UNIVERSITY OF SURREY©

Faculty of Engineering and Physical Sciences

Department of Mathematics

Undergraduate Programmes in Mathematics

Module MAT3004; 15 Credits

Introduction to Function Spaces

FHEQ Level 6 (Year 3) Examination

Time allowed: Two hours

Semester 2 2015/16

Answer **THREE** questions only.

If you attempt more than THREE questions, only your best THREE answers will be taken into account.

Each question carries 25 marks.

Where appropriate the mark carried by an individual part of a question is indicated in square brackets [ ].

Approved calculators are allowed.

Additional material:
0 Handouts

Question 1

(a) Let (X, d) be a metric space and $U \subset X$ be a subset of X.

(i) Give the definition of a *closure* $\bar{U}$ and an *interior* $\text{int}(U)$ of U in (X, d). [3]

(ii) Prove by first principles that the closure $\bar{U}$ is a closed set. [5]

(iii) Let V be another subset of X. Prove that

$$\overline{U \cap V} \subset \bar{U} \cap \bar{V}.$$

[4]

(iv) Give an example of a metric space (X, d) and sets $U, V \subset X$ such that

$$\overline{U \cap V} \neq \bar{U} \cap \bar{V}.$$

[3]

(v) Let $U := \{x_0\}$ be a one-point set. Could it be *open*? Justify your answer. [3]

(b) Let V be a vector space.

(i) Define what it means for two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V to be *equivalent*. [2]

(ii) Let $V := C[0, 1]$ be the space of continuous functions $f : [0, 1] \to \mathbb{R}$ and

$$\|f\|_1 := \int_0^1 (x - \sin(x))|f(x)|\, dx, \quad \|f\|_2 := \int_0^1 x^3|f(x)|\, dx.$$

Are these norms *equivalent*? Justify your answer. [5]

Question 2

(a) Let (X, d) be a metric space.

(i) Define what it means for X to be called *complete*, *totally bounded*, and *separable*. [3]

(ii) Define what it means for X to be called *compact*. [2]

(iii) Let $f : X \to \mathbb{R}$ be *continuous* and X be *compact*. Prove that $\max_{x \in X} f(x)$ exists. [4]

(iv) Give an example of a metric space X and a *continuous* function $f : X \to \mathbb{R}$ such that $\sup_{x \in X} f(x) < \infty$, but $\max_{x \in X} f(x)$ does not exist. [3]

(v) Let X and Y be compact metric spaces and $f : X \to Y$ be *continuous* and *bijective* (one-to-one and onto). Prove that $f^{-1} : Y \to X$ is also *continuous*. [6]

(b) State the definition of the Lebesgue space $L^p(0, 1)$, $1 \leq p < \infty$. [2]

(c) Let

$$f(x) := \frac{1}{(1 - x)\sqrt{x}}.$$

Does f belong to $L^1(0, 1)$? Justify your answer. [5]

Question 3

(a) Let (X, d) be a metric space and $F : X \to X$ be a function. Define what it means for F to be a *contraction* on X and state the Banach contraction theorem. [4]

(b) Let X be complete metric space and $F : X \to X$ be such that its nth iteration

$$F^{(n)}(x) := \underbrace{F(F(\cdots F}_{n\text{-times}}(x)\cdots))$$

is a contraction on X. Prove that F has a *unique* fixed point on X (you may use the Banach contraction theorem without proving it). [4]

(c) Let $F(x) := \frac{1}{1+x}$, $x \in \mathbb{R}_+ := [0, \infty)$.

(i) Prove that the function F is *not* a contraction on the metric space $X = \mathbb{R}_+$ with the usual metric $d(x, y) := |x - y|$. [3]

(ii) Prove that its second iteration $F^{(2)}(x)$ is a contraction. [4]

(iii) Find the fixed point of F. [2]

(d) Let $X := C[-1, 1]$ be the space of continuous functions $f : [-1, 1] \to \mathbb{R}$ and let the map $F : X \to X$ be defined as follows:

$$F(f)(x) := 1 + \int_0^x y f(y)\, dy.$$

(i) Prove that F is a *contraction* on X with the usual sup-norm. [5]

(ii) Find the fixed point of F. [3]

Question 4

(a) Let H be a vector space.

(i) Give the definition of the *inner* product on H. [3]

(ii) State the Cauchy-Schwarz inequality for the inner product space $(H, (\cdot, \cdot))$. [1]

(iii) Prove the triangle inequality for the norm $\|x\| := (x, x)^{1/2}$ on $(H, (\cdot, \cdot))$. [4]

(b) Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal system in a *Hilbert* space H and let

$$f \sim \sum_{n=1}^{\infty} f_n e_n, \quad f_n = (f, e_n)$$

be the abstract Fourier expansion of the element f.

(i) State the Bessel inequality. [2]

(ii) State the condition on $\{e_n\}$ which guarantees that this inequality is an *equality* for all $f \in H$. [2]

(c) Let $f(x) = x$, $x \in [-\pi, \pi]$ with the classical Fourier series

$$f(x) \sim -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$$

(the coefficients a_n and b_n are already found for you). Let also

$$f_N(x) := -2 \sum_{n=1}^{N} \frac{(-1)^n}{n} \sin nx$$

be the corresponding partial sums.

(i) What is the point-wise limit of $f_N(x)$ as $N \to \infty$? Is the convergence *uniform* with respect to $x \in [-\pi, \pi]$? Justify your answer. [5]

(ii) Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Hint: use the Parseval equality. [4]

(d) Consider the trigonometric series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin nx.$$

Is it a classical Fourier series for some function $f \in L^2(-\pi, \pi)$? Justify your answer. Hint: use the Parseval equality. [4]

END OF PAPER

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