COURSEWORK 2 FOR "FUNCTION SPACES" (MMATH AND BSC)

Deadline: 21 of November 2007

Problem 1:

a) (2 points): Prove the following generalization of the Banach contraction theorem: Let (X, d) be a complete metric space and let $T : X \to X$ be a map such that the nth iteration T^n of that map is a contraction for some $n \in \mathbb{N}$. Then the map T has a *unique* fixed point in X.
Hint: Use the standard Banach contraction theorem (its proof is not required).

b) (1 point): Consider the integral operator

$$(Ty)(x) := 1 + \int_0^x y(s)\, ds$$

on the Banach space $X = C[0, L]$ of continuous functions on the segment $[0, L]$ with the usual sup-norm ($L > 0$ is some fixed positive number). Prove that T is a contraction on X if $L < 1$ and that T is not a contraction on X if $L \geq 1$.

c) (1 point): Prove that the nth iteration of the map T satisfies

$$(T^n y)(x) = \sum_{k=0}^{n-1} \frac{x^k}{k!} + \int_0^x \frac{(x-s)^{n-1}}{(n-1)!} y(s)\, ds.$$

Hint: use induction.

d) (2 points): Using the previous formula, prove that, for every $L > 0$, there exists $n = n(L)$ such that the nth iteration T^n of the map T is a contraction on $X = C[0, L]$.

Problem 2: Let

$$f_N(x) := a_0 + \sum_{i=1}^{N} a_n \cos nx + b_n \sin nx$$

be the Nth Fourier sum for the function $f(x) = x^2$ on $[-\pi, \pi]$.

a) (1 point): Compute the coefficients a_0, a_n and b_n.

b) (2 points): Prove that $f_N(x) \to f(x)$ *uniformly* with respect to $x \in [-\pi, \pi]$ (i.e., that $f_N \to f$ in the space $C[-\pi, \pi]$ with the usual sup-norm).
Hint: verify that $\{f_N\}$ is a Cauchy sequence in $C[-\pi, \pi]$ and use the Dirichlet theorem to in order find the limit.

c) (1 point): Using the found Fourier expansions for $f(x) = x^2$, prove that

$$i) \ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}, \quad ii) \ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solutions

Problem 1a): By the Banach contraction theorem, the map T^n has a *unique* fixed point $p \in X$. Let us prove that p is a fixed point of T. Let $q = T(p)$. Then, $T^n(q) = T^{n+1}(p) = T(T^n(p)) = T(p) = q$. Thus, q is also a fixed point of T^n and, thanks to the uniqueness, we conclude that $q = p$ and p is a fixed point of T.

Let us prove that the fixed point p of the map T is unique. Indeed, let $p_1 \in X$ is another fixed point. Then, $T^n(p_1) = T^{n-1}(p_1) = \cdots = T(p_1) = p_1$. So, p_1 is a fixed point of T^n as well and, thanks to the uniqueness, $p = p_1$.

Warning: The fact that the nth iteration T^n of the map T is a contraction for some $n > 1$ *does not* imply that T is also a contraction!!

Problem 1b): The correct solution should contain *two* different parts: first, we need to prove that T is a contraction if $L < 1$ and, second, we need to verify that it is not a contraction if $L \geq 1$.

Part I: Let $L < 1$ and let $y_1, y_2 \in C[0, L]$. Then

$$|(Ty_1)(x) - (Ty_2)(x)| = |\int_0^x y_1(s) - y_2(s)\, ds| \leq$$
$$\leq \int_0^x |y_1(s) - y_2(s)|\, ds \leq \int_0^x ds \|y_1 - y_2\|_{L^\infty} = x\|y_1 - y_2\|_{L^\infty}$$

and, consequently,

$$\|Ty_1 - Ty_2\|_{L^\infty} := \sup_{x\in[0,L]} |(Ty_1)(x) - (Ty_2)(x)| \leq L\|y_1 - y_2\|_{L^\infty}.$$

Thus, T is a contraction with the contraction factor $\kappa = L < 1$.

Part II: Let now $L \geq 1$. We need to prove that T is *not* a contraction. Indeed, let $y_0 = 0$ and $y_1 = 1$. Then, $Ty_0 = 1$ and $(Ty_1)(x) = 1 + x$. We see that $\|y_0 - y_1\|_{L^\infty} = 1$ and

$$\|Ty_0 - Ty_1\|_{L^\infty} = \sup_{x\in[0,L]} |x| = L.$$

By this reason, T is not a contraction if $L \geq 1$.

Warning: It is *not sufficient* to verify, say, that

$$d(Tx, Ty) \leq 2d(x, y)$$

in order to conclude that T is *not* a contraction!! (the constant 2 in this formula may be not optimal and the inequality still may hold with better constant which is less than one.)

