Lecture: Banach theorems and Weyl criterion

Two of 3 famous Banach theorems will be considered. The third one (Hahn-Banach theorem) will be not discussed since we mainly consider here only Hilbert spaces where it is trivial.

We start with the uniform boundedness principle

Theorem (Banach-Steinhaus) Let H_1, H_2 be two Banach (Hilbert) spaces and a family $T_\alpha \in \mathcal{L}(H_1, H_2)$ be such that

$$\boxed{\sup_\alpha \|T_\alpha x\|_{H_2} < \infty} \quad \forall\, x \in H_1$$

Then

$$\boxed{\sup_\alpha \|T_\alpha\|_{op} < \infty}$$

Proof (not necessary for the exam!): Based on the Bair theorem. Let

$$\boxed{C_R := \{x \in H_1,\ \|T_\alpha x\| \le R,\ \forall \alpha\}}$$

Then, by the assumption of the theorem,

$$\boxed{H_1 = \bigcup_{n=1}^{\infty} C_n}$$

On the other hand, C_n are closed. Indeed

$$\boxed{C_n = \bigcap_\alpha C_{n,\alpha}}, \quad \boxed{C_{n,\alpha} := \{x \in H_1,\ \|T_\alpha x\| \le R\}}$$

Every $C_{n,\alpha}$ is closed as the inverse image of the closed set $\boxed{\overline{B_R(0)} \subset H_2}$ under the continuous map T_α and C_n is closed as intersection of closed sets.

By Baire theorem (H_1 is complete), at least one of C_n is not nowhere dense. Let it be C_{n_0}. Since C_{n_0} is closed it must contain a ball

$$\boxed{\bar{B}_\varepsilon(x_0) \subset C_{n_0}}$$

By definition, this means

$$\boxed{\|T_\alpha x\| \le n_0} \quad \forall x \in \bar{B}_\varepsilon(x_0)$$

Let now $y \in \bar{B}_\varepsilon(0)$, then $\boxed{y + x_0 \in \bar{B}_\varepsilon(x_0)}$

and

$$\boxed{\|T_\alpha y\| = \|T_\alpha(y+x_0) - T_\alpha x_0\| \le \|T_\alpha(y+x_0)\| + \|T_\alpha x_0\| \le n_0 + \sup_\alpha \|T_\alpha x_0\| \le N_0}$$

Finally

$$\boxed{\|T_\alpha\| := \sup_{\|y\|\le 1} \|T_\alpha y\| \le \frac{N_0}{\varepsilon}} \quad \forall \alpha \qquad \square$$

Remark: Completeness of H_2 is nowhere used and is not necessary

Exercise: Show that the theorem is not true if H_1 is not complete (give a counter example)

Second will be the Banach inverse theorem

Theorem (Banach) Let H_1, H_2 be two Banach spaces and $\boxed{A \in \mathcal{L}(H_1, H_2)}$ be one-to-one. Then the inverse $\boxed{A^{-1} \in \mathcal{L}(H_2, H_1)}$ is continuous.

Exercise: Give an example of two metric spaces X_1 and X_2 and a continuous one-to-one map $f: X_1 \to X_2$ such that f^{-1} is not continuous.

Proof (also not for the exam!)

Step 1 (similar to Banach-Steinhaus)

Let $\boxed{C_n := \{x \in H_2,\ \|A^{-1}x\| \le n\}}$

Since A is one-to-one, A^{-1} is well-defined and linear, so only its boundedness should be proved

Obviously, $\boxed{H_2 = \bigcup_{n=1}^{\infty} C_n}$

and by Baire theorem (here we use that H_2 is complete) $\exists\, n_0$ such that C_{n_0} is not nowhere dense. Thus there exists a ball

$$(*) \qquad \boxed{\overline{B_\varepsilon(x_0)} \subset \overline{C_{n_0}}}$$

NB: Here we do not know that C_{n_0} is closed, so we need to use closure.

Step 2: Similarly to Banach-Steinhaus, we move the center of the ball in (*) to zero

Indeed by the translation $\boxed{\overline{B_\varepsilon(0)} = \overline{B_\varepsilon(x_0)} - x_0}$

and since the translation map $y \to y - x_0$ is one-to-one and continuous

$$\boxed{\overline{B_\varepsilon(0)} = \overline{B_\varepsilon(x_0)} - x_0 \subset \overline{C_{n_0}} - x_0 = \overline{C_{n_0} - x_0}}$$

