UNIVERSITY OF SURREY©

Faculty of Engineering and Physical Sciences

Department of Mathematics

Undergraduate Programmes in Mathematics

Module MAT3004; 15 Credits

Introduction to Function Spaces

FHEQ Level 6 (Year 3) Examination

Time allowed: **2 hours** Semester 1 2018/19

Answer **THREE** questions only.

If a candidate attempts more than THREE questions, then only the best THREE solutions will be taken into account.

Each question carries 25 marks.

Where appropriate the mark carried by an individual part of a question is indicated in square brackets [ ].

Candidates may only use calculators that are non-programmable, with no alphanumeric memory and not wireless enabled.

Additional material:
0 Handouts

Question 1

(a) Give the definition of the *norm* on a vector space X. [1]

(b) Define what it means for two norms on a vector space X to be *equivalent*. [1]

(c) Let $X := C[0,1]$ be the space of all continuous functions $f : [0,1] \to \mathbb{R}$ and let

$$\|f\|_1 := \max_{x\in[0,1]} \left\{(x - x^2)|f(x)|\right\}, \quad \|f\|_2 := \max_{x\in[0,1]} \left\{\sin(x)|f(x)|\right\}.$$

Are these two norms on X equivalent? Justify your answer. [6]

(d) Let (X, d) be a metric space and let $U, V \subset X$.

(i) Give the definition of a *closure* $\bar{V}$ of the set V. [1]

(ii) Prove by first principles that the closure $\bar{V}$ is a closed set. [4]

(iii) Prove that $\overline{U \cup V} = \bar{U} \cup \bar{V}$. [4]

(e) Let (X, d) and (Y, d) be two metric spaces and $f : X \to Y$ be a function.

(i) Define what it means for a function $f : X \to Y$ to be *continuous* at $x_0 \in X$. [2]

(ii) Let $X = Y = l_\infty$ with the standard sup-metric and let

$$f(x) := (\cos x_1, \cos(2x_2), \cdots, \cos(nx_n), \cdots).$$

Is this function *continuous* at $x = 0$? Justify your answer. [6]

Question 2

(a) Let (X, d) be a metric space.

(i) Define what it means for (X, d) to be *compact*. [1]

(ii) Define what it means for a metric space (X, d) to be *separable*. [1]

(iii) Prove that any compact metric space is *separable*. [6]

(b) (i) Define what it means for the metric space (X, d) to be *complete*. [1]

(ii) Define what it means for the metric space $(\tilde{X}, \tilde{d})$ to be a *completion* of the metric space (X, d). [3]

(iii) Let $X := C[0, 1]$ be the space of all continuous functions $f : [0, 1] \to \mathbb{R}$ endowed with the following norm:

$$\|f\| := \sup_{x \in [0,1]} \{x|f(x)|\}.$$

Is this space *complete*? Justify your answer. *Hint:* Think about the function $f(x) := \frac{1}{\sqrt{x}}$. [6]

(c) (i) Define the Lebesgue space $L_p(0, 1)$ where $1 \le p < \infty$. [1]

(ii) Let $f(x) := \frac{1}{(1-x)\sqrt{x}}$. Does the function f belong to the space $L_1(0, 1)$? Justify your answer. [6]

Question 3

(a) Let (X, d) be a metric space and $F : X \to X$ be a function. Define what it means for F to be a *contraction* on X and state the Banach contraction theorem. [4]

(b) Let $X := \mathbb{R}^2$ and let $f : X \to X$ be defined by

$$f(x) := \begin{pmatrix} \frac{1}{2}x_1 + 5x_2 \\ \frac{1}{2}x_2 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2.$$

(i) Prove that $f : X \to X$ is *not* a contraction in X endowed with the standard Euclidean norm. [3]

(ii) Prove that f is a contraction in X endowed with the following norm:

$$\|x\| := |x_1| + 20|x_2|.$$

[5]

(c) Let the sequence $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ be defined by

$$x_1 = 0, \quad x_{n+1} = \sqrt{6 + x_n}, \quad n \in \mathbb{N}.$$

Prove that the sequence is convergent and find its limit.

