Function spaces (MMath): Coursework 1.
Deadline Monday Week 9 (November 29th).

Problem 1. Let $Lip[a,b]$ be a subspace of $C[a,b]$ consisting of all functions f for which the following norm is finite:

$$\|f\|_{Lip} := \|f\|_{sup} + \sup_{x,y\in[a,b],\, x\neq y} \frac{|f(x)-f(y)|}{|x-y|} < \infty$$

1. (1 point): Verify that $\|\cdot\|_{Lip}$ is indeed a *norm* on $Lip[a,b]$.

2. (3 points): Prove that the normed space $(Lip[a,b], \|\cdot\|_{Lip})$ is *complete.*

Problem 2. Let X and Y be two complete metric spaces and $f : X \to Y$ be a continuous *injective* function.

1. (1 point): Prove that, for any set $V \subset X$, $\operatorname{int}\{f(V)\} \subset f(\operatorname{int} V)$.

2. (1 point): Give an example where $f(\operatorname{int} V) \neq \operatorname{int}\{f(V)\}$.

3. (1 point): Give an example of X, Y, V and non-injective function f such that the assertion of 1) is wrong.

Problem 3 (3 points). Let X and Y be *compact* metric spaces and $f : X \to Y$ be a *one-to-one* map satisfying the following property: for every *closed* $V \subset X$ the direct image $f(V)$ is closed. Prove that both f and f^{-1} are continuous. Hint: you may use without proving the criterium of continuity via open sets as well as the fact that V is open iff $X - V$ is closed and that the continuous image of a compact set is compact.

Solutions

Problem 1: The fact that $\|f\|_{Lip}$ is a norm is straightforward. Let us check that the space is complete. Indeed, let f_n be a Cauchy sequence in $Lip[a,b]$. Indeed, by the definition of the Lip-norm, $\|f\|_{sup} \le \|f\|_{Lip}$, therefore, f_n is a Cauchy sequence in $C[a,b]$ and (since $C[a,b]$ is complete), $f_n \to f$ in $C[a,b]$. So, we only need to prove that $f \in Lip[a,b]$ and that there is a convergence $f_n \to f$ in $Lip[a,b]$. Let us check the first statement.

Indeed, since f_n is a Cauchy sequence in $Lip[a,b]$, the sequence of norms $\|f_n\|_{Lip}$ is a convergent sequence in $\mathbb{R}$ and therefore bounded. Thus, if $\|f_n\|_{Lip} \le L$, we have

$$\|f_n(x) - f_n(y)| \le L|x-y|, \quad \forall x, y \in [a,b].$$

Passing to the limit $n \to \infty$ in this inequality (we can do that since $f_n(x) \to f(x)$ uniformly on $[a,b]$), we see that the limit function f is Lipschitz continuous and, therefore, belongs to $Lip[a,b]$.

Let us prove the convergence. Indeed, since f_n is a Cauchy sequence, for every $\varepsilon > 0$, there is $N = N(\varepsilon)$ such that

$$|f_n(x) - f_{n+m}(x) - f_n(y) + f_{n+m}(y)| \le \varepsilon|x-y|, \quad \forall x, y \in [a,b]$$

and for all $n \ge N(\varepsilon)$ and $m \in \mathbb{N}$ (since the convergence in sup-norm is known, we only expand the definition of a Cauchy sequence in the Lip-part of the norm $\|f\|_{Lip}$. Now, passing to the limit $m \to \infty$ in the last inequality, we see that

$$|f_n(x) - f(x) - f_n(y) + f(y)| \le \varepsilon|x-y|, \quad \forall x, y \in [a,b]$$

and $n \ge N(\varepsilon)$. Thus, the convergence of f_n in the Lip-norm is proved and $Lip[a,b]$ is complete.

Problem 2: Let $f : X \to Y$ be an injective continuous function and let $V \subset X$.

a) Let $y_0 \in \operatorname{int} f(V)$. That means, there exists $x_0 \in V$ and $\varepsilon > 0$ such that $B_\varepsilon(f(x_0)) \subset f(V)$. Since f is continuous, there exists $\delta > 0$ such that

$$f(B_\delta(x_0)) \subset B_\varepsilon(f(x_0)) \subset f(V).$$

Since the function f is *injective*, $f(U) \subset f(V)$ implies $U \subset V$ (check if it is not evident!), we have $B_\delta(x_0) \subset V$ and $x_0 \in \operatorname{int} V$. Therefore, $y_0 = f(x_0) \in f(\operatorname{int} V)$.

b) Let, say, $X = \mathbb{R}$, $Y = \mathbb{R}^2$ (with Euclidean norms) and $f(x) = (x, 0)$. Then, the map f obviously injective and continuous, but $\operatorname{int} f(V) = \varnothing$ for every $V \subset X$.

c) Let, say, $X := \mathbb{R}$ and $Y := \mathbb{S}^1$ (a unit circle in $\mathbb{R}^2$ with the usual Euclidean metric) and let $f(x) := (\cos x, \sin x)$. Let also $V := [0, 2\pi]$. Then, $f(V) = \operatorname{int} f(V) = \mathbb{S}^1$, but $f(\operatorname{int} V) = f((0, 2\pi)) = \mathbb{S}^1 - \{(1,0)\}$.

Problem 3: Indeed, since $f : X \to Y$ is one-to-one, the inverse map $f^{-1} : Y \to X$ is well-defined. Moreover, since $f(X - V) = Y - f(V)$ (that also follows from the fact that f is one-to-one, check!) and the set is open iff its complement is closed, we conclude that $f(V)$ is open if V is open. Now, let us consider the inverse map $g = f^{-1} : Y \to X$. According to the continuity criterium, g is continuous if $g^{-1}(V)$

is open in Y for every open $V \subset X$. Since $g^{-1} = f$ and $f(V)$ is open, we conclude that $g = f^{-1}$ is continuous.

Let us now prove that $f = g^{-1}$ is continuous. To this end, we note that, for every closed $W \subset Y$, the image $g(W)$ is closed (since W is compact and the continuous image of a compact is compact). Thus, the map g possesses the same property as f and applying the above reasons to g, we see that $g^{-1} = f$ is continuous.