MAT3004: ClassTest I. November 8, 2012

Question 1.

a) (1 point) Define what does it mean that two norms on a vector space are equivalent.

b) (3 points) Let $V = C[0,1]$ be the space of continuous functions and let

$$\|f\|_1 := \int_0^1 |f(x)|\,dx \quad \text{and} \quad \|f\|_2 := \int_0^1 x|f(x)|\,dx.$$

Are these norms equivalent? Justify your answer.

Question 2.

a) (2 points) Compute the l_1 and l_2-norms of a sequence $x = (x_1, x_2, \cdots)$ where $x_n := 2^{-n}$.

b) (3 points) Let e_n be the n-coordinate vector in the space of square summable sequences l_2 ($e_n = (0, \cdots, 0, 1(\text{on } n\text{th position}), 0, \cdots)$). Let also $x = (x_1, x_2, \cdots) \in l_2$ be an arbitrary square summable sequence and

$$x_N := \sum_{n=1}^{N} x_n e_n \in l_2.$$

Prove that $x_N \to x$ as $N \to \infty$ in l_2.

Question 3.

a) (1 point) Define what does it mean that x_n is a Cauchy sequence in a metric space X.

b) (2 points) Prove that any convergent sequence is a Cauchy sequence.

c) (1 point) Give an example of a complete and an example of a non-complete metric space.

d) (2 points) Define what does it mean that a metric space $(\tilde{X}, \tilde{d})$ is a *completion* of a metric space (X, d).

Solutions

Question 1 a) Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space V are equivalent if there are *positive* numbers l, L such that

$$l\|x\|_1 \le \|x\|_2 \le L\|x\|_1$$

for all $x \in V$.

b) These norms are not equivalent. Indeed, let $f_n(x) = 1 - nx$ for $x \in [0, 1/n]$ and $f_n(x) = 0$ for $x > 1/n$. Then

$$\|f\|_1 = \int_0^{1/n} (1 - nx)dx = \frac{1}{2n}, \quad \|f_n\|_2 = \int_0^{1/n} x(1 - nx)dx = \frac{1}{6n^2},$$

so $\|f_n\|_2$ converges as $n \to \infty$ *faster* that $\|f_n\|_1$ and the inequality $l\|f_n\|_1 \le \|f_n\|_2$ is violated for any positive l if n is large enough.

Question 2 a) $\|x\|_{l_1} = \sum_{n=1}^{\infty} 2^{-n} = 1$, $\|x\|_{l_2} = \sqrt{\sum_{n=1}^{\infty} 2^{-2n}} = \sqrt{\frac{1}{3}}$.

b) Obviously $x_N = (x_1, x_2, \cdots, x_N, 0, \cdots)$ and $x - x_N = (0, \cdots, 0, x_{N+1}, x_{N+2}, \cdots)$. Therefore,

$$\|x - x_N\|_{l_2}^2 = \sum_{n=N+1}^{\infty} x_n^2.$$

Since $x \in l_2$, $\sum_{n=1}^{\infty} x_n^2 = \|x\|_{l_2}^2 < \infty$. By this reason, $\sum_{n=N+1}^{\infty} x_n^2 \to 0$ as $N \to \infty$. This proves that $\|x - x_N\|_{l_2} \to 0$ as $N \to \infty$.

Question 3 a) A sequence x_n in a metric space (X, d) is a Cauchy sequence if for any $\varepsilon > 0$ there is $N = N(\varepsilon)$ such that

$$d(x_n, x_{n+m}) < \varepsilon$$

for all $n \ge N$ and all $m \in \mathbb{N}$.

b) Let $x_n \to x_0$. Then, for every $\varepsilon > 0$ there is $N = N(\varepsilon)$ such that $d(x_n, x_0) < \varepsilon/2$ if $n > N$. By the triangle inequality

$$d(x_n, x_{n+m}) \le d(x_n, x_0) + d(x_{n+m}, x_0) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

if $n > N$ (and, consequently, $n + m > N$ as well). Thus, x_n is Cauchy.

c) For instance, the space $\mathbb{R}$ with a usual metric is complete and the space $\mathbb{Q}$ is not.

d) A metric space $(\tilde{X}, \tilde{d})$ is a *completion* of a metric space (X, d) if

1) $(\tilde{X}, \tilde{d})$ is complete;

2) $X \subset \tilde{X}$ and X is dense in $\tilde{X}$;

3) $d(x, y) = \tilde{d}(x, y)$ for all $x, y \in X$.