Problem 1c): Let $y_n(x) := (T^n y)(x)$. Then

$$y_{n+1}(x) = 1 + \int_0^x y_n(s)\, ds$$

which is equivalent to

$$\frac{d}{dx} y_{n+1}(x) = y_n(x), \quad y_{n+1}(0) = 0.$$

Let us prove that

$$(*) \qquad y_n(x) = \sum_{k=0}^{n-1} \frac{x^k}{k!} + \int_0^x \frac{(x-s)^{n-1}}{(n-1)!} y(s)\,ds$$

by induction. For $n = 1$,

$$y_1(x) = (Ty)(x) = 1 + \int_0^x y(s)\,ds$$

and $(*)$ is true. Assume that $(*)$ holds for some n and let us verify that it is true for $n+1$. Indeed, let

$$v(x) := \sum_{k=0}^{n} \frac{x^k}{k!} + \int_0^x \frac{(x-s)^n}{n!} y(s)\,ds.$$

Then,

$$\frac{d}{dx} v(x) = \sum_{k=0}^{n-1} \frac{x^k}{k!} + \int_0^x \frac{(x-s)^{n-1}}{(n-1)!} y(s)\,ds = y_n(x)$$

by the induction assumption. In addition, $v(0) = 0$. Thus, $v(x) = y_{n+1}(x)$ and $(*)$ is true for $n+1$. Thus, by induction, $(*)$ is true for all n.

Problem 1d): Let $y_1, y_2 \in C[0, L]$. Then

$$|(T^n y_1)(x) - (T^n y_2)(x)| = |\int_0^x \frac{(x-s)^{n-1}}{(n-1)!} (y_1(s) - y_2(s))\,ds| \le$$
$$\le \int_0^x \frac{(x-s)^{n-1}}{(n-1)!}\,ds \|y_1 - y_2\|_{L^\infty} = \frac{x^n}{n!} \|y_1 - y_2\|_{L^\infty}$$

and, consequently,

$$\|T^n y_1 - T^n y_2\|_{L^\infty} \le \frac{L^n}{n!} \|y_1 - y_2\|_{L^\infty}.$$

Thus, the map T^n will be a contraction if $\kappa(L, n) := \frac{L^n}{n!} < 1$. Since $\frac{L^n}{n!} \to 0$ as $n \to \infty$ for all fixed L, for any given L there exists $n = n(L)$ such that $\kappa(L, n) < 1$.

Problem 2a): Since the function $f(x) = x^2$ is even, all coefficients $b_n = 0$ and we only need to compute a_n:

$$a_0 := \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi} \frac{x^3}{3}\Big|_{x=-\pi}^{x=\pi} = \frac{\pi^2}{3}$$

and, for $n > 0$, integrating by parts two times

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx)\,dx = \frac{1}{\pi n} x^2 \sin(nx)\Big|_{x=-\pi}^{x=\pi} -$$
$$- \frac{2}{\pi n} \int_{-\pi}^{\pi} x \sin(nx)\,dx = -\frac{2}{\pi n} \int_{-\pi}^{\pi} x \sin(nx)\,dx = \frac{2}{\pi n^2} x \cos(nx)\Big|_{x=-\pi}^{x=\pi} -$$
$$- \frac{2}{\pi n^2} \int_{-\pi}^{\pi} \cos(nx)\,dx = \frac{4}{n^2} \cos(n\pi) = 4\frac{(-1)^n}{n^2}.$$

Thus, the Fourier series for $f(x) = x^2$ is

$$x^2 \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx).$$

Problem 2b) Let

$$f_N(x) := \frac{\pi^2}{3} + 4\sum_{n=1}^{N} \frac{(-1)^n}{n^2} \cos(nx)$$

be the partial Fourier sums for $f(x) = x^2$. Let us prove that f_N is a Cauchy sequence in $X = C[-\pi, \pi]$. Indeed,

$$|f_N(x) - f_{N+M}(x)| = 4|\sum_{n=N+1}^{N+M} \frac{(-1)^n}{n^2} \cos(nx)| \le 4\sum_{n=N+1}^{N+M} \frac{1}{n^2} \le 4\sum_{n=N+1}^{\infty} \frac{1}{n^2}$$

and, consequently,

$$\|f_N - f_{N+M}\|_X \le 4\sum_{n=N+1}^{\infty} \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, the right-hand side of the last inequality tends to zero as $N \to \infty$ and, therefore, f_N is a Cauchy sequence in X.

Since the space $C[-\pi, \pi]$ is complete, we conclude that the Fourier sums f_N converge uniformly (=in the sup-metric of $C[-\pi, \pi]$) to some function f_0:

$$f_0(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx).$$

Let us prove that $f_0(x) = f(x) = x^2$. Indeed, the 2π-periodic extension $f_{per}(x)$ is, obviously, piece-wise smooth on $\mathbb{R}$ and has no jumps. Thus, by the Dirichlet theorem, $f_N(x) \to f(x)$ point-wise. Since we have already known that $f_N \to f_0$ uniformly, we conclude that $f = f_0$.

Problem 2c): We know from the previous problem that

$$(**) \qquad x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx).$$

Let us take $x = 0$ here. Then,

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = 0$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Let us now take $x = \pi$ in $(**)$. Then

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$