Part III Hilbert spaces

Lecture Inner products and Cauchy-Schwartz inequality

Def.: Let H be a vector space (over $\mathbb{R}$)

a function $x, y \to (x,y) \in \mathbb{R}$, $(\cdot,\cdot): H \times H \to \mathbb{R}$

is an inner product on H if

1) It is *bi-linear* = linear with respect to both arguments

$$(aX_1 + bX_2, y) = a(X_1, y) + b(X_2, y), \quad a,b \in \mathbb{R},\ X_1, X_2, y \in H$$

and

$$(X, ay_1 + by_2) = a(X, y_1) + b(X, y_2), \quad a,b \in \mathbb{R},\ X, y_1, y_2 \in H$$

2) it is *symmetric*

$$(X, y) = (y, X)$$

3) it is positive definite:

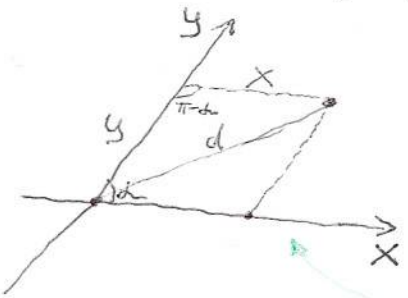
$$(X,X) \geq 0 \quad \text{and} \quad (X,X) = 0 \Leftrightarrow X = 0$$

Examples:

a) $H = \mathbb{R}^n$, $(X, y) := \sum_{i=1}^{n} x_i \cdot y_i$

check that all properties of the inner product are satisfied!!

b) $H = \mathbb{R}^2$



$$\|X\|^2 = d(0,X)^2 = x^2 + y^2 + 2xy \cdot \cos\alpha$$

$\cos$-law

Then

$$(X, y) = x_1 x_2 + y_1 y_2 + (x_1 y_2 + x_2 y_1)\cos\alpha$$

bi-linear and symmetric and positivity holds since $(x,x) = d(0,x)^2$ in that coordinate system

c) More generally: let $H=\mathbb{R}^n$ and

$$(x,y) := \sum_{i=1}^{n} x_i y_i$$

and let $[x,y]$ be another inner product in H. Then,

$$x=\sum_i x_i e_i \,, \quad y=\sum_j y_j e_j \,, \quad e_i \text{ - } i\text{th coordinate vector}$$

$$[x,y] = \Big[\sum_i x_i e_i, \sum_j y_j e_j\Big] = \sum_{ij} x_i y_j [e_i, e_j] =$$

$$= \sum_{ij} a_{ij} x_i x_j = (Ax, y) \,, \quad a_{ij} := [e_i, e_j]$$

$A=(a_{ij})$ - $n\times n$ matrix.

A is symmetric since $a_{ij}=[e_i,e_j]=[e_j,e_i]=a_{ji}$.

$\Rightarrow$ every inner product in $\mathbb{R}^n$ has the form

$(x,y)_A := (Ax,y)$ for some symmetric matrix A.

Not any symmetric matrix A generates an inner product in $\mathbb{R}^n$! (not necessarily positive)

Proposition: A generates an inner product in $\mathbb{R}^n$ by $(x,y)_A$ $\Leftrightarrow$ A is symmetric and all eigenvalues of A are positive

Proof: $\Rightarrow$ since $(x,y)_A=(Ax,y)=(y,x)_A=(Ay,x)$

$\Rightarrow A$ must be symmetric.

Fact from linear algebra: any symmetric matrix has real eigenvalues, eigenvectors corresponding to different eigenvectors are orthogonal and form a basis on $\mathbb{R}^n$.

Let v be an eigenvector of A

$$Av=\lambda v$$

$$0<(Av,v)=\lambda\|v\|^2 \Rightarrow \lambda>0$$

$\Downarrow$

all eigenvalues of A must be positive

$\Leftarrow$ Let A be symmetric and all eigenvalues $\lambda_1, \dots, \lambda_n$ are positive

$v_1, \dots, v_n$ – corresponding eigenvectors

$$Av_k = \lambda_k v_k, \quad \lambda_k > 0, \quad (v_k, v_j) = \delta_{k,j}$$

We need to check only positivity. Let $X \in H$. Then, since $\{v_k\}$ is a basis in $\mathbb{R}^n$

$$X = \sum' (X, v_k) v_k = \sum' \beta_k v_k$$

$$(AX, X) = \sum'_{i,j} \beta_i \cdot \beta_j \cdot (Av_i, v_j) = \sum'_i \lambda_i \beta_i^2 > 0$$

$$\lambda_i (v_i, v_j) = \lambda_i \delta_{ij}$$

d) inf-dim example:

let $H = C[a, b]$ and

$$(f, g) := \int_a^b f(x) g(x) dx$$

obviously symmetric and Bi-linear

Positivity:

$$(f, f) = \int_a^b f^2(x) dx \geq 0 \qquad (f, f) = 0 \Leftrightarrow f \equiv 0$$