Hint: Prove that the map $f(x) := \sqrt{6 + x}$ is a contraction on $X := [0, \infty)$ with the standard norm. [5]

(d) Let $X := C[0, 1]$ be the space of continuous functions $f : [0, 1] \to \mathbb{R}$ and let the map $F : X \to X$ be defined as follows:

$$F(f)(x) := \frac{1}{2}f(x) + 1 + \int_0^{x/2} sf(2s)\, ds.$$

(i) Prove that F is a *contraction* on X with the usual sup-norm. [5]

(ii) Find the fixed point of F. [3]

Question 4

(a) Let H be a real vector space.

(i) Give the definition of the inner product (x, y) on H. [3]

(ii) State the Cauchy-Schwarz inequality. [1]

(iii) Let (x, y) be an *inner product* on H. Using the Cauchy-Schwarz inequality prove that $\|x\| := \sqrt{(x, x)}$ is a *norm* on H. [4]

(iv) Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal base in H. State the Bessel equality. [1]

(v) Let (x, y) be an inner product on H and $\|x\|$ be the corresponding norm. Prove that

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

[2]

(vi) Let $H := C[0, 1]$ endowed with the standard sup-norm $\|f\|_{sup}$. Is it possible to find an *inner product* (f, g) on H such that $\|f\|_{sup} = \sqrt{(f, f)}$? Justify your answer. [4]

(b) Let $f(x) = x^3$, $x \in [-\pi, \pi]$, with the classical Fourier series

$$f(x) \sim -2\sum_{n=1}^{\infty} \frac{(-1)^n(\pi^2n^2 - 6)}{n^3} \sin(nx)$$

(the coefficients a_n and b_n are already found for you). Let also

$$f_N(x) = -2\sum_{n=1}^{N} \frac{(-1)^n(\pi^2n^2 - 6)}{n^3} \sin(nx)$$

be the corresponding partial sums.

(i) What is the point-wise limit of $f_N(x)$ as $N \to \infty$? Is the convergence *uniform* with respect to $x \in [-\pi, \pi]$? Justify your answer. [5]

(ii) Prove that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^3} = \frac{\pi^3}{32}.$$

Hint: Look at $x = \frac{\pi}{2}$ in the Fourier expansions of $f(x)$. You may also use that $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ without proving this. [5]

END OF PAPER

INTERNAL EXAMINER: Prof. S. Zelik
EXTERNAL EXAMINER: Dr. E. Crooks

Informal Solutions

Question 1.

a) A function $x \to \|x\|$ ($X \to \mathbb{R}$) is a norm on X iff:
1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ iff $x = 0$.
2) $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X$ and $\lambda \in \mathbb{R}$.
3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$. [book]

b) Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space X are equivalent if there are positive constants l and L such that

$$l\|x\|_1 \leq \|x\|_2 \leq L\|x\|_1$$

for all $x \in X$. [book]

c) These norms are not equivalent. Indeed, let us consider the function

$$Q(x) := \frac{x - x^2}{\sin x}.$$

This function is continuous and nonnegative for all $x \in [0, 1]$, but $Q(0) = 0$. Let us consider the sequence of spikes localized near $x = 1$, say, $f_n(x) := 1 - n(1 - x)$ for $x > 1 - 1/n$ and $x = 0$ otherwise. Then,

$$\lim_{n\to\infty} \|f_n\|_2 = \lim_{n\to\infty} \sup_{x\in[1-1/n,1]} \sin(x)|f_n(x)| = \sin(1).$$

