Introduction to Function Spaces (BsC): Coursework 1. Deadline Wednesday Week 8 (November 25th).

Problem 1.

1. (1 point): Prove that $d(x, y) := \min\{1, |x - y|\}$ is a metric on the space $\mathbb{R}$.

2. (1 point): Consider the space $\mathbb{R}^\infty$ of all sequences $x = (x_1, x_2, \cdots,)$ (not necessarily bounded!) and let

$$d(x, y) := \sum_{n=1}^{\infty} 2^{-n} \min\{1, |x_n - y_n|\}.$$

Prove that d is a metric on $\mathbb{R}^\infty$.

3. (2 points): Prove that $x^k \to x^0$ in $\mathbb{R}^\infty$ (with the metric introduced above) if and only if $x_n^k \to x_n^0$ as $k \to \infty$ for every $n \in \mathbb{N}$ (coordinate-wise convergence). Is this space *complete*? Justify your answer.

Problem 2. Let X and Y be two complete metric spaces and $f : X \to Y$ be a continuous function.

1. (1 point): Prove that, for any set $V \subset X$, $f(\bar{V}) \subset \overline{f(V)}$.

2. (2 points): Give an example where $f(\bar{V}) \neq \overline{f(V)}$.

Problem 3 (3 points). Let X and Y be two metric spaces and X is *compact.* Assume that the function $f : X \to Y$ is *locally* Lipschitz continuous, i.e., that for every $x_0 \in X$ there exists a ball $B_{r_{x_0}}(x_0)$ and a constant L_{x_0} such that

$$d(f(x), f(y)) \leq L_{x_0} d(x, y) \text{ for all } x, y \in B_{r_{x_0}}(x_0).$$

Prove that f is *globally* Lipschitz continuous, i.e., that there exists L such that

$$d(f(x), f(y)) \leq L d(x, y) \text{ for all } x, y \in X.$$

Solutions

Problem 1: 1) Typical *mistake* here is to use the "natural" inequality

$$\min\{A_i + B_i\} \le \min\{A_i\} + \min\{B_i\}$$

which is however WRONG (and, in a fact, the opposite inequality always holds!). Indeed, let $f_1(x) = (x-1)^2$ and $f_2(x) = (x+1)^2$. Then, $\min_{x\in\mathbb{R}} f_i(x) = 0$, but $\min_{x\in\mathbb{R}}\{f_1(x) + f_2(x)\} = \min_{x\in\mathbb{R}}\{2x^2+2\} = 2 > 0$.

So, one should be a bit more accurate in checking the triangle inequality (the only thing which requires attention here). Namely, let $x, y, z \in \mathbb{R}$. Then, consider two cases. Case I: one of numbers $|x-y|$ or $|y-z|$ is greater than one. Then,

$$\min\{1, |x-z|\} \le 1 \le \min\{1, |x-y|\} + \min\{1, |y-z|\}.$$

Case II: both of that numbers are less or equal one. Then,

$$\min\{1, |x-z|\} \le |x-z| \le |x-y| + |y-z| = \min\{1, |x-y|\} + \min\{1, |y-z|\}.$$

We see that the triangle inequality holds in both cases and, consequently, it is indeed a metric.

2) First, $d(x,y)$ is well-defined that the series is convergent for all $x, y \in \mathbb{R}^\infty$. Then, it is obviously non-negative and symmetric. Furthermore, $d(x,y) = 0$ implies that $\min\{1, |x_n - y_n|\} = 0$ for all n which gives $x_n = y_n$ and $x = y$. So, we only need to check the triangle inequality. To do that, we note that $\min\{1, |x_n - y_n|\}$ satisfies the triangle inequality for all n (by the first part of the problem) and summing them with coefficients 2^{-n}, we obtain the triangle inequality for $d(x,y)$.

3) Let $x^k \to x^0$ in $\mathbb{R}^\infty$. Then, $d(x^k, x^0) \to 0$. But, obviously,

$$2^{-n}\min\{1, |x_n^k - x_n^0|\} \le d(x^k, x^0) \tag{2}$$

for any n and, therefore, $|x_n^k - x_n^0| \to 0$ as $k \to \infty$ (for every fixed n). Thus, $x^k \to x^0$ coordinate-wise.

Assume now that $x^k \to x^0$ coordinate-wise. Fix arbitrary $\varepsilon > 0$ and fix $N = N(\varepsilon)$ in such way that

$$\sum_{n=N+1}^{\infty} 2^{-n} < \varepsilon/2.$$

Then,

$$d(x^k, x^0) \le \sum_{n=1}^{N} |x_n^k - x_n^0| + \varepsilon/2.$$

Now, since $x^k \to x^0$ coordiante-wise, we may fix $K = K(\varepsilon)$ such that $|x_n^k - x_n^0| \le \varepsilon/(2N)$ if $n \le N$ and $k \ge K$. Then,

$$d(x^k, x^0) \le \sum_{n=1}^{N} \varepsilon/(2N) + \varepsilon/2 = \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus, $x^k \to x^0$ in $\mathbb{R}^\infty$. So, we have verified that the coordinate-wise convergence is equivalent to the metric convergence in $\mathbb{R}^\infty$.

