MAT3004. Function spaces. MOCK-test 1. October 30th 2018.

Question 1

a) (1 point). Define what sets are called *open* in a metric space (X, d).

b) (1 point). Define what norms on a vector space V are called *equivalent*.

c) (1 point). Define what sequence in a metric space (X, d) is called Cauchy.

d) (1 point). Define what metric spaces are called *separable*.

Question 2

a) (2 points). Prove that, for every $A, B \subset (X, d)$,

$$\mathrm{int}(A) \cup \mathrm{int}(B) \subset \mathrm{int}(A \cup B).$$

b) (2 points). Give an example where the inclusion is *proper*.

Question 3 (4 points). Let $X := C[0, 1]$ be the space of continuous functions $f : [0, 1] \to \mathbb{R}$ and let

$$\|f\|_1 := \max_{x\in[0,1]} |f(x)|, \quad \|f\|_2 := \max_{x\in[0,1]} \{x|f(x)|\} + \max_{x\in[0,1]} \{(1 - x)|f(x)|\}.$$

Are these norms *equivalent*? Justify your answer.

Solutions

Question 1

a) The set $V \subset X$ is open if for every $x \in V$ there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset V$.

b) Two norms $\|x\|_1$ and $\|x\|_2$ are equivalent if there exist two positive numbers $l, L \in \mathbb{R}$ such that

$$l\|x\|_1 \leq \|x\|_2 \leq L\|x\|_1, \quad \forall x \in V.$$

c) A sequence $\{x_n\}_{n=1}^{\infty}$ is Cauchy if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{R}$ such that $d(x_n, x_{n+m}) < \varepsilon$ if $n > N$ and $m \in \mathbb{N}$.

d) A metric space (X, d) is separable if there is a countable dense subset in it.

Question 2

a) Let $x \in \operatorname{int}(A) \cup \operatorname{int}(B)$. Then, it belongs either to the first set or to the second one. Therefore, there exists $\varepsilon > 0$ such that either $B_\varepsilon(x) \subset A$ or $B_\varepsilon(x) \subset B$. In both cases, $B_\varepsilon(x) \subset A \cup B$ and $x \in \operatorname{int}(A \cup B)$.

b) Let $X = \mathbb{R}$ with the standard topology, $A := \mathbb{Q}$ and $B := \mathbb{R} \backslash \mathbb{Q}$. Then, $\operatorname{int} A = \operatorname{int} B = \varnothing$, but $\operatorname{int}(A \cup B) = \mathbb{R}$.

Question 3. The norms are equivalent. Indeed,

$$\|f\|_2 \leq \|f\|_1 + \|f\|_1 = 2\|f\|_1$$

since $\max_{x \in [0,1]} x = \max_{x \in [0,1]}(1 - x) = 1$. On the other hand,

$$\begin{aligned} \|f\|_1 = \max_{x \in [0,1]} |f(x)| = \max_{x \in [0,1]} \{x|f(x)| + (1-x)|f(x)|\} \leq \\ \leq \max_{x \in [0,1]} \{x|f(x)|\} + \max_{x \in [0,1]} \{(1-x)|f(x)|\} = \|f\|_2. \end{aligned}$$