Function spaces (MMath): Coursework 1.
Deadline Wednesday Week 8 (November 25th).

Problem 1. Let a function $\Phi : \mathbb{R}_+ \to \mathbb{R}$ be monotone *increasing*, *convex* and such that $\Phi(0) = 0$. Recall that the convexity means that the inequality

$$\Phi(\alpha x + \beta y) \le \alpha\Phi(x) + \beta\Phi(y)$$

holds for every non-negative α, β, x and y such that $\alpha + \beta = 1$.

1. (3 points): Prove that the following functional

$$\|f\|_{L_\Phi} := \inf\left\{k : \int_0^1 \Phi\left(\frac{|f(x)|}{k}\right) dx \le 1\right\} \tag{1}$$

defines the *norm* on the space $V = C[0,1]$ of continuous functions.

Hint: In order to verify the triangle inequality, prove first that

$$\Phi\left(\frac{|f_1(x) + f_2(x)|}{k_1 + k_2}\right) \le \frac{k_1}{k_1 + k_2}\Phi\left(\frac{|f_1(x)|}{k_1}\right) + \frac{k_2}{k_1 + k_2}\Phi\left(\frac{|f_2(x)|}{k_2}\right)$$

for all positive k_1 and k_2.

2. (1 point): Compute the norm (1) for the case $\Phi(x) = x^p$, $1 \le p < \infty$ and deduce that the space of continuous functions with the L^p-norm is a *normed* space.

Problem 2. Let X and Y be two complete metric spaces and $f : X \to Y$ be a continuous function.

1. (1 point): Prove that, for any set $V \subset X$, $f(\bar{V}) \subset \overline{f(V)}$.

2. (2 points): Give an example where $f(\bar{V}) \ne \overline{f(V)}$.

Problem 3 (3 points). Let X and Y be two metric spaces and X is *compact*. Assume that the function $f : X \to Y$ is *locally* Lipschitz continuous, i.e., that for every $x_0 \in X$ there exists a ball $B_{r_{x_0}}(x_0)$ and a constant L_{x_0} such that

$$d(f(x), f(y)) \le L_{x_0} d(x, y) \text{ for all } x, y \in B_{r_{x_0}}(x_0).$$

Prove that f is *globally* Lipschitz continuous, i.e., that there exists L such that

$$d(f(x), f(y)) \le L d(x, y) \text{ for all } x, y \in X.$$

Solutions

Problem 1: First, the function $\|f\|_{L_\Phi}$ is well defined. Indeed, let $f \in C[0,1]$, then $|f(x)| \le M$. By convexity of Φ,

$$\Phi(z) \le \frac{z}{M}\Phi(M) + \frac{M-z}{M}\Phi(0) = \frac{z}{M}\Phi(M), \quad z \le M$$

and, consequently, for $k \ge 1$,

$$\int_0^1 \Phi(|f(x)|/k)\,dx \le \frac{\Phi(M)}{Mk}\int_0^1 |f(x)|\,dx \le \frac{\Phi(M)}{k} \to 0$$

as $k \to \infty$. By this reason, there exists a finite $k > 0$ such that $\int_0^1 \Phi(|f(x)|/k)\,dx \le 1$ and $\|f\|_{L_\Phi}$ is well defined.

The function $\|f\|_{L_\Phi}$ is non-negative by definition. Let us check that $\|f\|_{L_\Phi} = 0$ implies that $f = 0$. Indeed, let $\|f\|_{L_\Phi} = 0$. Then,

$$\int_0^1 \Phi(|f(x)|/k)\,dx \le 1$$

for all $k > 0$. Of course, it is implicitly assumed that Φ is not identically zero, so, say, $\Phi(1) > 0$. Then, by convexity and taking into the account that $\Phi(0) = 0$,

$$\Phi(z) \ge \Phi(1)z, \quad z \ge 1$$

and, in particular $\Phi(z) \ge \Phi(1)(z-1)$ for all $z \ge 0$. Thus,

$$\int_0^1 \Phi(|f(x)|/k)\,dx \ge \Phi(1)\frac{\int_0^1 |f(x)|\,dx}{k} - \Phi(1).$$

The right-hand side of that inequality tends to $+\infty$ as $k \to 0$ if $\int_0^1 |f(x)|\,dx \ne 0$. Consequently, $\|f\|_{L_\Phi} = 0$ implies that $\|f\|_{L^1} = 0$ and $f \equiv 0$. Thus, the positivity is verified.

