COURSEWORK 1 FOR "INTRODUCTION TO FUNCTION SPACES" (BSC)

Deadline: 14 of October 2007

Problem 1 (2 points): Let $V = \mathbb{R}_+ := \{x \in \mathbb{R},\ x > 0\}$ and let

$$x \oplus y := xy, \quad x, y \in V, \quad xy \text{ is a usual multiplication}, \quad \alpha \cdot x = x^\alpha, \ \alpha \in \mathbb{R}$$

and $\vec{0} = 1$.

a) check that V is a vector space;

b) check that $\|x\| := |\log x|$ is a norm on V;

c) what is the dimension of V?

Problem 2 (3 points):
a) Give an example of a closed set G and an open set V (possibly in different metric spaces) such that

$$\overline{\operatorname{int} G} \neq G \quad \text{and} \quad \operatorname{int}(\bar{V}) \neq V.$$

b) Prove that, for any continuous function $f : X \to Y$ (X and Y are metric spaces) and any closed set G in Y, the inverse image $f^{-1}(G)$ is a closed set in X.

c) Give an example of a metric space X and a continuous function $f : X \to \mathbb{R}$ such that $f(X)$ is not closed in $\mathbb{R}$.

Problem 3 (3 points): Let $X = \mathbb{R}$ and $d(x, y) := \frac{|x-y|}{1+|x-y|}$.
Check that (X, d) is a metric space. Is it a normed space? (explain your answer).

Problem 4 (2 points): Consider the metric space (X, d) where X is an arbitrary set and

$$d(x, y) := \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases}$$

a) Is (X, d) a *complete* metric space?

b) Under what conditions on the set X, the metric space (X, d) will be compact? Explain your answer.

Solutions

Problem 1. The problem can be solved by direct checking the axioms of a vector space and a norm. But there exists a more elegant proof. Namely, define a map $f : V \to \mathbb{R}$ as follows

$$f(x) = \log x$$

Then, obviously, f is one-to-one and onto. Moreover,

$$f(x \oplus y) = f(xy) = \log(xy) = \log x + \log y = f(x) + f(y), \quad x, y \in V$$

and

$$f(\alpha \cdot x) = f(x^\alpha) = \log(x^\alpha) = \alpha \log x = \alpha f(x)$$

for all $\alpha \in \mathbb{R}$ and $x \in V$. Thus, f is a *linear* isomorphism between the space V and $\mathbb{R}$ (with the standard linear structure on $\mathbb{R}$). Since $\mathbb{R}$ is a vector space, $V = f^{-1}(\mathbb{R})$ is also a vector space. Moreover, since $\dim R = 1$, the dimension of $\mathbb{R}$ is also one.

In addition, for any $x \in V$

$$\|x\| = |\log x| = |f(x)|,$$

so f is an *isometry* between $(V, \|\cdot\|)$ and $(\mathbb{R}, |\cdot|)$ and, by this reason $\|x\|$ is a norm on V.

Problem 2.

a) There are a lot of such examples. The simplest ones are on the real line $X = \mathbb{R}$ with the standard metric: a closed set $G := \{0\}$ and an open set $V := (0,1) \cup (1,2)$. Then, $\operatorname{int} V = \varnothing$ and $\overline{\operatorname{int} V} = \varnothing \neq V$. Analogously, $\bar{V} = [0,2]$ and $\operatorname{int}(\bar{V}) = (0,2) \neq V$.

b) There are two natural ways to prove this fact. The first one is to use that a set is closed if and only if its complement is open together with the fact that the inverse image $f^{-1}(V)$ is open if V is open and f is continuous (the theorem proved in the lecture notes). The second one is a direct proof. Let me give more details for the second way.

Let $V \subset Y$ be closed. We need to check that

$$W = f^{-1}(V) := \{x \in X,\ f(x) \in V\}$$

is closed. Indeed, let $x_n \in W$ be an arbitrary sequence converging in X to some x_0. We need to prove that $x_0 \in W$. Since f is continuous, it maps convergent sequences to the convergent ones. So, $f(x_n) \to f(x_0)$ in Y. But V is closed in Y. Consequently, $f(x_0) \in V$ and, therefore, $x_0 \in W$ and W is closed.

c) Again, there are a lot of such examples. Let me give several possible ones:
i) $X = \mathbb{R}$ (with standard metric) and $f(x) = e^x$. Then $f(X) = (0, \infty)$ is not closed.
ii) $X = \mathbb{N}$ (with the usual metric $d(n,m) = |n - m|$) and $f(n) = 1/n$. Then, $f(X)$ is not closed (zero is a limit point which does not belong to $f(X)$).
iii) $X = (0,1)$ (with the standard metric) and $f(x) = x$. $f(X) = (0,1)$ is not closed as a subset of $\mathbb{R}$.

Problem 3. Let us check the axioms of the norm. Obviously, $d(x,y) = d(y,x)$, $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$. So, we only need to check the triangle inequality

$$\frac{|x-y|}{1+|x-y|} \leq \frac{|x-z|}{1+|x-z|} + \frac{|y-z|}{1+|y-z|} \tag{1}$$

for any $x, y, z \in \mathbb{R}$. The key ingredient of the proof is, of course, the triangle inequality for the usual norm on $\mathbb{R}$

$$|x-y| \leq |x-z| + |y-z| \tag{2}$$

and, in fact, one can check (1) directly by finding the common denominator, simplifying the obtained expression and using (2) in the proper place. But let me give a more elegant proof. Indeed,

$$\begin{aligned} \frac{|x-z|}{1+|x-z|} + \frac{|y-z|}{1+|y-z|} &= \frac{|x-z| + |y-z| + 2|x-z| \cdot |y-z|}{(1+|x-z|)(1+|y-z|)} \geq \\ &\geq \frac{|x-z| + |y-z| + |x-z| \cdot |y-z|}{1+|x-z|+|y-z|+|x-z| \cdot |y-z|} = \frac{|x-y| + W}{1+|x-y|+W} \end{aligned} \tag{3}$$

where $W = |x-z| + |y-z| + |x-z| \cdot |y-z| - |x-y| \geq 0$ thanks to (2). Let us now consider $f(z) := \frac{z}{z+1}$. This function is monotone increasing since $f'(z) = \frac{1}{(z+1)^2} > 0$ and, therefore, since $|x-y| + W \geq |x-y|$, we have

$$d(x,z) + d(y,z) \geq f(|x-y| + W) \geq f(|x-y|) = d(x,y)$$

and the triangle inequality is verified.

Problem 4.

a) The totally discrete space (X,d) is complete. Indeed, let $x_n \in X$ be a Cauchy sequence in X. This means that, for every $\varepsilon > 0$ there is $N = N(\varepsilon)$ such that

$$d(x_n, x_{n+m}) < \varepsilon, \quad n \geq N, \ m \in \mathbb{N}$$

Take $\varepsilon < 1$. Then, by the definition of the metric, x_n must coincide with x_{n+m} for any $m \in \mathbb{N}$ and $n \geq N$. Thus,

$$x_n \equiv x_N, \quad \forall n \geq N$$

and, therefore, x_N is a limit of that sequence.

b) The totally discrete space (X,d) is compact if and only if X consists of finitely many elements. Indeed, let X be compact. Then, by the Hausdorff criterium, it can be covered by finitely many ε-balls for any positive ε. But, the ball $B_\varepsilon(x_0) = \{x_0\}$ if $\varepsilon < 1$ and any $x_0 \in X$. So, X must be finite. Wise versa, let X be finite. Then, it is complete and totally bounded and therefore compact.