UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies

Level HE3 Examination

Module MAT3004 Introduction to Function Spaces

Time allowed – 2 hrs Autumn Semester 2009

Attempt THREE questions
If a candidate attempts more than THREE questions only the best THREE questions will be taken into account.

Question 1

(a) Give the definition of a *norm* in a vector space. [3]

(b) Prove that, in any normed space $(V, \|\cdot\|)$, the function $f(x) := \|x\|$ is continuous as a function from V to $\mathbb{R}$. [5]

(c) Let V be a normed space and X be a set in it

- Give the definitions of a *closure* $\bar{X}$, *interior* $\operatorname{int} X$ and of a *boundary* ∂X.
- Prove that the *boundary* of a unit ball $B_1(x_0)$ *coincides* with the unit sphere $S_1(x_0) := \{x \in V, \|x - x_0\| = 1\}$.
- Give an example of a *metric* space such that $\partial B_1(x_0) \neq S_1(x_0)$

[5]

(d) Prove that the set

$$V = \{f \in L^2[-1,1], \quad \int_{-1}^{1} \frac{f(x)}{\sqrt[3]{x}}\, dx < 1\}$$

is an *open* set in the space $L^2[-1,1]$ of the square integrable functions with the metric

$$\|f\|_{L^2} := \left(\int_{-1}^{1} |f(x)|^2\, dx\right)^{1/2}.$$

[6]

(e) Let V be the space of *analytic* functions $f : \mathbb{C} \to \mathbb{C}$ and let

$$\|f\| := \max_{x\in[-1,1]} |f(x)|.$$

Is it a *norm* on V? Justify your answer. [6]

Hint: A function f is analytic on $\mathbb{C}$ if and only if it's Taylor expansions

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

converge to $f(z)$ for every $z \in \mathbb{C}$.

Question 2

(a) Let (X, d) be a metric space.

- Give the definition of a dense set in a metric space (X, d).
- What does it mean to say that the metric space (X, d) is *separable*?
- Give an example a non-separable metric space.

[4]

(b) Let (X, d) be a metric space.

- Define what it means when X is called *compact*.
- State the Hausdorff criterium.

[4]

(c) Prove (without using the Hausdorff criterium) that any compact metric space is totally bounded. [7]

(d) Let X and Y be two metric spaces and $f : X \to Y$ be a function.

- What does it mean to say that the function f is *uniformly* continuous? [3]
- Let X be a *normed* space and $f : X \to Y$ be a uniformly continuous function such that it's *modulus of continuity* $\omega(z)$ satisfies

$$\lim_{z \to 0+} \frac{\omega(z)}{z} = 0.$$

Prove that $f(x) \equiv const$ on X.
Hint: Use that

$$d(f(x), f(0)) \le d(f(x), f((n-1)x/n))+ \\ + d(f((n-1)x/n), f((n-2)x/n)) + \cdots + d(f(x/n), f(0))$$

for any $x \in X$ and any $n \in \mathbb{N}$. [7]

Question 3

(a) What is a *contraction* of a metric space (X, d)? [3]

(b) State the Banach contraction theorem. [4]

(c) Let $f(x) := \sqrt{2+x}$, $x \in \mathbb{R}_+$.

- Prove that the function f is a contraction on a metric space $X := R_+$ with the usual metric $d(x, y) := |x - y|$ [5]
- Find the fixed point of this sequence. [2]
- Verify that the sequence

$$\sqrt{2},\quad \sqrt{2+\sqrt{2}},\quad \sqrt{2+\sqrt{2+\sqrt{2}}},\quad \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}},\quad \cdots$$

is *convergent*. What is the limit of this sequence? [4]

(d) The following differential equation

$$\frac{d}{dt}y(t) = 3[y(t)]^{2/3}$$

with the initial data $y(0) = 0$ has at least two solutions $y_1(t) \equiv 0$ and $y_2(t) = t^3$.

- Does it contradict the local existence and uniqueness theorem for ordinary differential equations? Explain your answer.
- The function $y_3(t) := (t+1)^3$ solves this equation with the initial data $y(0) = 1$ on a *half-line* $t \geq 0$. Is that solution unique? Justify your answer.
- The function $y_3(t) := (t+1)^3$ solves this equation with the initial data $y(0) = 1$ also on the *whole line* $t \in \mathbb{R}$. Is that solution unique? Explain your answer. [7]

Question 4

(a) What is an *inner product* on a real vector space H? [2]

(b) State the Cauchy-Schwartz inequality. [3]

(c) Prove that any Hilbert space is a *normed* space with the norm $\|x\| := \sqrt{(x,x)}$ (you may use the Cauchy-Schwartz inequality without proving it). [5]

(d) Let $f(x) = |x|$, $x \in [-\pi, \pi]$. Find the Fourier coefficients a_n and b_n for the Fourier sums

$$f_N(x) := a_0 + \sum_{i=1}^{N} a_n \cos(nx) + b_n \sin(nx)$$

of the function $f(x)$. [5]

(e) What is the point-wise limit of $f_N(x)$ as $N \to \infty$? Is the convergence *uniform* with respect to $x \in [-\pi, \pi]$? Explain your answer. [5]

(f) Prove that

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Deduce from this equality that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

[5]

INTERNAL EXAMINER: S. Zelik
EXTERNAL EXAMINER: D. Chillingworth