On the other hand,

$$\lim_{n\to\infty} \|f_n\|_1 = \lim_{n\to\infty} \sup_{x\in[1-1/n,1]} x(1 - x)|f_n(x)| = 0$$

and the norms are not equivalent. [seen similar]

d)
(i) The point $x \in X$ belongs to $\bar{V}$ iff there exists a sequence $x_n \in V$ such that $x_n \to x$ in (X, d). [book]
(ii) Let $x_n \in \bar{V}$ be a sequence such that $x_n \to x$ in (X, d). We need to prove that $x \in \bar{V}$, so we need to find $y_n \in V$ such that $y_n \to x$. Since x_n is a limit point of V, we may find $y_n \in V$ such that $d(x_n, y_n) < 1/n$. Then, by triangle inequality

$$d(x, y_n) \leq d(x, x_n) + d(x_n, y_n) < \frac{1}{n} + d(x, x_n)$$

and $d(x, y_n) \to 0$ which means $x \in \bar{V}$ and $\bar{V}$ is closed. [book]
(iii) Let $x \in \overline{U \cup V}$. Then there exists a sequence $x_n \in U \cup V$ such that $x_n \to x$. This means that at least one of the sets U or V (say, U for definiteness) contains infinitely many points of x_n, i.e., there is

a subsequence $x_{n_k} \in U$ which is convergent to x. Thus, $x \in \bar{U}$ and $\overline{U \cup V} \subset \bar{U} \cup \bar{V}$.

Assume now that $x \in \bar{U} \cup \bar{V}$. Then $x \in \bar{U}$ or $x \in \bar{V}$ and there exists a sequence $x_n \in U$ or $x_n \in V$ such that $x_n \to x$. In both cases $x_n \in U \cup V$ and therefore $x \in \overline{U \cup V}$. [seen similar]

e)

(i) The function $f : X \to Y$ is continuous at $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $d(f(x), f(x_0)) < \varepsilon$ for all $x \in X$ satisfying $d(x, x_0) < \delta$. [book]

(ii) This function is *not* continuous at $x = 0$. Indeed, let e_n be the nth coordinate vector in l_∞ and let $x^n := \frac{1}{n} e_n$. Then, obviously, $x^n \to 0$ as $n \to \infty$. But

$$f(x^n) = (1, 1, \cdots, \cos 1, 1, \cdots), \quad \|f(x^n) - f(0)\|_{l_\infty} = 1 - \cos 1.$$

Thus, $f(x^n)$ does not tend to $f(0)$ and the function is not continuous. [seen similar]

Question 2.

a)

(i) The metric space (X, d) is compact if any sequence in X has a convergent subsequence. [book]

(ii) The metric space (X, d) is separable if there exists a dense countable subset of it. [book

(iii) Let (X, d) be compact. Then, due to the Hausdorff criterion, for any $\varepsilon > 0$ there exists a finite ε-net $C_n \subset X$. Let us take $V := \cup_{n=1}^{\infty} C_n$. Then this set is countable (as a countable union of finite sets) and is dense. Indeed, let $x \in X$ be arbitrary. Then, by the definition of C_n, there exists $x_n \in C_n \subset C$ such that $d(x, x_n) < 1/n$. Thus $x_n \to x$ and $x \in \bar{C}$. [seen similar]

b)

(i) The space (X, d) is complete if any Cauchy sequence in it is convergent. [book]

(ii) The space $(\tilde{X}, \tilde{d})$ is a completion of the metric space (X, d) if

a) $(\tilde{X}, \tilde{d})$ is complete.

b) X is a dense subset of $(\tilde{X}, \tilde{d})$.

c) $\tilde{d}(x, y) = d(x, y)$ for all $x, y \in X$. [book]

(iii) This space is not complete. Indeed, let us consider the continuous approximations $f_n(x)$ to $f(x) = \frac{1}{\sqrt{x}}$ defined by

$$f_n(x) := \begin{cases} f(x), & x > \frac{1}{n}, \\ \sqrt{n}, & x \leq \frac{1}{n}. \end{cases}$$

Then

$$\|f - f_n\| = \sup_{x \leq 1/n} \{x(\frac{1}{\sqrt{x}} - \sqrt{n})\} = \frac{1}{\sqrt{n}} \sup_{y \leq 1} \{\sqrt{y} - y\} = \frac{1}{4\sqrt{n}}$$

and therefore $\{f_n\}$ is a Cauchy sequence in X:

$$\|f_n - f_{n+m}\| \leq \|f - f_n\| + \|f - f_m\| = \frac{1}{4\sqrt{n}} + \frac{1}{4\sqrt{n+m}} \leq \frac{1}{2\sqrt{n}}.$$

