UNIVERSITY OF SURREY©

Faculty of Engineering and Physical Sciences

Department of Mathematics

Undergraduate Programmes in Mathematics

Module MAT3004; 15 Credits

Introduction to Function Spaces

FHEQ Level 6 (Year 3) Examination

Time allowed: **2 hours** **Semester 1 2013/14**

Answer **THREE** questions only.

If you attempt more than THREE questions, only your best THREE answers will be taken into account.

Each question carries 25 marks.

Where appropriate the mark carried by an individual part of a question is indicated in square brackets [ ].

Approved calculators are allowed.

Additional material:
None

Question 1

(a) Give the definition of a *metric* space (X, d). [3]

(b) Let $X := \mathbb{R}^2$ and let

$$d(x, y) := |x_1 - y_1|^{1/2} + |x_2 - y_2|^{1/2}, \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad y = (y_1, y_2) \in \mathbb{R}^2.$$

(i) Prove that $d(x, y)$ is a *metric* on X. Hint: use that $\sqrt{A + B} \leq \sqrt{A} + \sqrt{B}$ for all $A, B \geq 0$. [4]

(ii) Is the function $x \to d(x, 0)$ a *norm* on X? Justify your answer. [3]

(c) Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be a function.

(i) Define what it means for the function f to be *continuous* at a point $x_0 \in X$. [2]

(ii) Define what it means for a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ to be *convergent*. [2]

(iii) Let $x_n \to x_0$ in X and f be continuous at $x_0 \in X$. Prove by first principles that

$$\lim_{n \to \infty} f(x_n) = f(\lim_{n \to \infty} x_n).$$

[4]

(d) Let X be a vector space. Define what it means that for two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ on X to be *equivalent*. [2]

(e) Let $X = C[0, 1]$ be the space of continuous functions $f : [0, 1] \to \mathbb{R}$ and let

$$\|f\|_1 := \max_{x \in [0,1]} \{x^2 |f(x)|\}, \quad \|f\|_2 := \max_{x \in [0,1]} \{(x^3 + x^2)|f(x)|\}.$$

Are these norms *equivalent*? Justify your answer. [5]

Question 2

(a) Let (X, d) be a metric space.

(i) Define what it means for $\{x_n\}_{n=1}^{\infty}$ to be a *Cauchy sequence* in X. Define what it means for a metric space to be *complete*. [2]

(ii) Prove that any *convergent* sequence is a *Cauchy* sequence. [3]

(iii) Let $U \subset X$ be a subset of X which is not *closed* in X. Prove that the metric space (U, d) is not *complete* [4]

(iv) Define what it means for a metric space $(\tilde{X}, \tilde{d})$ to be a *completion* of the metric space (X, d). [3]

(b) Prove that the space $C[0,1]$ of continuous functions $f : [0,1] \to \mathbb{R}$ with the norm

$$\|f\|_{L^1} := \int_0^1 |f(x)|\, dx$$

is *not* complete. [5]

(c) State the definition of the Lebesgue space $L^1(0,1)$. [2]

(d) Let $f(x) := \dfrac{1}{x + \sqrt{x}}$. Does this function belong to $L^1(0,1)$? Justify your answer. [6]

Question 3

(a) What is a *contraction* on a metric space (X, d)? [2]

(b) State the Banach contraction theorem. [3]

(c) Let $X := [-10, 10] \subset \mathbb{R}$ with the metric $d(x, y) := |x - y|$ and let the function $f : X \to \mathbb{R}$ be defined by $f(x) := \sqrt{x^2 + 1}$.

(i) Prove that

$$d(f(x), f(y)) \leq \frac{10}{\sqrt{101}} d(x, y)$$

for all $x, y \in X$. [5]

(ii) Prove that f does not have any fixed points in X. [2]

(iii) Does (ii) contradict the Banach contraction theorem? Justify your answer. [3]

(d) Let $X := C[0, 1/3]$ be the space of continuous functions $f : [0, 1/3] \to \mathbb{R}$ and let the map $F : X \to X$ be defined as follows:

$$F(f)(x) := 1 + xf(x) + \int_0^x f(y)\, dy.$$

(i) Prove that F is a *contraction* on X with the usual sup-norm. [6]

(ii) Find the fixed point of F. [4]

Question 4

(a) Let H be an inner product space with the inner product (x, y).

(i) State the *Cauchy-Schwarz* inequality. [1]

(ii) Give the proof of the Cauchy-Schwartz inequality. [3]

(iii) Define the *angle* between two vectors x and y in H. [1]

(b) Let $H := l_2$ be the space of square summable sequences with the usual norm

$$\|a\|_{l_2}^2 := \sum_{n=1}^{\infty} a_n^2, \quad a = (a_1, a_2, \cdots) \in H.$$

Find the *angle* between sequences $a := \{2^{-n}\}_{n=1}^{\infty}$ and $b := \{3^{-n}\}_{n=1}^{\infty}$. [3]

(c) Let $\zeta(x) := \sum_{n=1}^{\infty} \frac{1}{n^x}$, $x \in (1, \infty)$, be the Riemann zeta function. Prove that

$$\zeta\left(\frac{1}{2}x + \frac{1}{2}y\right) \le \sqrt{\zeta(x)\zeta(y)}$$

for any $x, y \in (1, \infty)$. Hint: use the Cauchy-Schwarz inequality in the space l_2 of square summable sequences. [5]

(d) Let $f(x) := \cosh(x)$, $x \in [-\pi, \pi]$, with the classical Fourier series

$$f(x) \sim \frac{\sinh(\pi)}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}\cos(nx)\right)$$

(the coefficients a_n and b_n are already computed for you). Let also

$$f_N(x) = \frac{\sinh(\pi)}{\pi}\left(1 + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2+1}\cos(nx)\right)$$

be the corresponding partial sums.

(i) What is the point-wise limit of $f_N(x)$ as $N \to \infty$? Is the convergence *uniform* with respect to $x \in [-\pi, \pi]$? Justify your answer. [5]

(ii) Compute

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1}.$$

[3]

(iii) Using the Parseval equality, compute

$$\sum_{n=0}^{\infty} \frac{1}{(n^2+1)^2}.$$

Hint: $\int_{-\pi}^{\pi} \cosh^2(x)\, dx = \frac{1}{2}\sinh(2\pi) + \pi$. [4]

END OF PAPER

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