Let $\boxed{y \in C_{n_0} - x_0} \iff \boxed{\|A^{-1}(y + x_0)\| \le n_0}$ and

$$\boxed{\|A^{-1}y\| \le \|A^{-1}(y+x_0)\| + \|A^{-1}x_0\| \le n_0 + C := n_1}$$

Thus, $\boxed{C_{n_0} - x_0 \subset C_{n_1}}$ and

$$(**) \qquad \boxed{\overline{B_\varepsilon(0)} \subset \overline{C_{n_1}}}$$

Step 3 (new!) Scaling (**) we see that

$$\boxed{\overline{B_1(0)} \subset \overline{C_K}} \qquad K = n_1/\varepsilon$$

and we want to remove "closure" from the RHS

One more scaling gives

(***) $$\boxed{\overline{B_{2^{-n}}(0)} \subset \overline{C_{K\cdot 2^{-n}}}} \quad \forall\, n = 0, 1, 2 \ldots$$

Let $\boxed{y \in \overline{B_1(0)}}$. Then by (***) with $n=0$

$\exists\, \boxed{y_1 \in C_K}$ such that $\boxed{\|y - y_1\| \le \frac{1}{2}} \Leftrightarrow \boxed{y - y_1 \in \overline{B_{2^{-1}}(0)}}$

Set $\boxed{x_1 := A^{-1} y_1} \Rightarrow \boxed{\|x_1\| \le K}$

Since $y - y_1 \in \overline{B_{2^{-1}}(0)}$, (***) with $n=1$ gives that

$\exists\, \boxed{y_2 \in C_{K\cdot 2^{-1}}}$ such that $\boxed{\|y - y_1 - y_2\| \le \frac{1}{4}} \Leftrightarrow \boxed{y - y_1 - y_2 \in \overline{B_{2^{-2}}(0)}}$

Set $\boxed{x_2 := A^{-1} y_2} \Rightarrow \boxed{\|x_2\| \le K\cdot 2^{-1}}$

Continuing this procedure, we construct a sequence $y_n \in H_2$ such that

$$\boxed{\|y - y_1 - \cdots - y_n\| \le \frac{1}{2^n}}, \qquad \boxed{\|x_n\| \le K\cdot 2^{-n+1}}, \quad \boxed{x_n := A^{-1} y_n}$$

Obviously $$\boxed{y = \lim_{N\to\infty} \sum_{n=1}^{N} y_n = \sum_{n=1}^{\infty} y_n}$$

Since H_1 is complete, the series $\boxed{x := \sum_{n=1}^{\infty} x_K}$ is convergent in H_1

and $$\boxed{\|x\| \le \sum_{n=1}^{\infty} \|x_K\| \le K \sum_{n=1}^{\infty} 2^{-n+1} = 2K}$$

Thus, since A is continuous, $\boxed{Ax = \Sigma' Ax_n = \Sigma' y_n = y}$

and $\boxed{\overline{B_1(0)} \subset C_{2K}} \Rightarrow \boxed{\|A^{-1}\| \le 2K}$

□

Obtain the so-called closed graph theorem as a corollary.

Def: Let ~~$A \in \mathcal{L}(X,Y)$~~ $A: H_1 \to H_2$ be linear

Then $\boxed{\text{graph}(A) := \{(x,y) \in H_1 \times H_2,\ y = Ax\}}$

Theorem: Let H_1, H_2 be Banach spaces and the map $\boxed{A: H_1 \to H_2}$ be linear. Then

$$\boxed{A \in \mathcal{L}(H_1, H_2)} \Leftrightarrow \boxed{\text{graph}(A) \text{ is closed in } H_1 \times H_2}$$

(Equivalent formulation of the Banach inverse Th)

Proof: $\Rightarrow$ is obvious and we only need to proof $\Leftarrow$. Consider the closed subspace $\boxed{H = \text{graph}(A)}$ in $H_1 \times H_2 \Rightarrow H$ is a Banach space with the norm ~~$\|x\|_Y$~~

$$\boxed{\|(x,Ax)\|_H := \|x\| + \|Ax\|} \text{ - graph norm}$$

Let also the map $\boxed{P_1: H \mapsto H_1}$ be $\boxed{P_1(x,Ax) := x}$

Then it is linear, one-to-one and continuous.

By Banach theorem, P_1^{-1} is bounded

$$\Rightarrow \boxed{\sup_{\|x\| \le 1} \{\|x\| + \|Ax\|\} \le L < \infty} \Rightarrow \boxed{\|A\| \le L < \infty}$$ □

This reformulation will be used for study the unbounded operators.

Now - return to the spectral theory

Theorem (Weyl criterion) Let H - be Hilbert and $A \in \mathcal{L}(H)$. Then $\lambda \in \sigma(A)$ iff

1) $\lambda \in \sigma_p(A)$ - point spectrum
2) $\overline{\lambda} \in \sigma_p(A^*)$ - point spectrum of A^* ~ Residual spectrum
3) $\exists\, x_n \in H$, $\|x_n\| = 1$ such that $Ax_n - \lambda x_n \to 0$

Def: $\lambda \in \sigma_{ap}(A)$ approximate point spectrum iff $\exists\, x_n \in H$, $\|x_n\| = 1$ and

$$\boxed{Ax_n - \lambda x_n \to 0},$$

Weyl criterion can be reformulated as

$$\boxed{\sigma(A) = \sigma_p(A) \cup \overline{\sigma_p(A^*)} \cup \sigma_{ap}(A)}$$

NB: Not a disjoint union! Moreover,

$$\boxed{\sigma_p(A) \subset \sigma_{ap}(A)}, \quad \boxed{\sigma_{res}(A) = \overline{\sigma_p(A^*)} \setminus \sigma_{ap}(A)}$$

check!