The space $\mathbb{R}^\infty$ is *complete* . Indeed, let $x^k \in \mathbb{R}^\infty$ be a Cauchy sequence. Then, by inequality (2), $x_n^k \in \mathbb{R}$ is a Cauchy sequence in $\mathbb{R}$ for all n. Since $\mathbb{R}$ is complete, $x^k \to x^0$ coordinate-wise and then, as proved before, $x^k \to x^0$ in $\mathbb{R}^\infty$.

Problem 2: a) Let $y_0 \in f(\bar{V})$. By definition, there exists $x_0 \in \bar{V}$ such that $y_0 = f(x_0)$ and there exists a sequence $x_n \in V$ such that $x_n \to x_0$. Since f is continuous, $f(x_n) \to f(x_0) = y_0$. But $f(x_n) \in f(V)$ which implies that $y_0 = f(x_0) \in \overline{f(V)}$. Thus, $f(\bar{V}) \subset \overline{f(V)}$.

b) Let $X = Y = V = \mathbb{R}$ with the standard topology and let $f(x) = \arctan x$. Then $f(V) = f(\bar{V}) = (-\pi/2, \pi/2)$ and $\overline{f(V)} = [-\pi/2, \pi/2]$.

Problem 3: I will give two alternative proofs: one using the sequential definition of compactness (which we used on the lectures and which is expected to be used here) and the second one using the "covering compactness" which some students tried to use.

Proof I (sequential). Assume that the assertion is wrong. Then, there exist two sequences $x_n, y_n \in X$ such that

$$d(f(x_n), f(y_n)) > nd(x_n, y_n). \tag{1}$$

Since X is compact, we may extract convergent subsequences $x_{n_k} \to x_0$ and $y_{n_k} \to y_0$. Let us prove that $x_0 = y_0$. Indeed, since X is compact and f is continuous, $f(X)$ is also compact. Since the metric d is a continuous function on $f(X)$, it is bounded. Therefore, there exists M such that $d(f(x), f(y)) \le M$ for all $x, y \in X$. Then, from (1) we see that $d(x_{n_k}, y_{n_k}) \le M/n_k \to 0$ as $k \to \infty$. Thus, indeed, $x_0 = y_0$.

So, the points x_{n_k} and y_{n_k} will belong to the ball $B_{r_{x_0}}(x_0)$ if k is large enough. Then, we should have the local Lipschitz continuity

$$d(f(x_{n_k}), f(y_{n_k})) \le L_{x_0} d(x_{n_k}, y_{n_k})$$

which contradicts (1) with n replaced by n_k and this contradiction proves the global Lipschitz continuity.

Proof II (via coverings). Idea: the open balls $B_{r_{x_0}}(x_0)$, $x_0 \in X$ cover X. Using that any covering of a compact set contains a *finite* subcovering, we may fix $x_1, \cdots, x_N$ such that $X \subset \cup_{i=1}^N B_{r_{x_i}}(x_i)$ and fix after that $L := \max\{L_{x_1}, \cdots, L_{x_n}\}$ which looks as a desired global Lipschitz constant.

However, there is a *difficulty* here, namely, a priori x and y may be very close to each other, but belong to *different* balls of that covering and then we are unable to estimate the distance $d(f(x), f(y))$ using only the local Lipschitz continuity in that balls.

Standard *trick* which solves this problem is to consider (instead of covering $X = \cup_{x \in X} B_{r_x}(x)$) another covering $X = \cup_{x \in X} B_{r_x/2}(x)$ and to extract the finite subcovering

$$X = \cup_{i=1}^M B_{r_{x_i}/2}(x_i)$$

and take $\varepsilon_0 := \min\{r_{x_i}/2\}$. The difference is that NOW if $x \in B_{r_{x_i}/2}(x_i)$ and $d(x,y) \leq \varepsilon_0$, we have (due to the triangle inequality) $y \in B_{r_{x_i}}(x_i)$ and we may use the local Lipschitz estimate.

Thus, we have proved that

$$d(f(x), f(y)) \leq \bar{L} d(x,y), \quad \text{if } d(x,y) \leq \varepsilon_0$$

with $\bar{L} = \max L_{x_i}$. The case when $d(x,y) > \varepsilon_0$ can be easily estimated using that $f(X)$ is bounded (see previous proof), namely,

$$\frac{d(f(x), f(y))}{d(x,y)} \leq \frac{M}{\varepsilon_0}.$$

Finally, the global Lipschitz continuity holds with $L := \max\{\bar{L}, \frac{M}{\varepsilon_0}\}$.