Introduction to Function spaces (BsC): Coursework 1. Deadline Monday Week 9 (November 29th).

Problem 1. Let $C^2[a,b]$ be a space of twice continuously differentiable functions f with the norm:

$$\|f\|_{C^2} := \|f\|_{sup} + \|f'\|_{sup} + \|f''\|_{sup}$$

1. **(1 point):** Verify that $\|\cdot\|_{C^2}$ is indeed a *norm* on $C^2[a,b]$.
2. **(3 points):** Prove that the normed space $(C^2[a,b], \|\cdot\|_{C^2})$ is *complete*.

Problem 2. Let (X,d) be a metric space and $V_1, V_2 \subset X$ be two subset in it.

1. **(1 point):** Prove that $\text{int}(V_1) \cup \text{int}(V_2) \subset \text{int}(V_1 \cup V_2))$.
2. **(1 point):** Give an example where $\text{int}(V_1) \cup \text{int}(V_2) \neq \text{int}(V_1 \cup V_2))$.

Problem 3. Let $C[-1,1]$ be the space of continuous functions $f : [-1,1] \to \mathbb{R}$ with the standard sup-norm and let the function $l : C[-1,1] \to \mathbb{R}$ be defined as follows

$$l(f) := \int_{-1}^{1} \text{sgn}(x) f(x)\, dx.$$

1) (1 point): Prove that the function l is *continuous*.

2) (2 points): Let $\bar{B}_1(0)$ be the closed unit ball in $C[a,b]$. Prove that

$$\sup_{f \in \bar{B}_1(0)} |l(f)| = 2. \tag{1}$$

Hint: Use the standard approximations f_n of the function $\text{sgn}(x)$ by continuous functions.

3) (1 point): Prove that the *maximum* in (1) does not exist.

Solutions

Problem 1: The fact that $\|f\|_{C^2}$ is a norm is immediate (since $\|\cdot\|_{sup}$ is a norm). Let us check completness. Indeed, let f_n be a Cauchy sequence in C^2. Then, by the definition of the norm in C^2, f_n, f_n' and f_n'' are Cauchy sequences in $C[a,b]$ (with the usual sup-norm), since this space is complete, we conclude that $f_n \to f$, $f_n' \to g$ and $f_n'' \to h$ for some continuous functions f, g, h. To complete the proof, we only need to check that $f \in C^2$, $g(x) = f'(x)$ and $h(x) = f''(x)$. Note that these assertions are equivalent to

$$(*) \qquad f(x) - f(a) = \int_a^x g(s)\,ds, \quad g(x) - g(a) = \int_a^x h(s)\,ds.$$

Note that,

$$f_n(x) - f_n(a) = \int_a^x f_n'(s)\,ds, \quad f_n'(x) - f_n'(a) = \int_a^x f_n''(s)\,ds$$

and passing to the limit $n \to \infty$ in that formulas (it is possible to do due to the uniform convergences $f_n \to f$, $f_n' \to g$ and $f_n'' \to h$), we end up with $(*)$.

Problem 2:

a) Let $x_0 \in \operatorname{int}(V_1) \cup \operatorname{int}(V_2)$. Then, $x_0 \in \operatorname{int}(V_1)$ or $x_0 \in \operatorname{int}(V_2)$. By the definition of an interior point, there exists $\delta > 0$ such that $B_\delta(x_0) \subset V_1$ or $B_\delta(x_0) \subset V_2$. In both cases, $B_\delta(x_0) \in V_1 \cup V_2$ and $x_0 \in \operatorname{int}(V_1 \cup V_2)$.

b) Let $X = \mathbb{R}$ with standard norm and $V_1 = [0,1]$, $V_2 = [1,2]$. Then,

$$\operatorname{int}(V_1 \cup V_2) = (0,2) \neq \operatorname{int}(V_1) \cup \operatorname{int}(V_2) = (0,1) \cup (1,2).$$

Problem 3:

a) Let f_1 and f_2 be two functions from $C[-1,1]$. We need to estimate

$$|l(f_1) - l(f_2)| = |l(f_1 - f_2)| = |\int_{-1}^1 \operatorname{sgn}(x)(f_1(x) - f_2(x))\,dx| \leq$$
$$\leq \int_{-1}^1 |f_1(x) - f_2(x)|\,dx \leq 2\|f_1 - f_2\|_{sup}.$$

Thus, the function $l : C[-1,1] \to \mathbb{R}$ is continuous (and even Lipschitz continuous with Lipschitz constant 2).

b) The desired supremum is less or equal 2 due to the previous estimate with $f_2 \equiv 0$, so we only need to prove the equality. To this end we consider the standard approximations of $\operatorname{sgn}(x)$ by continuous functions f_n:

$$f_n(x) = \begin{cases} \operatorname{sgn}(x), & |x| > 1/n \\ nx, & |x| \leq 1/n \end{cases}$$

Then, $f_n \in \bar{B}_1(0)$ (since $\|f_n\|_{sup} = 1$) and

$$2 - l(f_n) = l(\operatorname{sgn}(x)) - l(f_n) = 2\int_0^{1/n} (1 - nx)\,dx = 1/n \to 0$$

as $n \to \infty$. This proves that the supremum equals 2.

c) Indeed, let the maximum exists and is achieved on a *continuous* function $f \in C[0,1]$. Then, $|f(x)| \le 1$ everywhere,

$$\int_0^1 f(x)\,dx \le 1$$

and the equality is achieved iff $f(x) \equiv 1$ for all $x \in [0,1]$. Analogously,

$$-\int_{-1}^0 f(x)\,dx \le 1$$

and the equality holds iff $f(x) = -1$ for all $x \in [-1,0]$. Thus, the equality

$$l(f) = \int_0^1 f(x)\,dx - \int_{-1}^0 f(x)\,dx = 2$$

is possible only if $f(x) = \operatorname{sgn}(x)$ which is not continuous. Thus, the maximum does not exist.