$\Rightarrow (f, g)$ is an inner product in $C[a, b]$

e) Space of summable sequences l_2:

$$(x, y) := \sum_{i=1}^{\infty}{}' x_i y_i$$

All properties of the inner product are obvious, but we need to verify that (x, y) is <u>finite</u>. Namely, that

$$\sum_{i=1}^{\infty}{}' x_i^2 < \infty \quad + \quad \sum_{i=1}^{\infty}{}' y_i^2 < \infty \Rightarrow \sum_{i=1}^{\infty}{}' x_i y_i \text{ is finite and well-defined}$$

for that we need the Cauchy – Schwartz inequality!

Theorem: (Cauchy - Schwartz inequality)

Let H be a vector space and let (x,y) be an inner product on it. Then

$$|(x,y)| \le \|x\| \cdot \|y\|, \quad x,y \in H$$

where

$$\|x\| := (x,x)^{1/2}$$

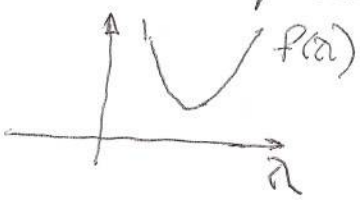
Proof: Let $\lambda \in \mathbb{R}$ be arbitrary and let $x, y \in H$, $x,y \neq 0$

Consider a function

$$f(\lambda) := \|x - \lambda y\|^2 = (x-\lambda y, x-\lambda y) = (x,x) - 2\lambda(x,y) + \lambda^2(y,y) =$$

$$= \lambda^2 \cdot \|y\|^2 - 2\lambda \cdot (x,y) + \|x\|^2$$

$f(\lambda)$ is a quadratic polinom on λ - parabola



$+$ $f(\lambda)$ is non-negative

$\Rightarrow$ $f(\lambda)$ cannot have different real roots

$$\Rightarrow \quad D = b^2 - 4ac = 4(x,y)^2 - 4\|x\|^2 \cdot \|y\|^2 \le 0$$

$$\Rightarrow \quad (x,y)^2 \le \|x\|^2 \cdot \|y\|^2$$

$$\Rightarrow \quad |(x,y)| \le \|x\| \cdot \|y\|. \qquad \Delta$$

Alternatively,

$$f(\lambda) = \left(\lambda \cdot \|y\| - \frac{(x,y)}{\|y\|}\right)^2 + \|x\|^2 - \frac{(x,y)^2}{\|y\|^2} \ge 0$$

$$\Rightarrow \quad \|x\|^2 - \frac{(x,y)^2}{\|y\|^2} \ge 0$$

Corollary: Let H be a vector space and (x,y) be an inner product

$$\|x\| := (x,y)^{1/2}$$

is a norm on a vector space H

Proof: We only need to check the triangle inequality

$\|x+y\| \le \|x\| + \|y\|$ OR $(x+y, x+y) \le (\|x\| + \|y\|)^2$

OR $\|x\|^2 + \|y\|^2 + 2(x,y) \le \|x\|^2 + \|y\|^2 + 2\|x\| \cdot \|y\|$

OR $(x,y) \le \|x\| \cdot \|y\|$ = Cauchy - Schwartz inequality

$\square$

Def: A Hilbert space H is a complete inner product space.

In other words: H is a Hilbert space if

1) H is a vector-space (linear structure)
2) The inner product (x, y) on H is given
3) The normed space $(H, \|\cdot\|)$ where $\|x\| := (x, x)^{1/2}$ is complete.

Sometimes, one adds infinite-dimensional or/and separable to that definition.

Examples of Hilbert spaces:

a) $H = \mathbb{R}^n$ with $(x, y) := \sum_{i=1}^{n} x_i \cdot y_i$ is an H-space.

b) Let us consider $l_2 := \left\{ x = (x_1, \dots, x_n, \dots), \sum_{i=1}^{\infty} x_i^2 < \infty \right\}$

- the space of square summable sequences (main model example in the theory of Hilbert spaces!)

