# UNIVERSITY OF SURREY©

**M. Math. Undergraduate Programmes in Mathematical Studies**

**Level HE3 Examination**

Module MAT3010 Function Spaces

Time allowed – 2 hrs Autumn Semester 2008

Attempt THREE questions
If a candidate attempts more than THREE questions only the best THREE questions will be taken into account.

**Question 1**

(a) Give the definition of a *metric* space. [3]

(b) Prove that, for any 4 points $x, y, u, v$ of a metric space $(X, d)$, the following inequality holds:

$$|d(x,y) - d(u,v)| \leq d(x,u) + d(y,v).$$

[4]

(c) Let $(X, d)$ be a metric space

- Give the definitions of an *open* set and of a *closed* set in $X$.
- Prove that a union of any number of open sets is open.

[4]

(d) Give the definition of a *norm* on a vector space $V$. [3]

(e) Let $X$ be a space of all continuously differentiable functions $f : [-1, 1] \to \mathbb{R}$ and let

$$\|f\| := \max_{x \in [-1,1]} |f'(x)|.$$

- Is this a *norm* on the vector space $X$? Justify your answer.
- Is this a *norm* on the vector space

$$X_0 := \{f \in X, \int_{-1}^{1} f(x)\, dx = 0\}?$$

Justify your answer.

[5]

(f) Prove that the unit ball

$$V = \{f \in C^1[-1,1],\ \max\{\|f\|_\infty, \|f'\|_\infty\} \leq 1\}, \quad \|f\|_\infty := \max_{x \in [-1,1]} |f(x)|$$

in the space of continuously differentiable functions is *not closed* as a subset of $C[-1, 1]$ with the $\|\cdot\|_\infty$ norm.

*Hint*: prove that the function $f_0(x) := |x|$ belongs to the closure of $V$, but does not belong to $V$. [6]

**Question 2**

(a) Let $(X, d)$ be a metric space.

- Define what it means when $X$ is called *compact*.
- Give an example of a compact metric space and a non compact metric space. [3]

(b) Give the definition of a totally bounded metric space and state the Hausdorff criterium. [4]

(c) Using the Hausdorff criterium, check that any compact metric space is *separable* (i.e., contains a countable dense subset). [7]

(d) Let $X = \mathbb{R}$ and $d(x, y) := \min\{1, |x - y|\}$.

- Prove that $(X, d)$ is a metric space.
- Is $(X, d)$ *compact*? Explain your answer. [5]

(e) Let $X$ and $Y$ be two *compact* metric spaces and let $f : X \to Y$ be continuous, one-to-one and onto. Prove that the inverse function

$$f^{-1} : Y \to X$$

is also continuous. [6]

**Question 3**

(a) What is a *contraction* of a metric space $(X, d)$? [3]

(b) State the Banach contraction theorem. [4]

(c) Let $f(x) := \frac{\pi}{2} + x - \arctan x$, $x \in \mathbb{R}$.

- Using the finite increments formula (Mean Value Theorem), prove that

$$|f(x) - f(y)| < |x - y|$$

for all $x, y \in \mathbb{R}$ such that $x \neq y$.
- Prove that the function $f$ does not have any fixed points in $\mathbb{R}$.
- Is $f$ a contraction on $\mathbb{R}$? Explain your answer.

[5]

(d) Prove that the map $T : C[0, 1] \to C[0, 1]$ defined by

$$(Tf)(x) := 1 + \int_0^x \frac{f(s)}{s+1}\, ds, \quad f \in C[0, 1]$$

is a *contraction* on the space $C[0, 1]$ of continuous functions with the sup-norm. [6]

(e) Prove that the fixed point $Y(x)$ of the map $T$ solves the differential equation

$$\frac{dY(x)}{dx} - \frac{Y(x)}{x+1} = 0, \quad Y(0) = 1.$$

[4]

(f) Solve the above equation by separating variables and find the explicit expression for the fixed point $Y(x)$ of the map $T$. [3]

**Question 4**

(a) What is an *inner* product on a real vector space $H$? [3]

(b) Let $H$ be a Hilbert space.

- What is an orthonormal system in $H$?
- What is an orthonormal basis in $H$?

[4]

(c) State the Bessel inequality for an orthonormal system in a Hilbert space. When does it become an equality? [3]

(d) Let $f(x) = x$, $x \in [-\pi, \pi]$. Find the Fourier coefficients $a_n$ and $b_n$ for the Fourier sums

$$f_N(x) := a_0 + \sum_{i=1}^{N} a_n \cos(nx) + b_n \sin(nx)$$

of the function $f(x)$. [5]

(e) What is the point-wise limit of $f_N(x)$ as $N \to \infty$? Is the convergence *uniform* with respect to $x \in [-\pi, \pi]$? Explain your answer. [5]

(f) Prove that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$$

[5]

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