UNIVERSITY OF SURREY©

Faculty of Engineering and Physical Sciences

Department of Mathematics

Undergraduate Programmes in Mathematical Studies

Module MAT3010 — 15 Credits

Function Spaces

Level HE3 Examination

Time allowed: Two hours

Semester 1, 2011/12

Answer **THREE** questions only.

If you attempt more than THREE questions, only your best THREE answers will be taken into account.

Each question carries 25 marks.

Where appropriate the mark carried by an individual part of a question is indicated in square brackets [ ].

Approved calculators are allowed.

Additional material:
None

Question 1

(a) Give the definition of a *metric* space (X, d). [3]

(b) Let (X, d) be a metric space. State what does it mean that

- A sequence $x_n \in X$ is *convergent* to $x_0 \in X$. [1]
- A function $f : X \to \mathbb{R}$ is *continuous* at $x_0 \in X$. [1]

(c) Let (X, d) be the space of totally disjoint points ($d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$). Prove that any *continuous* function $f : \mathbb{R} \to X$ ($\mathbb{R}$ is endowed by the usual metric) is a constant: $f(x) \equiv y_0$ for some $y_0 \in X$. Hint: use the continuity criterium via inverse images of open and closed sets. [7]

(d) Let (X, d) be a metric space and let $V, U \subset X$ be subsets of it.

- Give the definitions of the *closure* $\bar{V}$, *interior* $\operatorname{int} V$ and *boundary* ∂V. [2]
- Prove by first principles that $\bar{V}$ is a *closed* set. [5]

(e) Let V be a vector space. Define what does it mean that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are *equivalent*. [2]

(f) Let $V := C[0, 1]$ be the space of continuous functions $f : [0, 1] \to \mathbb{R}$ and let

$$\|f\|_1 := \max_{x\in[0,1]}\{x^2|f(x)|\}, \quad \|f\|_2 := \max_{x\in[0,1]}\{x|f(x)|\}.$$

Are these norms *equivalent*? Justify your answer. [4]

Question 2

(a) Let l_p be the space of summable with power p sequences:

$$l_p = \{x = (x_1, x_2, \cdots), \sum_{i=1}^{\infty} |x_i|^p < \infty\}, \quad 1 \leq p < \infty.$$

- Let $x = (1, 1/\sqrt{2}, 1/\sqrt{3}, \cdots)$, $(x_i := 1/\sqrt{i})$. Find all $p \in [1, \infty)$ such that $x \in l_p$. [3]
- Prove that $l_p \subset l_q$ if $p < q$. [5]

(b) Let (X, d) be a metric space.

- Define what it means when X is called *separable*. [2]
- State the Hausdorff criterium for (X, d) to be compact. [4]

(c) Prove that any compact metric space is *separable*. Hint: use the Hausdorff criterium. [7]

(d) State the definition of the Lebesgue space $X := L^2([0, 1])$. Does the function $f(x) := \frac{1}{\sqrt{x}}$ belong to X? Justify your answer. [4]

Question 3

(a) What is a *contraction* on a metric space (X, d)? [3]

(b) State the Banach contraction theorem. [4]

(c) Let $f(x) := \frac{1}{1+x}$.

- Prove that the function f is a contraction on a metric space $X := [\frac{1}{2}, 1]$ with the usual metric $d(x, y) := |x - y|$. [4]
- Find the fixed point of this function. [2]
- Prove that the sequence

$$1, \quad \frac{1}{1+1}, \quad \frac{1}{1+\frac{1}{1+1}}, \quad \frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}, \quad \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}}, \quad \cdots$$

is *convergent*. What is the limit of this sequence? [4]

(d) Let $X := C[0, 1]$ be the space of continuous functions $f : [0, 1] \to \mathbb{R}$ (with the usual sup-norm) and let the map $F : X \to X$ be defined as follows:

$$F(f)(x) := \frac{x^2}{2} + \int_0^x y f(y)\, dy.$$

- Prove that F is a *contraction* on X. [4]
- Find the fixed point of F. [4]

Question 4

(a) Let H be an inner product space with the inner product (x, y).

- State the Cauchy-Schwartz inequality. [1]
- Prove the triangle inequality for the norm $\|x\| := (x, x)^{1/2}$. Hint: you may use Cauchy-Schwartz inequality without proving it. [4]

(b) Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal system in a *Hilbert* space H and let

$$f \sim \sum_{n=1}^{\infty} f_n e_n$$

be the abstract Fourier expansion of the element f.

- State the Bessel inequality. [2]
- State the condition on $\{e_n\}$ which guarantees that this inequality is an *equality* for all $f \in H$. [2]

(c) Let $f(x) = x$, $x \in [-\pi, \pi]$. Find the Fourier coefficients a_n and b_n for the Fourier sums

$$f_N(x) := a_0 + \sum_{i=1}^{N} a_n \cos(nx) + b_n \sin(nx)$$

of the function $f(x)$. [4]

(d) What is the point-wise limit of $f_N(x)$ as $N \to \infty$? Is the convergence *uniform* with respect to $x \in [-\pi, \pi]$? Explain your answer. [4]

(e) Prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^k}{(2k+1)} = \frac{\pi}{4}.$$

[4]

(f) Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Hint: use the Parseval equality. [4]

END OF PAPER

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