We already know that l_2 with $\|x\|_{l_2}^2 := \sum_{i=1}^{\infty} x_i^2$ is complete. Need to find the inner product. Analogously to finite-dimensional case, try

$$(x, y) := \sum_{i=1}^{\infty} x_i \cdot y_i$$

Problem: in contrast to finite-dimensional case, we need to verify that the infinite sum exists: i.e. we need to prove the existence of the limit

$$S = \lim_{N \to \infty} S_N \text{ in } \mathbb{R} \quad \text{where} \quad S_N := \sum_{i=1}^{N} x_i \, y_i$$

if we know that

$$\sum_{i=1}^{\infty} x_i^2 < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} y_i^2 < \infty$$

Since $\mathbb{R}$ is complete, it is sufficient to verify that S_N is a Cauchy sequence

$$|S_N - S_{N+M}| = \left|\sum_{i=N+1}^{N+M} x_i \cdot y_i\right| \leq \left(\sum_{i=N+1}^{N+M} x_i^2\right)^{1/2} \cdot \left(\sum_{i=N+1}^{M+N} y_i^2\right)^{1/2}$$

finite sum $\Rightarrow$ can use Cauchy-Schwarz

$$\leq \left(\sum_{i=N+1}^{\infty} x_i^2\right)^{1/2} \left(\sum_{i=N+1}^{\infty} y_i^2\right)^{1/2}.$$

But $\sum_{i=1}^{\infty} x_i^2 < \infty \Rightarrow \sum_{i=N+1}^{\infty} x_i^2 \to 0$ as $N \to \infty$

and the same is true for y_i.

$\Rightarrow |S_N - S_{N+M}| \to 0$ as $N \to \infty$

$\Rightarrow$ S_N is a Cauchy sequence in $\mathbb{R}$

Conclusion: ℓ_2 with $(x,y) := \sum_{i=1}^{\infty} x_i \cdot y_i$ is a Hilbert space

c) The Lebesgue space $L^2([a,b])$ of square integrable functions $f: [a,b] \to \mathbb{R}$

= completion of $C[a,b]$ with respect to the norm

$$\|f\|_{L^2}^2 := \int_a^b |f(x)|^2 dx$$

The inner product

$$(f,g) := \int_a^b f(x)\, g(x)\, dx$$

it is obviously the inner product on $C[a,b]$ (the integral is understood as a Riemann one)

For general $f, g \in L^2[a,b]$ - Lebesgue integration is needed + we again need to verify the existence of the integral.

Conclusion:

$L^2([a,b])$ is a Hilbert space.

Remark: As we will see below,

$L^2([a,b]) \cong \ell_2$

linear isomorphism preserving the inner products.

Lecture: Orthonormal systems in Hilbert spaces.

Def. Let H be a Hilbert space.

A system (sequence) of vectors $\{e_n\}_{n\in\mathbb{N}} \subset H$ is called orthonormal system if

$$(e_n, e_m) = \delta_{n,m}$$ – Kronecker delta

In other words: $\{e_n\}$ is orthonormal if

ortho- 1) $e_n \perp e_m$; $n \neq m$

-normal 2) $\|e_n\| = 1$

if only (1) holds then $\{e_n\}$ is orthogonal

if $\{e_n\}$ is orthogonal $\Rightarrow$ $\left\{\frac{e_n}{\|e_n\|}\right\}$ is orthonormal

$\neq 0$

Example: $H = \ell_2$, $e_n = (0, \dots 0, \underset{n\text{th position}}{1}, 0 \dots)$

Then $\{e_n\}$ is an orthonormal system

We want to consider series

$$\sum_{n=1}^{\infty} a_n e_n \qquad a_n \in \mathbb{R}$$

associated with orthonormal system $\{e_n\}$ and write any element of H in such form

$$(*) \qquad x \sim \sum_{n=1}^{\infty} a_n e_n$$

How to find the coefficients a_n - ?

Formally, taking the inner product with e_k

$$(x, e_k) \sim \sum_{n=1}^{\infty} a_n (e_n, e_k) = a_k$$

δ_{nk}

$\Rightarrow$ if $(*)$ exists, we should have

$$a_n = (x, e_n)$$

Def: H is a Hilbert space, $\{e_n\}$ is orthonormal system

Let $x \in H$. A Fourier series of x w.r.t. $\{e_n\}$

$$x \sim \sum_{n=1}^{\infty} (x, e_n) e_n$$

Abstract Fourier series!

Two natural questions:

1) convergence of a sum?

