UNIVERSITY OF SURREY©

Faculty of Engineering & Physical Sciences

Department of Mathematics

Undergraduate Programmes in Mathematical Studies

Module MAT3010; 15 Credits

Function Spaces

Level HE3 Examination

Time allowed: Two hours | Semester 1, 2010/11

Answer **THREE** questions only.

If you attempt more than THREE questions, only your best THREE answers will be taken into account.

Each question carries 25 marks.

Where appropriate the mark carried by an individual part of a question is indicated in square brackets [ ].

Approved calculators are allowed.

Additional material:
None

Question 1

(a) Give the definition of a *metric space* (X, d). [3]

(b) Let (X, d) be a metric space. Prove that the following inequality

$$|d(x, y) - d(u, v)| \leq d(x, u) + d(y, v)$$

holds for every $x, y, u, v \in X$. Hint: use the triangle inequality. [4]

(c) Let (X, d) be a metric space and let $V \subset X$ be a subset of it.

- Give the definitions of the *closure* $\bar{V}$, *interior* $\operatorname{int} V$ and *boundary* ∂V. [2]
- Prove by first principles that the *closure* $\bar{V}$ is a *closed* set in X. [5]
- Can the boundary ∂V of the set V be an *open* set? Justify your answer. [4]

(d) Let V be a vector space. Define what does it mean to say that two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ on V are *equivalent*. [2]

(e) Let $V := C[0, 1]$ be the space of continuous functions $f : [0, 1] \to \mathbb{R}$ and let

$$\|f\|_1 := \max_{x \in [0,1]} |f(x)|, \quad \|f\|_2 := \max_{x \in [0,1]} \{x|f(x)|\}.$$

Are these norms *equivalent*? Justify your answer. [5]

Question 2

(a) Let (X, d) be a metric space.

- Give the definition of a *Cauchy* sequence in (X, d).
- What does it mean to say that (X, d) is *complete*?
- Give an example a complete metric space and an example of a non-complete metric space.

[5]

(b) Let (X, d) be a metric space.

- Define what it means when X is called *totally bounded*.
- State the Hausdorff criterium.

[4]

(c) Prove that any compact metric space is *separable*. Hint: use the Hausdorff criterium. [7]

(d) State the definition of the Lebesgue space $L^p([-1, 1])$, $1 \leq p < \infty$. [4]

(e) Find all exponents $p \in [1, \infty)$ such that the function

$$f(x) := \frac{1}{\sqrt[4]{|x|}}$$

belongs to $L^p([-1, 1])$. Justify your answer. [5]

Question 3

(a) What is a *contraction* of a metric space (X, d)? [3]

(b) State the Banach contraction theorem. [4]

(c) Let $f(x) := \sqrt[3]{6 + x}$, $x \in \mathbb{R}_+$.

- Prove that the function f is a contraction on a metric space $X = \mathbb{R}_+ := [0, \infty)$ with the usual metric $d(x, y) := |x - y|$. [4]
- Find the fixed point of this function. [2]
- Prove that the sequence

$$\sqrt[3]{6}, \quad \sqrt[3]{6 + \sqrt[3]{6}}, \quad \sqrt[3]{6 + \sqrt[3]{6 + \sqrt[3]{6}}}, \quad \sqrt[3]{6 + \sqrt[3]{6 + \sqrt[3]{6 + \sqrt[3]{6}}}}, \quad \cdots$$

  is *convergent*. What is the limit of this sequence? [3]

(d) Let $V := \mathbb{R}^2$ and let $f : V \to V$ be defined as

$$f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} \frac{1}{2}x_1 + 3x_2 \\ \frac{1}{5}x_2 \end{pmatrix}$$

- Prove that the function $f : V \to V$ is a contraction if V is endowed with the following norm:

$$\|x\|_s := |x_1| + 10|x_2|, \quad x = (x_1, x_2) \in V$$

  [3]
- Is f a contraction on V endowed with the usual Euclidean norm? Justify your answer. [3]
- Let $x_0 \in \mathbb{R}^2$ be arbitrary and let $x_n := f(x_{n-1})$, $n \in \mathbb{N}$. Does the limit $\lim_{n\to\infty} x_n$ in the *Euclidean* norm exist? Justify your answer. [3]

Question 4

(a) Define what is an *orthonormal* system in an inner product space H. [2]

(b) Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal system in a *Hilbert* space H and let

$$f \sim \sum_{n=1}^{\infty} f_n e_n$$

be the abstract Fourier expansion of the element f.

- Write down the formula for the Fourier coefficients f_n. [1]
- State the Bessel inequality. [2]
- State the condition on $\{e_n\}$ which guarantees that this inequality is an *equality* for all $f \in H$. [2]

(c) Let $f(x) = x^2$, $x \in [-\pi, \pi]$. Find the Fourier coefficients a_n and b_n for the Fourier sums

$$f_N(x) := a_0 + \sum_{i=1}^{N} a_n \cos(nx) + b_n \sin(nx)$$

of the function $f(x)$. [4]

(d) What is the point-wise limit of $f_N(x)$ as $N \to \infty$? Is the convergence *uniform* with respect to $x \in [-\pi, \pi]$? Explain your answer. [5]

(e) Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

[5]

(f) Using the Parseval equality, prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

[4]

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