We claim that it does not have a limit in X. Indeed, let $g \in C[0, 1]$ is a limit of it. Then,

$$\|f - g\| \leq \lim_{n \to \infty} (\|f - f_n\| + \|f_n - g\|) = 0.$$

Thus, $f = g$ which is impossible since f is discontinuous at $x = 0$. [unseen]

c)

(i) The Lebesgue space $L_p(0,1)$, $1 \le p < \infty$ is a completion of the space $C[0,1]$ with respect to the norm

$$\|f\|_{L^1} := \left(\int_0^1 |f(x)|^p \, dx. \right)^{1/p}$$

[book]

(ii) The function f is continuous on $[0,1]$ except of two points $x = 0$ and $x = 1$. According to the criterion, we need to check whether or not the following improper Riemann integral is finite:

$$I := \lim_{\varepsilon \to 0} \int_\varepsilon^{1-\varepsilon} |f(x)| \, dx = \lim_{\varepsilon \to 0} \int_\varepsilon^{1/2} f(x) \, dx + \lim_{\varepsilon \to 0} \int_{1/2}^{1-\varepsilon} f(x) \, dx := I_1 + I_2$$

Let us start with I_2. Since $1 \le \frac{1}{\sqrt{x}} \le \sqrt{2}$, we have

$$J \le I_2 \le \sqrt{2} J, \quad J := \lim_{\varepsilon \to 0} \int_{1/2}^{1-\varepsilon} \frac{1}{1-x} \, dx.$$

Integral J can be found using the Newton-Leibnitz formula:

$$J = \lim_{\varepsilon \to 0} \ln \frac{1}{1-x} \Big|_{x=1/2}^{1-\varepsilon} = \lim_{\varepsilon \to 0} (\ln \frac{1}{\varepsilon} - \ln 2) = \infty$$

[seen similar]

and $f \notin L^1(0,1)$.

Question 3.

a) A function $f : X \to X$ on a metric space (X, d) is a contraction if there exists a number $\kappa < 1$ such that

$$d(f(x), f(y)) \leq \kappa d(x, y).$$

for all $x, y \in X$.

The Banach Contraction Theorem: If (X, d) is a complete metric space and f is a contraction on (X, d) then f has a unique fixed point p (i.e., the equation $f(x) = x$ has a unique solution $x = p$). This point can be obtained as a limit of iterations $x_{n+1} = f(x_n)$ starting from any point $x_0 \in X$. [book]

b)

(i) Let $x = 0$ and $y = (0, 1)$. Then $f(0) = 0$ and $f(y) = (5, 1/2)$. Therefore, $\|x - y\|_2 = 1$ and $\|f(x) - f(y)\|_2 = \frac{1}{2}\sqrt{101} > 1$ and f is not a contraction. [seen similar]

(ii) Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then

$$\|f(x) - f(y)\| = \|f(x - y)\| = |\frac{1}{2}(x_1 - y_1) + 5(x_2 - y_2)| + 10|x_2 - y_2| \leq$$
$$\leq \frac{1}{2}|x_1 - y_1| + 15|x_2 - y_2| \leq \frac{3}{4}(|x_1 - y_1| + 20|x_2 - y_2|) = \frac{3}{4}\|x - y\|$$

and f is a contraction with $\kappa = \frac{3}{4}$. [seen similar]

c) The map f is a contraction on $X = [0, \infty)$ since

$$|f'(x)| = \frac{1}{2}(6 + x)^{-1/2} \leq \frac{1}{2\sqrt{6}} < 1$$

and, due to the mean value theorem,

$$|f(x) - f(y)| \leq \kappa|x - y| \quad \text{with} \quad \kappa := \frac{1}{2\sqrt{6}} < 1.$$

Thus, by the Banach contraction theorem, the sequence x_n is convergent to p which is a unique fixed point for f on X. The fixed point satisfies $p^2 = p + 6$ and $p = 3$. [unseen]

d)