Let us check homogeneity. That is nothing more than scaling $k \to k|\lambda|$ in the definition of the norm. Indeed,

$$\inf\{k, \int_0^1 \Phi(|\lambda f(x)|/k)\,dx \le 1\} = \inf\{|\lambda|k, \int_0^1 \Phi(\frac{|\lambda f(x)|}{|\lambda|k})\,dx \le 1\} =$$
$$= |\lambda| \inf\{k, \int_0^1 \Phi(|f(x)|/k)\,dx \le 1\}.$$

Finally, we need to check the triangle inequality. First, we check the inequality stated in the hint. Indeed, for any $k_1, k_2 > 0$,

$$\frac{|f_1(x) + f_2(x)|}{k_1 + k_2} \le \frac{k_1}{k_1+k_2}\frac{|f_1(x)|}{k_1} + \frac{k_2}{k_1+k_2}\frac{|f_2(x)|}{k_2}.$$

Using the monotonicity and convexity of Φ, we conclude from here that

$$\Phi(\frac{|f_1(x) + f_2(x)|}{k_1 + k_2}) \le \frac{k_1}{k_1+k_2}\Phi(\frac{|f_1(x)|}{k_1}) + \frac{k_2}{k_1+k_2}\Phi(\frac{|f_2(x)|}{k_2})$$

and the desired inequality holds.

Let now $K_1 := \|f_1\|_{L_\Phi}$ and $K_2 := \|f_2\|_{L_\Phi}$. Then, for every $\delta > 0$,

$$\int_0^1 \Phi(\frac{|f_1(x)|}{K_1+\delta})\,dx \le 1 \quad \text{and} \quad \int_0^1 \Phi(\frac{|f_2(x)|}{K_2+\delta})\,dx \le 1.$$

Using the inequality with $k_i = K_i + \delta$, we have

$$\int_0^1 \Phi(\frac{|f_1(x)+f_2(x)|}{k_1+k_2})\,dx \le \frac{k_1}{k_1+k_2}\int_0^1 \Phi(\frac{|f_1(x)|}{k_1})\,dx + \\ + \frac{k_2}{k_1+k_2}\int_0^1 \Phi(\frac{|f_2(x)|}{k_2})\,dx \le \frac{k_1}{k_1+k_2} + \frac{k_2}{k_1+k_2} = 1.$$

Therefore, $\|f_1+f_2\|_{L_\Phi} \le k_1+k_2 = K_1+K_2+2\delta = \|f_1\|_{L_\Phi} + \|f_2\|_{L_\Phi} + 2\delta$ and the triangle inequality holds (since $\delta > 0$ is arbitrary). Thus, the function $f \to \|f\|_{L_\Phi}$ is indeed a norm on $C[0,1]$.

Let us now consider the particular case $\Phi(z) = z^p$ with $1 \le p < \infty$. Then Φ satisfies all of the above conditions and, therefore $(C, \|\cdot\|_{L_\Phi})$ is a normed space. Let us compute this norm explicitly:

$$\inf\{k,\ \int_0^1 \left(\frac{|f(x)|}{k}\right)^p \le 1\} = \inf\{k,\ \|f\|_{L^p}^p \le k^p\} = \inf\{k,\ \|f\|_{L^p} \le k\} = \|f\|_{L^p}.$$

Thus, $\|f\|_{L_\Phi} = \|f\|_{L^p}$ and $L^p(0,1)$ is a normed space.