2) "$=$" instead of "$\sim$"?

Start with question 1):

Theorem (Bessel inequality) Let H be H-space and $\{e_n\}$ is an orthonormal system in H. Then, for any $x \in H$

$$\sum_{n=1}^{\infty} (x, e_n)^2 \leq \|x\|^2$$

Bessel inequality

Proof: Consider the finite sum

$$x_N = \sum_{n=1}^{N} (x, e_n) e_n = \sum_{n=1}^{N} a_n e_n$$

and compute

$$\|x - x_N\|^2 = (x - x_N, x - x_N) = \|x\|^2 + (x_N, x_N) -$$

$$-2(x, x_N) = \|x\|^2 + \left(\sum a_n e_n, \sum a_\ell e_\ell\right) - \quad (e_n, e_\ell) = \delta_{n\ell}$$

$$-2\left(x, \sum a_n e_n\right) = \|x\|^2 + \sum_{n=1}^{N} a_n^2 - 2 \sum_{n=1}^{\infty} a_n \cdot \underbrace{(x, e_n)}_{a_n}$$

$$= \|x\|^2 - \sum_{n=1}^{N} a_n^2 \geq 0. \qquad (**)$$

Thus, for any N

$$\sum_{n=1}^{N} (x, a_n)^2 \leq \|x\|^2$$

Passing to the limit $N \to \infty$ $\Rightarrow$ Bessel inequality

□

Corollary: Any Fourier series is convergent:
$\forall X \in H$, $\{e_n\}$ - orthonormal system

- $\sum_{n=1}^{\infty} (X, e_n) e_n = X_0$ - exists.

Proof: Since H is complete, we only need to proof that

$$X_N = \sum_{n=1}^{N} (X, e_n) e_n$$

is a Cauchy sequence in H

Compute: $(e_n, e_k) = \delta_{nk}$

$$\|X_N - X_{N+M}\|^2 = \| \sum_{n=N+1}^{N+M} a_n e_n \|^2 = \left(\sum a_n e_n, \sum a_k e_k \right) =$$

$$= \sum_{n=N+1}^{N+M} a_n^2 \leq \sum_{n=N+1}^{\infty} a_n^2$$

Since, $\sum_{n=1}^{\infty} a_n^2 \leq \|X\|^2 < \infty \Rightarrow \sum_{n=N+1}^{\infty} a_n^2 \to 0$ as $N \to \infty$

$\Rightarrow X_N$ is a Cauchy sequence $\Rightarrow$ convergened. □

Passing to the limit $N \to \infty$ in (**)

$$\sum_{n=1}^{\infty} (X, e_n)^2 = \|X\|^2 - \|X - X_0\|^2$$

the sum of the Fourier series.

Question 2:

$$X \stackrel{?}{=} X_0 = \sum_{n=1}^{\infty} (X, e_n) e_n$$ - always?

Answer: No!

Example: $H = \ell_2$ and e_n - nth coordinate vector.
Let $v_n = e_{2n}$. Then $\{v_n\}$ is an orthonormal,

But for $X = e_1$ the Fourier series w.r.t. $\{v_n\}$ is identically zero!

We ~~may~~ need more vectors in $\{$~~v_n~~$\}$ = orthonormal system in order to have the equality

$\{v_n\}$ is NOT complete!

Observation: Let $x \in H$ and

$$x_0 = \sum_{n=1}^{\infty} (x, e_n) e_n$$

Then

$$x - x_0 \perp e_n, \text{ for any } n$$

Indeed

$$(x - x_0, e_n) = (x, e_n) - \sum_{k=1}^{\infty} (x, e_k)(e_k, e_n) = (x, e_n) - (x, e_n) = 0$$

Def: An orthonormal system $\{e_n\} \subset H$ is complete if

$$(x, e_n) = 0 \ \forall n \iff x = 0$$

A complete orthonormal system in H = orthonormal Basis in H.

Proposition: Let $\{e_n\}$ be an orthonormal basis in H. Then

$$x = \sum_{n=1}^{\infty} (x, e_n) e_n$$

Proof: if $x \neq x_0$, $\Rightarrow$ $x - x_0 \perp e_n$ $\forall n$ $\{e_n\}$ is complete $\Rightarrow x = x_0$. □

More formulas: Let $\{e_n\}$ is an orthonormal basis.

$$\sum_{n=1}^{\infty} (x, e_n)^2 = \|x\|^2 - \|x - x_0\|^2 = \|x\|^2$$

$\|_0$

Parseval equality (Bessel inequality becomes equality if the system is complete).

