MAT3004. COURSEWORK 2. AUTUMN 2012.
DEADLINE: DECEMBER 4, 2012

Problem 1. Let $X := C[0,\tau]$, $T > 0$ be the space of continuous functions on $[0,\tau]$ and let the function $T : X \to X$ be defined as follows:

$$(T\phi)(x) = 1 + \int_0^x \frac{\phi(s)}{2(1+s^2)}\, ds.$$

a) Prove that, for every $\tau > 0$, T is a *contraction* on $X = C[0,\tau]$ with the usual sup-norm.

b) Find the fixed point of this map.

Problem 2. Prove that it is *impossible* to find an inner product $(\cdot,\cdot)$ on $C[a,b]$ such that $\|f\|^2_{sup} = (f,f)$.

Problem 3. Orthogonalize functions $\{1, x, x^2, x^3\} \subset L^2(-1,1)$ with respect to the usual inner product $(f,g) := \int_{-1}^1 f(x)g(x)\, dx$.

Problem 4. Let $f(x) = \sin(x/2)$, $x \in [-\pi,\pi]$.

a) Find the coefficients a_n and b_n for the Fourier expansions of f:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

b) Find the point-wise limit of these series. Is the convergence *uniform*? Justify your answer.

c) Compute the sum $\sum_{n=1}^{\infty} \frac{(-1)^n n}{4n^2-1}$.

Solutions

Problem 1. For two functions $\phi_1, \phi_2 \in C[0,\tau]$, we have

$$|(T\phi_1)(x) - (T\phi_2)(x)| = |\int_0^x \frac{\phi_1(s)-\phi_2(s)}{2(1+s^2)}\,ds| \le \int_0^\tau \frac{|\phi_1(s)-\phi_2(s)|}{2(1+s^2)}\,ds \le$$
$$\le \|\phi_1-\phi_2\|_{sup}\int_0^\tau \frac{ds}{2(1+s^2)} \le \|\phi_1-\phi_2\|_{sup}\int_0^\infty \frac{ds}{2(1+s^2)} = \frac{\pi}{4}\|\phi_1-\phi_2\|_{sup}.$$

Since $\pi/4 < 1$, T is a contraction with the contraction factor $\pi/4$.

To find the fixed point $y \in C[0,\tau]$, we need to solve the equation

$$y(x) = 1 + \int_0^x \frac{y(s)}{2(1+s^2)}\,ds.$$

From this equation, we see that $y \in C^1[0,\tau]$ and $y(0) = 1$. Differentiating this equation in x, we have

$$y'(x) = \frac{y(x)}{2(1+x^2)}, \quad \frac{dy}{y} = \frac{dx}{2(1+x^2)}, \quad y(x) = Ce^{1/2\arctan(x)}$$

and using $y(0) = 0$, we see that $y(x) = e^{1/2\arctan(x)}$.

Problem 2. Assume that such scalar product exists. Then, due to the parallelogram law, we should have

$$2(\|f\|^2_{sup} + \|g\|^2_{sup}) = \|f+g\|^2_{sup} + \|f-g\|^2_{sup}.$$

Without loss of generality, we may consider the case of $C[-1,1]$ only. Indeed, there is an isometry $f \to \tilde{f}$

$$\tilde{f}(x) := f(\frac{b-a}{2}x + \frac{b+a}{2}), \quad f \in C[-1,1]$$

between $C[-1,1]$ and $C[a,b]$ which reduces the general case to the case of space $C[-1,1]$.

Take now $f(x) = x$ and $g(x) = 1$. Then,

$$\|f\|_{sup} = \|g\|_{sup} = 1, \quad \|f+g\|_{sup} = \|f-g\|_{sup} = 2$$

and the parallelogram law is violated. Thus, $C[a,b]$ is not a Hilbert space.

Problem 3. Note that $\{1, x^2\}$ is already orthogonal to $\{x, x^3\}$, so we only need to orthogonalize this two systems. We see that

$$\|1\|^2 = 2,\ (1,x^2) = \frac{2}{3};\quad \|x\|^2 = \frac{2}{3},\ (x,x^3) = \frac{2}{5}$$

and

$$e_1 = \frac{1}{\sqrt{2}},\ \tilde{e}_3 = x^2 - \frac{(1,x^2)}{\|1\|^2}\cdot 1 = x^2 - \frac{1}{3},\ \|\tilde{e}_3\|^2 = \int_{-1}^1 (x-1/3)^2\,dx = \frac{8}{45}.$$

Thus, $e_3 = \frac{3}{2}\sqrt{\frac{5}{2}}(x^2 - \frac{1}{3})$. Analogously, $e_2 = \sqrt{\frac{3}{2}}x$ and

$$\tilde{e}_4 = x^3 - \frac{(x, x^3)}{\|x\|^2}x = x^3 - \frac{3}{5}x, \ \|\tilde{e}_4\|^2 = \int_{-1}^{1}(x^3 - \frac{3}{5}x)^2\,dx = \frac{8}{175}$$

and $e_4 = \frac{5}{2}\sqrt{\frac{7}{2}}(x^3 - \frac{3}{5}x)$.

Problem 4. The function is odd, so all $a_n = 0$. To compute b_n, we use that $\sin(x/2)\sin(nx) = \frac{1}{2}(\cos(n-1/2)x - \cos(n+1/2)x)$. Therefore,

$$\begin{aligned} b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(x/2)\sin(nx)\,dx = \frac{1}{\pi}\int_{0}^{\pi}\cos(n+1/2)x - \cos(n-1/2)x\,dx = \\ = \frac{1}{\pi}\left(\frac{1}{n+1/2}\sin(n+1/2)\pi - \frac{1}{n-1/2}\sin(n-1/2)\pi\right) = \\ = \frac{-\cos(\pi n)}{\pi}(\frac{1}{n+1/2} + \frac{1}{n-1/2}) = -\frac{8n(-1)^n}{\pi(4n^2-1)}. \end{aligned}$$

Thus,

$$\sin(x/2) \sim -\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{4n^2-1}\sin(nx). \tag{1}$$

The function $f(x) = \sin(x/2)$ is piece-wise C^1 and its periodic extension $f_{per}(x)$ has jumps at $x = 2\pi n + \pi$, $n \in \mathbb{Z}$. Due to the Dirichlet theorem, we have *point-wise* convergence of the partial sums

$$f_N(x) \to \begin{cases} f_{per}(x), & x \neq (2n+1)\pi \\ 0, & x = (2n+1)\pi. \end{cases}$$

The convergence is not *uniform* since $f_{per}(x)$ is not continuous.

c) Oops! It seems that you cannot compute the series by just taking the proper value of x in (1). However, the series can be reduced to the telescopic one and computation of b_n gives a hint how to do that. Indeed, $\frac{n}{4n^2-1} = \frac{1}{8}(\frac{1}{n-1/2} + \frac{1}{n+1/2})$ and therefore

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{(-1)^n n}{4n^2-1} = \frac{1}{8}\left(-(\frac{1}{1/2} + \frac{1}{3/2}) + (\frac{1}{3/2} + \frac{1}{5/2}) - (\frac{1}{5/2} + \frac{1}{7/2}) + \cdots\right) = \\ = -\frac{1}{8}\frac{1}{1/2} = -\frac{1}{4}. \end{aligned}$$