UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies

Level HE3 Examination

Module MAT3004 Introduction to Function Spaces

Time allowed – 2 hrs

Autumn Semester 2008

Attempt THREE questions
If a candidate attempts more than THREE questions only the best THREE questions will be taken into account.

Question 1

(a) Give the definition of a *metric* space. [3]

(b) Prove that, for any 4 points x, y, u, v of a metric space (X, d), the following inequality holds:

$$|d(x,y) - d(u,v)| \leq d(x,u) + d(y,v).$$

[4]

(c) Let (X, d) be a metric space

- Give the definitions of an *open* set and and a *closed* set in X.
- Prove that a union of any number of open sets is open.

[4]

(d) Prove that the set

$$V = \{f \in C[-1,1], \quad \int_{-1}^{1} f(x) \sin x \, dx < 1\}$$

is an *open* set in the space of continuous functions $C[-1, 1]$ with the sup-metric. [5]

(e) Give the definition of a *norm* on a vector space V. [3]

(f) Let $\|x\|_1$ and $\|x\|_2$ be two norms on a vector space V.

- What does it mean when these two norms are *equivalent*?
- Prove that the following two norms:

$$\|f\|_{L^1} := \int_{-1}^{1} |f(x)| \, dx, \quad \|f\|_{L^\infty} := \sup_{x \in [-1,1]} |f(x)|$$

are not equivalent on the vector space $C[-1, 1]$ of continuous functions.

Hint: construct a sequence f_n of continuous functions such that $\|f_n\|_{L^\infty} = 1$, but $\|f_n\|_{L^1} \to 0$ as $n \to \infty$. [6]

Question 2

(a) Let (X, d) be a metric space.

- Give the definition of a Cauchy sequence in a metric space (X, d). What metric spaces are called *complete*?
- Give an example a non-complete metric space.

[3]

(b) Let (X, d) be a metric space.

- Define what it means when X is called *compact*.
- Give an example of a complete *non-compact* metric space.

[4]

(c) Prove that any compact metric space is complete. [6]

(d) Prove that the closed unit ball

$$\bar{B}_1(0) := \{x \in l_\infty,\ \|x\|_{l_\infty} = \sup_{i \in \mathbb{N}} |x_i| \leq 1\}$$

of the space of bounded sequences l_∞ is *not compact*. [5]

(e) Let X and Y be two *compact* metric spaces and let $f : X \to Y$ be continuous, one-to-one and onto. Prove that the inverse function

$$f^{-1} : Y \to X$$

is also continuous. [7]

Question 3

(a) What is a *contraction* of a metric space (X, d)? [3]

(b) State the Banach contraction theorem. [4]

(c) Let $f(x) := \frac{\pi}{2} + x - \arctan x$, $x \in \mathbb{R}$

- Using the finite implements formula, prove that

$$|f(x) - f(y)| < |x - y|$$

  for all $x, y \in \mathbb{R}$ such that $x \neq y$.

- Prove that the function f does not have any fixed points in $\mathbb{R}$.
- Is f a contraction on $\mathbb{R}$? Explain your answer.

[5]

(d) Prove that the map $T : C[0, 1] \to C[0, 1]$ defined by

$$(Tf)(x) := 1 - \int_0^x sf(s)\, ds, \quad f \in C[0, 1]$$

is a *contraction* on the space $C[0, 1]$ of continuous functions with the sup-norm. [6]

(e) Prove that the fixed point $Y(x)$ of the map T solves the differential equation

$$\frac{dY(x)}{dx} + xY(x) = 0, \quad Y(0) = 1.$$

[4]

(f) Solve the above equation by separating variables and find the explicit expression for the fixed point $Y(x)$ of the map T. [3]

Question 4

(a) What is an *inner* product on a real vector space H space? [3]

(b) Let H be a Hilbert space.

- What is an orthonormal system in H?
- What is an orthonormal basis in H?

[4]

(c) State the Bessel inequality for the orthonormal system in a Hilbert space. When does it become an equality? [3]

(d) Let $f(x) = x$, $x \in [-\pi, \pi]$. Find the Fourier coefficients a_n and b_n for the Fourier sums

$$f_N(x) := a_0 + \sum_{i=1}^{N} a_n \cos(nx) + b_n \sin(nx)$$

of the function $f(x)$. [5]

(e) What is the point-wise limit of $f_N(x)$ as $N \to \infty$? Is the convergence *uniform* with respect to $x \in [-\pi, \pi]$? Explain your answer. [5]

(f) Prove that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$$

[5]

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