Function spaces (BSc): Coursework 2.
Deadline: Friday Week 12 (January 15th).

Problem 1. Let $V := C[0, 1/2]$ with the standard sup-norm and let $F : V \to V$ be the map defined by

$$F(f)(x) := 1 + \frac{f(x)}{3} + \int_0^x f(s)\,ds.$$

a) (2 points). Prove that F is a *contraction* on V with the contraction factor $\kappa = 5/6$.

b) (2 points). Find the unique fixed point of that map (which exists due to the Banach contraction theorem).

Problem 2. Let $V := C[-1, 1]$ and let

$$(f, g) := \int_{-1}^{1} (1 - x^2) f(x) g(x)\,dx.$$

a) (1 point). Prove that (f, g) is an inner product on V.

b) (2 points). Orthogonalize the system $\{1, x, x^2, x^3\}$ with respect to this inner product using the Gram-Schmidt orthogonalization process.

Problem 3.

a) (1 point). Expand the function $f(x) := \operatorname{sgn}(x)$ into the Fourier series at $[-\pi, \pi]$.

b) (1 point). Describe *point-wise* limit of the partial Fourier sums $f_N(x)$, $x \in \mathbb{R}$, as $N \to \infty$. Is that convergence *uniform* with respect to $x \in \mathbb{R}$? Justify your answer.

c) (1 point). Using the Fourer expansions of $f(x)$, prove that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k + 1} = \pi/4.$$

Solutions

Problem 1: Let $f_1, f_2 \in V$ be arbitrary. Then,

$$|F(f_1)(x) - F(f_2)(x)| \leq \frac{1}{3}|f_1(x) - f_2(x)| + \int_0^{1/2} |f_1(s) - f_2(s)|\,ds \leq$$
$$\leq \frac{1}{3}|f_1(x) - f_2(x)| + \sup_{s\in[0,1/2]} |f_1(s) - f_2(s)| \int_0^{1/2} dx =$$
$$= \frac{1}{3}|f_1(x) - f_2(x)| + \frac{1}{2}\|f_1 - f_2\|_V.$$

Taking the supremum over x, we have

$$\|F(f_1) - F(f_2)\|_V \leq \frac{1}{3}\|f_1 - f_2\|_V + \frac{1}{2}\|f_1 - f_2\|_V = \frac{5}{6}\|f_1 = f_2\|_V$$

and F is a contraction with the contraction factor $5/6$. Thus, there is a *unique* fixed point of this map which satisfies

$$f(x) = 1 + \frac{1}{3}f(x) + \int_0^x f(s)\,ds \;\;<=>\;\; f(x) = \frac{3}{2} + \frac{3}{2}\int_0^x f(s)\,ds.$$

We see that f is differentiable, $f'(x) = \frac{3}{2}f(x)$ and $f(0) = \frac{3}{2}$. Thus, $f(x) = \frac{3}{2}e^{\frac{3}{2}x}$.

Problem 2. For a) one just needs to check all axioms which is straightforward. Let us do b). Very effective here is to use the odd/even structure which reduces greatly the number of calculations. Indeed, since $v_1 = 1$ and $v_2 = x$ are odd and even respectively, they are already orthogonal and we may take $e_1 = 1$ and $e_2 = x$.

Now, $v_3 = x^2$ is orthogonal to v_2 by the same reason, so, we may seek for the third vector in the form $e_3 = x^2 - \alpha v_1 = x^2 - \alpha$ for some $\alpha \in \mathbb{R}$ to be found from $(e_3, e_1) = 0$, i.e.,

$$(e_3, e_1) = 2\int_0^1 (1 - x^2)(x^2 - \alpha)\,dx = 2(\frac{1}{3} - \frac{1}{5} - \alpha + \frac{\alpha}{3}) = 0 \;\;=>\;\; \alpha = \frac{1}{5}.$$

Thus, $e_3 = x^2 - \frac{1}{5}$.

Finally, $v_4 = x^3$ is orthogonal to $v_1 = e_1$ and $e_3 = x^2 - \frac{1}{5}e_1$. So, we may seek for $e_4 := x^3 - \beta x$ with some $\beta \in \mathbb{R}$ to be found from $(e_4, e_1) = 0$, i.e.,

$$(e_4, e_1) = 2\int_0^1 (1 - x^2)x^2(x^2 - \beta)\,dx = 2(\frac{1}{5} - \frac{1}{7} - \frac{\beta}{3} + \frac{\beta}{5}) = 0 \;\;=>\;\; \beta = \frac{3}{7}.$$

Thus, $e_4 = x^3 - \frac{3}{7}x$ and the othogonilized system is $\{1, x, x^2 - \frac{1}{5}, x^3 - \frac{3}{7}x\}$. After that, one may make the system *orthonormal* by dividing the vectors by their norms, but it is NOT ASKED in the statement of the problem.

Problem 3. The function is *odd*, so all a_n's are zero and we only need to compute b_n.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \operatorname{sgn} x \sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin nx\,dx = -\frac{2}{\pi n}(\cos n\pi - 1) = \frac{2(1 - (-1)^n}{\pi n}$$

and

$$\operatorname{sgn} x \sim \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin(2k+1)x.$$

The 2π-periodic extension of $\operatorname{sgn} x$ is piece-wise smooth and have jumps at $x = n\pi$, $n \in \mathbb{Z}$. So, by the Dirichlet theorem, the partial sums $f_N(x)$ converge point-wise to that extension for all $x \neq n\pi$ and to zero for $x = n\pi$. The convergence is not uniform since there are jumps.

Take $x = \pi/2$ in that expansions. Then, since $\sin((2k+1)\pi/2) = (-1)^k$, we have

$$1 = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \quad => \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$