Let $x = \sum a_n e_n$, $y = \sum b_n e_n$

$$(x, y) = \left(\sum_{n=1}^{\infty} a_n e_n, y\right) = \sum_{n=1}^{\infty} a_n (y, e_n) = \sum_{n=1}^{\infty} a_n b_n$$

since the inner product is a continuous function.

[check using the Cauchy-Schwarz inequality!]

Conclusion: If a Hilbert space H has an orthonormal basis $\{e_n\}$ $\Rightarrow$ it is isomorphic to ℓ_2.

Indeed, the map

$$F: H \to \ell_2 \qquad F(x) = a = ((x, e_1), (x, e_2), \dots)$$

((x, e_n): Fourier coefficients)

is a **linear** one-to-one and onto map which preserves the inner product

$$(x, y)_H = (F(x), F(y))_{\ell_2}$$

think a little more about this construction!

In particular, any H-space with an orthonormal **basis** is **separable** (since ℓ_2 is separable).

Vise versa

Theorem: Any **separable** (inf-dim) Hilbert space has an orthonormal basis.

i.e. Any separable Hilbert space (inf-dim) $= \ell_2$!

Proof: Let $\{x_n\}_{n \in \mathbb{N}}$ be a countable dense set in H. Orthogonalize it using the Gram orthogonalization procedure; namely,

let $e_1 := \frac{x_1}{\|x_1\|}$ - base of induction

let $\{e_1, \dots, e_k\}$ are already constructed - step of induction

if $x_{k+1} \in \operatorname{span}\{e_1, \dots e_k\}$ - just drop out x_{k+1}

so, we may assume that $x_{k+1} \notin \operatorname{span}\{e_1, \dots e_k\}$

let $v_{k+1} = x_{k+1} - \sum_{n=1}^{k} (x_{k+1}, e_n) e_n \neq 0$ and $e_{k+1} := \frac{v_k}{\|v_k\|}$

Then, $(e_{k+1}, e_n) = 0 \quad \forall n \leq k$

By **induction**, the orthonormal system $\{e_n\}$ is constructed with the additional property any of x_n is a **finite** linear combination of $\{e_k\}$. $\Rightarrow$ **finite** linear combinations of $\{e_k\}$ are **dense** in H.

Let us prove that $\{e_k\}$ is complete. By contradiction. Assume that it is not complete.
Then $\exists\ x \in H$ such that $(x, e_k) = 0\ \forall k$
$\Rightarrow (x, x_n) = 0$ for all n (x_n are finite linear combinations of e_n). But $\{x_n\}$ is dense $\Rightarrow x = \lim_{k\to\infty} x_{n_k}$

$$(x, x) = (x, \lim_{k\to\infty} x_{n_k}) = \lim_{k\to\infty} (x, x_{n_k}) = 0$$

$\Rightarrow x = 0$ and $\{e_n\}$ is a basis. □

All of the theory of separable Hilbert spaces is now reduced to the case $H = l_2$.

Some preparations for classical Fourier series.

Theorem (the completness criterium) Let H - be a Hilbert space and let $\{e_n\}$ be an orthonormal system $\Rightarrow$ $\{e_n\}$ is complete if

$$x = \sum_{n=1}^{\infty}{}' (x, e_n)\, e_n \qquad (*)$$

for all x from a dense set in H.

Proof: By contradiction, Assume $\{e_n\}$ is NOT complete
$\Rightarrow \exists\ x_0 \in H$, $(x_0, e_n) = 0\ \forall n \Rightarrow (x_0, x) = 0$ $\forall x$ satisfying $(*)$.
$\Rightarrow (x_0, x) = 0$ $\forall x$ from a dense set $\Rightarrow x = 0$ □

Formulas for the orthogonal (not orthonormal) basis in H.

$\{v_n\}$ is orthogonal and complete $\Rightarrow e_n = \frac{v_n}{\|v_n\|}$ is an orthonormal basis

$$x = \sum_{n=1}^{\infty}{}' a_n v_n = \sum{}' (x, e_n) e_n = \sum_{n=1}^{\infty}{}' \frac{(x, v_n)}{\|v_n\|^2} v_n$$

$$\Rightarrow a_n = \frac{(x, v_n)}{\|v_n\|^2} = \frac{(x, e_n)}{\|v_n\|}.$$

Parseval equality:

$$\sum_{n=1}^{\infty}{}' \|v_n\|^2 \cdot a_n^2 = \|x\|^2$$