Proof of the theorem: Obviously 1) 2) 3) $\Rightarrow \lambda \in \sigma(A)$ and we only need to check the inverse. For simplicity, we check it only for $\lambda = 0 \in \sigma(A)$. General case is reduced to that one by the shift

$$A \to A - \lambda \mathrm{Id}.$$

Let $\boxed{0 \in \sigma(A)}$. Then if A is not injective $\Rightarrow 0 \in \sigma_p(A)$ and assumption 1) holds.

Assume that A is injective. Then again two possibilities:

a) $\overline{\text{Range}(A)} \neq H$ or b) $\overline{\text{Range}(A)} = H$

In the case a) there exists a non-zero vector $v \in H$ such that $v \perp \overline{\text{Range}(A)}$

This means

$$(Ax, v) = (x, A^* v) = 0 \quad \forall x \Rightarrow A^* v = 0$$

Thus $0 \in \sigma_p(A)$ and assumption 2) holds

Assume now that

$$\overline{\text{Range}(A)} = H$$

Then if $\text{Range}(A) = H$ the operator A is one-to-one and $0 \notin \sigma(A)$ by the Banach inverse theorem.

Thus, only one possibility remains: A is injective, but $\text{Range}(A)$ is not closed although is dense in H.

Then $A^{-1}: \text{Range}(A) \to H$ is well-defined but not bounded. Indeed if it is bounded then

$y_n \in \text{Range}(A)$, $y_n \to y_0$ in $H \Rightarrow y_n$ is Cauchy in $\text{Range}(A) \Rightarrow A^{-1} y_n$ is Cauchy in H

$\Rightarrow x_n := A^{-1} y_n \to x_0 \in H \Rightarrow A x_0 = y_0$ and $\text{Range}(A)$ is closed.

Since A^{-1} is unbounded, there exists $y_n \in \text{Range}(A)$ such that $\|y_n\| = 1$ $\|A^{-1} y_n\| \geq n$

Then set $x_n := \frac{A^{-1} y_n}{\|A^{-1} y_n\|}$ and $\|x_n\| = 1$ but

$\|A x_n\| \leq \frac{1}{n}$ which gives $0 \in \sigma_{ap}(A)$

□

Especially simple form for the self-adjoint operators : $A = A^*$

Corollary : Let $A \in \mathcal{L}(H)$ and $A = A^*$

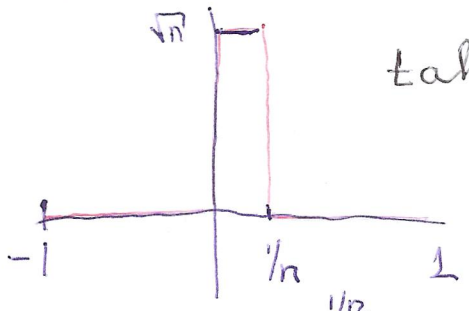
Then $\boxed{\sigma(A) = \sigma_{ap}(A)}$

may write $\sigma(A) = \sigma_p(A) \cup \sigma_{ap}(A)$ but it is the same since $\sigma_p(A) \subset \sigma_{ap}(A)$.

Important: $\boxed{\sigma_{Res}(A) = \emptyset}$ for any self-adjoint operator.

Often allows to find the spectrum or at least the so-called essential spectrum.

Example $H = L^2(-1,1)$ and $(T_x f)(x) := x f(x)$ be the multiplication operator by x. We already know that $\sigma(T_x) = [-1,1]$. Let us show that by Weyl criterion. Take, say, $\lambda = 0$ (any point $\lambda \in [-1,1]$ can be treated in the same way)



take these $f_n \in L^2(-1,1)$

$\|f_n\|^2_{L^2} = 1$ But

$$\|T_x f_n\|^2 = \int_0^{1/n} x^2 |f_n|^2 dx \le \frac{1}{n^2} \int_0^{1/n} |f_n|^2 dx = \frac{\|f_n\|^2}{n^2} = \frac{1}{n^2} \to 0$$

$\Rightarrow \boxed{0 \in \sigma_{ap}(T_x)}$ You see that there are no convergent subsequences in $\{f_n\}$. In a fact $f_n \xrightarrow[\text{weakly}]{} 0$ (typical situation when the Range is not closed).