Function Spaces (MMATH): Coursework 2.
Deadline Tuesday Week 12 (January 11th).

Problem 1. Let $X := C[0, 10]$ be the space of continuous functions on a segment $[0, 10]$ and let $F : X \to X$ be defined as follows:

$$F(f)(x) := \frac{1}{2}f(x) + 1 + \int_0^x f(s)\, ds$$

1. (1 point): Is F a contraction on X with the usual sup-norm? Justify your answer.

2. Let us introduce the following weighted norm on X:

$$\|f\|_w := \sup_{x\in[0,10]} \{e^{-2x}|f(x)|\}$$

a) (0.5 points): Check that this norm is equivalent to the usual sup-norm.
b) (3 points): Prove that F is a contraction in X with the $\|f\|_w$-norm.

3. (0.5 points): Find the fixed point of this map.

Problem 2. Let $f_a(x) := \cos(\frac{x}{a})$, $x \in [-\pi, \pi]$, where $a > 0$ is a fixed real number.

1. (1.5 points): Find the coefficients a_n and b_n in the Fourier expansions

$$f_a(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

in terms of the parameter a.

2. (1 point): Let $f_N(x) := a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx)$, $x \in \mathbb{R}$ (now x is not only in $[-\pi, \pi]$!) be the Nth partial sum of that series. Find the *point-wise* limit $N \to \infty$ of $f_N(x)$. Is the convergence *uniform*? Justify your answer.

3. (1 point): Prove that

$$\sum_{n=1}^{\infty} \frac{1}{a^2n^2 - 1} = \frac{1}{2} \cdot \frac{a - \pi\cot(\pi/a)}{a}.$$

4. (1.5 points): Deduce from the last formula that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Hint: use that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{a\to\infty} \sum_{n=1}^{\infty} \frac{1}{n^2 - 1/a^2}.$$