look at $x = 0$ since the function is already not integrable near $x = 1$.

Question 3.

a) A function $f : X \to X$ on a metric space (X, d) is a contraction if there exists a number $\kappa < 1$ such that

$$d(f(x), f(y)) \leq \kappa d(x, y)$$

for all $x, y \in X$.

The Banach Contraction Theorem: If (X, d) is a complete metric space and f is a contraction on (X, d) then f has a unique fixed point p (i.e., the equation $f(x) = x$ has a unique solution $x = p$).

b) Since $F^{(n)}$ is a contraction on a complete metric space (X, d) there is a unique fixed point $p \in X$ of it. Assume that p is not a fixed point of F, then $q = F(p) \neq p$ and on the other hand $F^{(n)}(q) = F(F^{(n)}(p)) = F(p) = q$ is also a fixed point of $F^{(n)}$, so by the uniqueness part $q = p$ and p is a fixed point of F. Let now $q \neq p$ be another fixed point of F, then q is also a fixed point of $F^{(n)}$ and must coinside with p. Thus, p is unique.

c) (i) According to the mean value theorem,

$$|F(x) - F(y)| = |F'(\xi)| \cdot |x - y|$$

for some point $\xi \in (x, y)$. Note that in our case $F'(\xi) = -\frac{1}{(1+\xi)^2}$ and $\lim_{\xi \to 0} |F'(\xi)| = 1$. By this reason, taking $x = 0$, we see that the inequality

$$|F(y) - F(0)| \leq \kappa |y|, \quad y \in [0, 1]$$

cannot be satisfied if $\kappa < 1$. Thus, F is not a contraction.

(ii) The second iteration $F^{(2)}(x) = F(F(x)) = \frac{1}{1+\frac{1}{1+x}} = \frac{x+1}{x+2}$ and $\frac{d}{dx} F^{(2)}(x) = \frac{1}{(x+2)^2} \leq \frac{1}{4}$. Thus, by the mean value theorem $F^{(2)}$ is a contraction with the contraction factor $\kappa = 1/4$.

(iii) The unique fixed point p of this function satisfies $p = \frac{1}{1+p}$ or $p^2 + p - 1 = 0$. Thus, $p = \frac{\sqrt{5}-1}{2}$ (the second root is negative and does not belong to X.

d) (i) Let $f_1, f_2 \in C[-1, 1]$. Then

$$|F(f_1)(x) - F(f_2)(x)| = |\int_0^x s(f_1(s) - f_2(s))\, ds| \leq$$

$$\leq \max_{s \in [0,x]} |f_1(s) - f_2(s)| \int_0^{|x|} s\, ds \leq 1/2 \|f_1 - f_2\|_{sup}.$$

Therefore, $\|F(f_1) - F(f_2)\|_{sup} \leq 1/2 \|f_1 - f_2\|_{sup}$ and F is contraction.

(ii) The unique fixed point $y \in C([-1,1])$ should satisfy the equation

$$y(x) = 1 + \int_0^x sy(s)\,ds.$$

From this equation we see that y must be continuously differentiable and $y(0) = 1$. Differentiating this equation by x, we find that $y'(x) - xy(x) = 0$ and $y(x) = e^{x^2/2}$.

Question 4.

a) (i) A *bi-linear* form (x, y) on a real vector space V is an inner product iff

1) It is symmetric: $(x, y) = (y, x)$;

2) Positive definite: $(x, x) \geq 0$ and $(x, x) = 0$ iff $x = 0$.

(ii) Cauchy-Schwarz inequality: $|(x, y)| \leq \|x\|\|y\|$, for all $x, y \in V$.

(iii) We need to prove that $\|x + y\| \leq \|x\| + \|y\|$. Squaring this inequality, we get

$$\|x\|^2 + \|y\|^2 + 2(x, y) \leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|$$

and it reduces to $(x, y) \leq \|x\|\|y\|$ which is Cauchy-Schwarz inequality.

b) (i) Bessel inequality: $\sum_{n=1}^{\infty} f_n^2 \leq \|f\|^2$.

(ii) This inequality is an equality for all $x \in H$ if and only if the orthonormal system $\{e_n\}$ is complete (the equality $(f, e_n) = 0$ for all n implies that $f = 0$).

c) (i) Extend the function $f(x) = x$ *periodically* from $x \in [-\pi, \pi]$ to the whole real line. Then the obtained function $f_{per}(x)$ will be piece-wise continuous and piece-wise smooth. By the Dirichlet theorem, the Fourier sums converge point-wise to the limit function

$$(1) \qquad f_{lim}(x) = f_{per}(x), \quad x \neq n\pi \quad \text{and} \quad f_{lim}(n\pi) = 0, \quad n \in \mathbb{Z}.$$

This convergence is not uniform since the limit function is not continuous.

(ii) By the Parseval equality

$$2\pi^3/3 = \int_{-\pi}^{\pi} x^2\, dx = \|f\|^2 = \pi \sum_{n=1}^{\infty} b_n^2 = 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

c) Assume that such $f \in L^2(-\pi, \pi)$ exists. Then, by Parseval equality

$$\pi \sum_{n=1}^{\infty} b_n^2 = 4\pi \sum_{n=1}^{\infty} \frac{1}{n} = \|f\|^2 < \infty.$$

But $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ and this contradiction proves that such $f \in L^2(-\pi, \pi)$ does not exist.