Coursework 1 for "Function Spaces" (Mmath)

Deadline: 14 of October 2007

Problem 1 (2 points): Let $V = \mathbb{R}_+ := \{x \in \mathbb{R},\ x > 0\}$ and let

$$x \oplus y := xy, \quad x, y \in V, \quad xy \text{ is a usual multiplication}, \quad \alpha \cdot x = x^\alpha, \ \alpha \in \mathbb{R}$$

and $\vec{0} = 1$.

a) check that V is a vector space;

b) check that $\|x\| := |\log x|$ is a norm on V;

c) what is the dimension of V?

Problem 2 (3 points):
a) Give an example of a closed set G and an open set V (possibly in different metric spaces) such that

$$\overline{\operatorname{int} G} \neq G \quad \text{and} \quad \operatorname{int}(\bar{V}) \neq V.$$

b) Prove that, for any continuous function $f : X \to Y$ (X and Y are metric spaces) and any closed set G in Y, the inverse image $f^{-1}(G)$ is a closed set in X.

c) Give an example of a metric space X and a continuous function $f : X \to \mathbb{R}$ such that $f(X)$ is not closed in $\mathbb{R}$.

Problem 3 (2 points): Consider the metric space (X, d) where X is an arbitrary set and

$$d(x, y) := \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases}$$

a) Is (X, d) a *complete* metric space?

b) Under what conditions on the set X, the metric space (X, d) will be compact? Explain your answer.

Problem 4 (3 points): Let $X = C[0, 1]$ be the space of all continuous functions $f : [0, 1] \to \mathbb{R}$. Prove that the point-wise convergence in X cannot be generated by a norm on X, i.e., that it is impossible to find a norm $\|\cdot\|$ on X such that $f_n \to f$ point-wise if and only if $\|f_n - f\| \to 0$.

Hint: Argue by contradiction: assume that such a norm $\|\cdot\|$ exists. Then

a) Find a sequence of continuous functions $f_n \in C[0, 1]$ (which are not identically zero) such that

$$f_n(x) \cdot f_m(x) \equiv 0, \quad x \in [0, 1], \quad n \neq m.$$

b) Consider the sequence $\phi_n(x) := A_n f_n(x)$ where $A_n \in \mathbb{R}$ is an arbitrary sequence of real numbers. What is the point-wise limit ϕ of this sequence?

c) Fix numbers A_n in such a way that $\|\phi_n - \phi\|$ does not tend to zero.

Solutions

Problem 1. The problem can be solved by direct checking the axioms of a vector space and a norm. But there exists a more elegant proof. Namely, define a map $f : V \to \mathbb{R}$ as follows

$$f(x) = \log x$$

Then, obviously, f is one-to-one and onto. Moreover,

$$f(x \oplus y) = f(xy) = \log(xy) = \log x + \log y = f(x) + f(y), \quad x, y \in V$$

and

$$f(\alpha \cdot x) = f(x^\alpha) = \log(x^\alpha) = \alpha \log x = \alpha f(x)$$

for all $\alpha \in \mathbb{R}$ and $x \in V$. Thus, f is a *linear* isomorphism between the space V and $\mathbb{R}$ (with the standard linear structure on $\mathbb{R}$). Since $\mathbb{R}$ is a vector space, $V = f^{-1}(\mathbb{R})$ is also a vector space. Moreover, since $\dim R = 1$, the dimension of $\mathbb{R}$ is also one.

In addition, for any $x \in V$

$$\|x\| = |\log x| = |f(x)|,$$

so f is an *isometry* between $(V, \|\cdot\|)$ and $(\mathbb{R}, |\cdot|)$ and, by this reason $\|x\|$ is a norm on V.

Problem 2.

a) There are a lot of such examples. The simplest ones are on the real line $X = \mathbb{R}$ with the standard metric: a closed set $G := \{0\}$ and an open set $V := (0,1) \cup (1,2)$. Then, $\operatorname{int} V = \varnothing$ and $\overline{\operatorname{int} V} = \varnothing \neq V$. Analogously, $\bar{V} = [0,2]$ and $\operatorname{int}(\bar{V}) = (0,2) \neq V$.

b) There are two natural ways to prove this fact. The first one is to use that a set is closed if and only if its complement is open together with the fact that the inverse image $f^{-1}(V)$ is open if V is open and f is continuous (the theorem proved in the lecture notes). The second one is a direct proof. Let me give more details for the second way.

