Function spaces (MMath): Coursework 2.
Deadline: Friday Week 12 (January 15th).

Problem 1. Let $V := C[0, 1/2]$ with the standard sup-norm and let $F : V \to V$ be the map defined by

$$F(f)(x) := 1 + \int_0^x f(x-s)\, ds.$$

a) (2 points). Prove that F is a *contraction* on V with the contraction factor $\kappa = 1/2$.

b) (2 points). Find the unique fixed point of that map (which exists due to the Banach contraction theorem).

Problem 2. Let $V := C[-1, 1]$ and let

$$(f, g) := \int_{-1}^1 (1-x^2) f(x) g(x)\, dx.$$

a) (1 point). Prove that (f, g) is an inner product on V.

b) (2 points). Orthogonalize the system $\{1, x, x^2, x^3\}$ with respect to this inner product using the Gram-Schmidt orthogonalization process.

Problem 3.

a) (1 point). Expand the function $f(x) := \cosh(x) = \frac{e^x+e^{-x}}{2}$ into the Fourier series at $[-\pi, \pi]$.

b) (1 point). Do the partial Fourier sums $f_N(x)$ converge *point-wise* and/or *uniformly* to the initial function f? Justify your answer.

c) (1 point). Using the Fourer expansions of $f(x)$, compute

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1}.$$

Solutions

Problem 1: Let $f_1, f_2 \in V$ be arbitrary. Then,

$$|F(f_1)(x) - F(f_2)(x)| \le \int_0^x |f_1(x-s) - f_2(x-s)|\, ds \le$$

$$\le \sup_{s\in[0,x]} |f_1(x-s) - f_2(x-s)| \int_0^x dx = x \sup_{y\in[0,x]} |f_1(y) - f_2(y)| \le 1/2 \|f_1 - f_2\|_V.$$

Taking the supremum over x, we have

$$\|F(f_1) - F(f_2)\|_V \le \frac{1}{2}\|f_1 - f_2\|_V$$

and F is a contraction with the contraction factor $1/2$. Thus, there is a *unique* fixed point of this map which satisfies

$$f(x) = 1 + \int_0^x f(x-s)\, ds.$$

Changing the dependent variable $s = x - y$ in the integral, we get the standard equation for the exponent

$$f(x) = 1 + \int_0^x f(y)\, dy$$

and $f(x) = e^x$.

Problem 2. For a) one just needs to check all axioms which is straightforward. Let us do b). Very effective here is to use the odd/even structure which reduces greatly the number of calculations. Indeed, since $v_1 = 1$ and $v_2 = x$ are odd and even respectively, they are already orthogonal and we may take $e_1 = 1$ and $e_2 = x$.

Now, $v_3 = x^2$ is orthogonal to v_2 by the same reason, so, we may seek for the third vector in the form $e_3 = x^2 - \alpha v_1 = x^2 - \alpha$ for some $\alpha \in \mathbb{R}$ to be found from $(e_3, e_1) = 0$, i.e.,

$$(e_3, e_1) = 2\int_0^1 (1 - x^2)(x^2 - \alpha)\, dx = 2(\frac{1}{3} - \frac{1}{5} - \alpha + \frac{\alpha}{3}) = 0 \ \ => \ \ \alpha = \frac{1}{5}.$$

Thus, $e_3 = x^2 - \frac{1}{5}$.

Finally, $v_4 = x^3$ is orthogonal to $v_1 = e_1$ and $e_3 = x^2 - \frac{1}{5}e_1$. So, we may seek for $e_4 := x^3 - \beta x$ with some $\beta \in \mathbb{R}$ to be found from $(e_4, e_1) = 0$, i.e.,

$$(e_4, e_1) = 2\int_0^1 (1 - x^2)x^2(x^2 - \beta)\, dx = 2(\frac{1}{5} - \frac{1}{7} - \frac{\beta}{3} + \frac{\beta}{5}) = 0 \ \ => \ \ \beta = \frac{3}{7}.$$

Thus, $e_4 = x^3 - \frac{3}{7}x$ and the othogonilized system is $\{1, x, x^2 - \frac{1}{5}, x^3 - \frac{3}{7}x\}$. After that, one may make the system *orthonormal* by dividing the vectors by their norms, but it is NOT ASKED in the statement of the problem.

Problem 3. The function is *even*, so only a_n's are non-zero. To compute the coefficients, one may proceed as follows:

$$\int_{-\pi}^{\pi} \cosh x \cos nx\, dx = 2 \int_{-} \pi^{\pi} e^x \cos x\, dx = \operatorname{Re} \int_{-\pi}^{\pi} e^x e^{inx}\, dx =$$

$$\operatorname{Re} \int_{-\pi}^{\pi} e^{(1+in)x}\, dx = \operatorname{Re} \frac{1}{1+in} e^{(1+in)x}\Big|_{-\pi}^{\pi} =$$

$$= 2(-1)^n \sinh \pi \operatorname{Re} \frac{1}{1+ni} = \frac{2(-1)^n \sinh \pi}{1+n^2}.$$

That gives $a_0 = \frac{\sinh \pi}{\pi}$, $a_n = \frac{2(-1)^n}{\pi(1+n^2)}$ and

$$\cosh x \sim \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} \cos nx.$$

The periodic extension of this function is piece-wise smooth and *continuous* (since $f(-\pi) = f(\pi)$), so, by Dirichlet theorem, we have the point-wise convergence to that periodic extension and this convergence is uniform.

Finally, to compute the sum, we need to take $x = \pi$ in the obtained expansion. This gives

$$\cosh \pi = \frac{2 \sinh \pi}{\pi} (1/2 + \sum_{n=1}^{\infty} \frac{1}{1+n^2}) = \frac{2 \sinh \pi}{\pi} (-1/2 + \sum_{n=0}^{\infty} \frac{1}{1+n^2})$$

which gives

$$\sum_{n=0}^{\infty} \frac{1}{1+n^2} = \frac{1}{2} + \frac{\pi}{2 \tanh \pi}.$$