UNIVERSITY OF SURREY©

Faculty of Engineering and Physical Sciences

Department of Mathematics

Undergraduate Programmes in Mathematics

Module MAT3004; 15 Credits

Introduction to Function Spaces

FHEQ Level 6 (Year 3) Examination

Time allowed: **2 hours** **Semester 1 2012/13**

Answer **THREE** questions only.

If you attempt more than THREE questions, only your best THREE answers will be taken into account.

Each question carries 25 marks.

Where appropriate the mark carried by an individual part of a question is indicated in square brackets [ ].

Approved calculators are allowed.

Additional material:
None

Question 1

(a) Give the definition of a *metric* space (X, d). [2]

(b) Let (X, d) be a metric space. Prove that the following inequality

$$|d(x, y) - d(u, v)| \leq d(x, u) + d(y, v)$$

holds for any $x, y, u, v \in X$. Hint: use the triangle inequality. [4]

(c) Let (X, d) be a metric space and let V and U be subsets of X.

- Give the definitions of the *closure* $\bar{V}$ and *interior* $\operatorname{int} V$ of the set $V \subset X$. [2]
- Prove that $\overline{U \cup V} = \bar{U} \cup \bar{V}$. [4]

(d) Let $X = C[0, 1]$ be the space of continuous functions $f : [0, 1] \to \mathbb{R}$ endowed by the usual sup-norm and let

$$V := \left\{ f \in X, \quad \sup_{x \in (0,1]} \frac{f(x) - f(0)}{x} < 1 \right\}.$$

Prove that $\operatorname{int}(V) = \varnothing$. [6]

(e) Let X be a vector space. Define what it means that for two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X to be *equivalent*. [2]

(f) Let $X = C[0, 1]$ be the space of continuous functions $f : [0, 1] \to \mathbb{R}$ and let

$$\|f\|_1 := \max_{x \in [0,1]} \{x^2 |f(x)|\}, \quad \|f\|_2 := \max_{x \in [0,1]} \{(x + x^2)|f(x)|\}.$$

Are these norms *equivalent*? Justify your answer. [5]

Question 2

(a) Let (X, d) be a metric space.

- Define what it means for X to be called *totally bounded*. [2]
- Define what it means for $\{x_n\}_{n=1}^{\infty}$ to be a *Cauchy sequence* in X. Define what it means for a metric space to be *complete*. [2]
- Define what it means for X to be called *compact* and state the Hausdorff criterion. [3]
- Prove that the closed unit ball in the space l_2 of square summable sequences with the usual norm is not compact. [4]

(b) Prove using first principles that any compact metric space is *complete*. [7]

(c) State the definition of the Lebesgue space $L^1(-1, 1)$. [2]

(d) Let $f(x) := \frac{1}{x}$. Does this function belong to $L^1(-1, 1)$? Justify your answer. [5]

Question 3

(a) What is a *contraction* on a metric space (X, d)? [2]

(b) State the Banach contraction theorem. [3]

(c) Let $f(x) := \frac{\pi}{2} + x - \arctan(x)$ and let $X = \mathbb{R}$ with the usual metric $d(x, y) := |x - y|$.

- Prove that
$$d(f(x), f(y)) < d(x, y)$$
for all $x, y \in X$ and $x \neq y$. Hint: $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$. [4]
- Prove that f does not have any fixed points in X. [2]
- Does it contradict the Banach contraction theorem? Justify your answer. [3]

(d) Let $X := C[0, 1]$ be the space of continuous functions $f : [0, 1] \to \mathbb{R}$ (with the usual sup-norm) and let the map $F : X \to X$ be defined as follows:

$$F(f)(x) := 1 + \int_0^x \sin(y) f(y)\, dy.$$

- Prove that F is a *contraction* on X. [4]
- Find the fixed point of F. [3]
- Is the map F a contraction on the space $C[0, \pi]$ with the sup-norm? Justify your answer. [4]

Question 4

(a) Let $\{e_n\}_{n=1}^\infty$ be an orthonormal system in a *Hilbert* space H and let

$$f \sim \sum_{n=1}^{\infty} f_n e_n, \quad f_n = (f, e_n)$$

be the abstract Fourier expansion of the element f.

- State the Bessel inequality. [2]
- State the condition on $\{e_n\}$ which guarantees that this inequality is an *equality* for all $f \in H$. [2]
- Prove that the limit

$$\bar{f} := \lim_{N\to\infty} \sum_{n=1}^{N} f_n e_n$$

exists in the metric of the space H. [5]

(b) Let $f(x) = |x|$, $x \in [-\pi, \pi]$.

- Find the Fourier coefficients a_n and b_n for the Fourier sums

$$f_N(x) := a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx)$$

of the function $f(x)$. [5]
- What is the point-wise limit of $f_N(x)$ as $N \to \infty$? Is the convergence *uniform* with respect to $x \in [-\pi, \pi]$? Explain your answer. [5]
- Prove that

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}. \qquad (*)$$

[3]

- Using the identity $(*)$ above, prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

[3]

END OF PAPER

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