Problem 2: a) Let $y_0 \in f(\bar V)$. By definition, there exists $x_0 \in \bar V$ such that $y_0 = f(x_0)$ and there exists a sequence $x_n \in V$ such that $x_n \to x_0$. Since f is continuous, $f(x_n) \to f(x_0) = y_0$. But $f(x_n) \in f(V)$ which implies that $y_0 = f(x_0) \in \overline{f(V)}$. Thus, $f(\bar V) \subset \overline{f(V)}$.

b) Let $X = Y = V = \mathbb{R}$ with the standard topology and let $f(x) = \arctan x$. Then $f(V) = f(\bar V) = (-\pi/2, \pi/2)$ and $\overline{f(V)} = [-\pi/2, \pi/2]$.

Problem 3: I will give two alternative proofs: one using the sequential definition of compactness (which we used on the lectures and which is expected to be used here) and the second one using the "covering compactness" which some students tried to use.

Proof I (sequential). Assume that the assertion is wrong. Then, there exist two sequences $x_n, y_n \in X$ such that

$$d(f(x_n), f(y_n)) > n d(x_n, y_n). \tag{1}$$

Since X is compact, we may extract convergent subsequences $x_{n_k} \to x_0$ and $y_{n_k} \to y_0$. Let us prove that $x_0 = y_0$. Indeed, since X is compact and f is continuous, $f(X)$ is also compact. Since the metric d is a continuous function on $f(X)$, it is bounded. Therefore, there exists M such that $d(f(x), f(y)) \le M$ for all $x, y \in X$. Then, from (1) we see that $d(x_{n_k}, y_{n_k}) \le M/n_k \to 0$ as $k \to \infty$. Thus, indeed, $x_0 = y_0$.

So, the points x_{n_k} and y_{n_k} will belong to the ball $B_{r_{x_0}}(x_0)$ if k is large enough. Then, we should have the local Lipschitz continuity

$$d(f(x_{n_k}), f(y_{n_k})) \le L_{x_0} d(x_{n_k}, y_{n_k})$$

which contradicts (1) with n replaced by n_k and this contradiction proves the global Lipschitz continuity.

Proof II (via coverings). Idea: the open balls $B_{r_{x_0}}(x_0)$, $x_0 \in X$ cover X. Using that any covering of a compact set contains a *finite* subcovering, we may fix $x_1, \cdots, x_N$ such that $X \subset \cup_{i=1}^N B_{r_{x_i}}(x_i)$ and fix after that $L := \max\{L_{x_1}, \cdots, L_{x_n}\}$ which looks as a desired global Lipschitz constant.

However, there is a *difficulty* here, namely, a priori x and y may be very close to each other, but belong to *different* balls of that covering and then we are unable to estimate the distance $d(f(x), f(y))$ using only the local Lipschitz continuity in that balls.

Standard *trick* which solves this problem is to consider (instead of covering $X = \cup_{x\in X} B_{r_x}(x)$) another covering $X = \cup_{x\in X} B_{r_x/2}(x)$ and to extract the finite subcovering

$$X = \cup_{i=1}^M B_{r_{x_i}/2}(x_i)$$

and take $\varepsilon_0 := \min\{r_{x_i}/2\}$. The difference is that NOW if $x \in B_{r_{x_i}/2}(x_i)$ and $d(x, y) \le \varepsilon_0$, we have (due to the triangle inequality) $y \in B_{r_{x_i}}(x_i)$ and we may use the local Lipschitz estimate.

Thus, we have proved that

$$d(f(x), f(y)) \le \bar{L} d(x, y), \quad \text{if } d(x, y) \le \varepsilon_0$$

with $\bar{L} = \max L_{x_i}$. The case when $d(x, y) > \varepsilon_0$ can be easily estimated using that $f(X)$ is bounded (see previous proof), namely,

$$\frac{d(f(x), f(y))}{d(x, y)} \le \frac{M}{\varepsilon_0}.$$

Finally, the global Lipschitz continuity holds with $L := \max\{\bar{L}, \frac{M}{\varepsilon_0}\}$.