(i) Let $f_1, f_2 \in C[0,1]$. Then

$$|F(f_1)(x) - F(f_2)(x)| = \frac{1}{2}|(f_1(x) - f_2(x)) + \int_0^{x/2} s(f_1(2s) - f_2(2s))\,ds| \leq$$

$$\leq \frac{1}{2}|f_1(x) - f_2(x)| + \max_{s\in[0,1/2]} |f_1(2s) - f_2(2s)| \int_0^{x/2} s\,ds \leq$$

$$\leq (1/2 + 1/8)\|f_1 - f_2\|_{sup} = \frac{5}{8}\|f_1 - f_2\|_{sup}.$$

[seen similar] Therefore, $\|F(f_1) - F(f_2)\|_{sup} \leq 3/4\|f_1 - f_2\|_{sup}$ and F is contraction.

(ii) The unique fixed point $p \in C[0,1]$ should satisfy the equation

$$p(x) = \frac{1}{2}p(x) + 1 + \int_0^{x/2} sp(2s)\,ds.$$

From this equation we see that p must be continuously differentiable and $p(0) = 2$. Differentiating this equation by x, we find $\frac{1}{2}p'(x) - \frac{1}{4}xp(x) = 0$ and $p(x) = 2e^{x^2/4}$. [seen similar]

Question 4.

a)

(i) A *bi-linear* form (x, y) on a real vector space V is an inner product iff

1) It is symmetric: $(x, y) = (y, x)$;
2) Positive definite: $(x, x) \geq 0$ and $(x, x) = 0$ iff $x = 0$. [book]

(ii) Cauchy-Schwarz inequality: $|(x, y)| \leq \|x\|\|y\|$. [book]

(iii) Positivity and homogeneity are obvious, so we only need to check the triangle inequality. Let $x, y, z \in H$. Then, we need to check that $\|x + y\| \leq \|x\| + \|y\|$. Squaring this inequality and using the Cauchy-Schwarz, we get

$$\|x+y\|^2 = (x+y, x+y) = \|x\|^2+\|y\|^2+2(x, y) \leq \|x\|^2+\|y\|^2+2\|x\|\|y\| = (\|x\|+\|y\|)^2.$$

[book]

(iv) Bessel inequality: $\sum_{n=1}^{\infty}(x, e_n)^2 \leq \|x\|^2$. [book]

(v)

$$\|x+y\|^2+\|x-y\|^2 = \|x\|^2+\|y\|^2+2(x, y)+\|x\|^2+\|y\|^2-2(x, y) = 2(\|x\|^2+\|y\|^2)$$

[unseen]

(vi) Let $f(x) \equiv 1$ and $g(x) = x$. Then, $\|f\|_{sup} = \|g\|_{sup} = 1$, $\|f + g\|_{sup} = 2$ and $\|f - g\|_{sup} = 1$ and the parallelogram law is not satisfied. [unseen]

b)

(i) Extend the function $f(x) = x^3$ *periodically* from $x \in [-\pi, \pi]$ to the whole real line. Then the obtained function $f_{per}(x)$ will be piece-wise continuous and piece-wise smooth with jumps at points $x_n = \pi + 2\pi n$, $n \in \mathbb{Z}$. At jump points we define f_{per} by

$$f_{per}(x_n) = \frac{f_{per}(x_n+) + f_{per}(x_n-)}{2} = 0.$$

By the Dirichlet theorem, the Fourier sums converge point-wise to the limit function

$$f_{lim}(x) = f_{per}(x), \quad x \in \mathbb{R}. \tag{1}$$

This convergence is not uniform since the limit function is not continuous. [seen similar]

(ii) Let us put $x = \frac{\pi}{2}$ into the Fourier expansions of $f(x) = x^3$. Then, using $\sin(\pi k) = 0$ and $\sin((k+1/2)\pi) = (-1)^k$, we get

$$\frac{\pi^3}{8} = 2\sum_{k=0}^{\infty}(-1)^k\left(\frac{\pi^2}{2k+1} - \frac{6}{(2k+1)^3}\right) = 2\pi^2\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} - 12\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}.$$

[seen similar]

Since $\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} = \frac{\pi}{4}$, we get the desired formula.