Let $V \subset Y$ be closed. We need to check that

$$W = f^{-1}(V) := \{x \in X,\ f(x) \in V\}$$

is closed. Indeed, let $x_n \in W$ be an arbitrary sequence converging in X to some x_0. We need to prove that $x_0 \in W$. Since f is continuous, it maps convergent sequences to the convergent ones. So, $f(x_n) \to f(x_0)$ in Y. But V is closed in Y. Consequently, $f(x_0) \in V$ and, therefore, $x_0 \in W$ and W is closed.

c) Again, there are a lot of such examples. Let me give several possible ones:

i) $X = \mathbb{R}$ (with standard metric) and $f(x) = e^x$. Then $f(X) = (0, \infty)$ is not closed.

ii) $X = \mathbb{N}$ (with the usual metric $d(n,m) = |n - m|$) and $f(n) = 1/n$. Then, $f(X)$ is not closed (zero is a limit point which does not belong to $f(X)$).

iii) $X = (0,1)$ (with the standard metric) and $f(x) = x$. $f(X) = (0,1)$ is not closed as a subset of $\mathbb{R}$.

Problem 3.

a) The totally discrete space (X, d) is complete. Indeed, let $x_n \in X$ be a Cauchy sequence in X. This means that, for every $\varepsilon > 0$ there is $N = N(\varepsilon)$ such that

$$d(x_n, x_{n+m}) < \varepsilon, \quad n \geq N, \ m \in \mathbb{N}$$

Take $\varepsilon < 1$. Then, by the definition of the metric, x_n must coincide with x_{n+m} for any $m \in \mathbb{N}$ and $n \geq N$. Thus,

$$x_n \equiv x_N, \quad \forall n \geq N$$

and, therefore, x_N is a limit of that sequence.

b) The totally discrete space (X, d) is compact if and only if X consists of finitely many elements. Indeed, let X be compact. Then, by the Hausdorff criterium, it can be covered by finitely many ε-balls for any positive ε. But, the ball $B_\varepsilon(x_0) = \{x_0\}$ if $\varepsilon < 1$ and any $x_0 \in X$. So, X must be finite. Wise versa, let X be finite. Then, it is complete and totally bounded and therefore compact.

Problem 4. Assume that such a norm $f \to \|f\|$ exists. Then this norm should satisfy the following property: $f_n \to f$ point-wise ($f_n, f \in C[0,1]$) if and only if $\|f_n - f\| \to 0$.

Let us consider the function $f_0 : \mathbb{R} \to \mathbb{R}$ like

$$f_0(x) = \begin{cases} x(1-x), & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

and let

$$f_n(x) := f_0(2^n x - 1).$$

for $n \in \mathbb{N}$. Then, all these functions are continuous ($f_n \in C[0,1]$), satisfy the additional property

$$f_n(x) \neq 0 \ \text{ if and only if } \ x \in (2^{-n}, 2^{-n+1})$$

and, consequently, for any $x \in [0,1]$ at most one number of $\{f_n(x)\}$ is not zero. (in a fact, a sequence f_n such that, for any fixed $x \in [0,1]$, at most finitely many numbers of $\{f_n(x)\}$, $n \in \mathbb{N}$, are non-zero is enough for the proof)

Let us consider a sequence $\varphi_n(x) := A_n f_n(x)$ where $A_n \in \mathbb{R}$ be arbitrary. Then, on the one hand, this sequence tend to zero point-wise since, for every $x \in [0,1]$ only one of the numbers $\varphi_n(x)$ is non-zero. So, we should have

$$\|\varphi_n\| \to 0$$

as $n \to \infty$. On the other hand, if we fix $A_n := 1/\|f_n\|$ (these A_n are well-defined since f_n are not zero identically), then,

$$\|\varphi_n\| = \|f_n / \|f_n\|\| = 1$$

and do not tend to zero. Contradiction.